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THE RATIO OF INVARIANT METRICS ON THE ANNULUS AND THETA FUNCTIONS

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Introduction. Let $A = A_q$ be the annulus with parameter $q \in (0,1)$:

$$A_q = \{\lambda \in \mathbb{C}; q < |\lambda| < 1\}.$$

Let C^A, K^A, and P^A be the Carathéodory, the Kobayashi, and the P-metric on A, respectively (for the definition of P^A see Section 1). Since all the metrics C^A, K^A, and P^A are invariant for biholomorphic mappings and since A is one-dimensional, the functions $CP^A(\lambda) := C^A(X)/P^A(X)$ and $KP^A(\lambda) := K^A(X)/P^A(X)$ for X a non-zero holomorphic tangent vector at $\lambda \in A$ are well-defined as functions on A and invariant for holomorphic automorphisms of A.

The main purpose of this paper is to show the following.

THEOREM A. *Let $r \in (0,1)$ be defined by*

$$\frac{\log q}{\pi i} = \frac{\pi i}{-\log r}. \tag{0.1}$$

For every $\lambda \in A = A_q$ with $v \in (0,1)$ such that

$$|\lambda| = q^v, \tag{0.2}$$

we have

$$CP^A(\lambda) = \prod_{n\ge 1} \frac{|e^{2\pi i v} + r^{2n-1}|^2}{(1 + r^{2n-1})^2}, \tag{0.3}$$

$$KP^A(\lambda) = \prod_{n\ge 1} \frac{|e^{2\pi i v} - r^{2n}|^2}{(1 - r^{2n})^2}. \tag{0.4}$$

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Consequently, the functions $\alpha : (0,1) \ni v \mapsto CP^A(q^v) \in [0,+\infty)$ *and* $\beta : (0,1) \ni v \mapsto KP^A(q^v) \in [0,+\infty)$ *are unimodal; moreover,* α (*resp.* β) *is strictly decreasing* (*resp. increasing*) *in* $(0,1/2)$ *and strictly increasing* (*resp. decreasing*) *in* $(1/2,1)$; *therefore*,

$$\min CP^A = \min \alpha = \alpha(1/2) = \prod_{n\ge 1} \frac{(1-r^{2n-1})^2}{(1+r^{2n-1})^2},$$

$$\max KP^A = \max \beta = \beta(1/2) = \prod_{n\ge 1} \frac{(1+r^{2n})^2}{(1-r^{2n})^2}.$$

Assertion (0.4) appeared in the proof of Proposition 3.4 in [2] and its proof in this paper is different from that in [2], which comes from Myrberg's theorem on the Green function of a hyperbolic Riemann surface. The argument of this paper is based on the theory of theta functions attached to the tori $\mathbb{T}(1,\tau) = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and $\mathbb{T}(1,-1/\tau) = \mathbb{C}/(\mathbb{Z}+(-1/\tau)\mathbb{Z})$, where $\tau \in H = \{\tau \in \mathbb{C};\ \operatorname{Im}\tau > 0\}$ is the number given by

$$\tau = \frac{\log q}{\pi i}, \qquad -\frac{1}{\tau} = \frac{\log r}{\pi i} \tag{0.5}$$

(see (0.1)). In fact, the functions CP^A and KP^A are directly represented by a ratio of theta functions attached to the torus $\mathbb{T}(1,-1/\tau)$ (Theorem C in Section 3).

Theorem A is important because as its consequence we get the following well-known fact: All holomorphic automorphisms of A consist of the functions $(\lambda \mapsto e^{i\theta}\lambda)_{\theta\in\mathbb{R}}$ and $(\lambda \mapsto e^{i\theta}q/\lambda)_{\theta\in\mathbb{R}}$. Indeed, let $C_s = \{\lambda \in A; |\lambda| = s\}$ for $s \in (q,1)$. Since the functions $r : A \ni \lambda \mapsto q/\lambda \in A$ and $A \ni \lambda \mapsto e^{i\theta}\lambda \in A$ for $\theta \in \mathbb{R}$ are automorphisms of A, we see that CP^A is constant on each C_s and that $CP^A(C_s) = CP^A(C_{q/s})$. Let φ be a holomorphic automorphism of A. Theorem A implies that for every $s \in (q,1)$, $\varphi(C_s)$ coincides with C_s or $C_{q/s}$. Since the function $(q,1) \ni s \mapsto |\varphi(s)| \in (q,1)$ is a homeomorphism, it follows that either $\varphi(C_s) = C_s$ for all s, or $\varphi(C_s) = C_{q/s}$ for all s. Assume first that $\varphi(C_s) = C_s$ for all s. Then the function $\varphi(\lambda)/\lambda$ has modulus 1 on A so that $\varphi(\lambda) = e^{i\theta}\lambda$, $\lambda \in A$ for some real θ. If $\varphi(C_s) = C_{q/s}$ for all s, then the last argument implies that $r \circ \varphi(\lambda) = e^{i\theta}\lambda, \lambda \in A$ for some real θ, as desired.

We also obtain the representation of CP^A in terms of the Green function of A.

THEOREM B. *If* $G^A(\cdot,\lambda)$ *is the Green function on* A *with pole at* $\lambda \in A$, *then*

$$CP^A(\lambda) = \exp\left(-G^A(-q/\overline{\lambda}, \lambda)\right) \tag{0.6}$$

for $\lambda \in A$.

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1. Invariant metrics on the annulus. For a point $p \in M$ of a complex manifold M, we define a subspace $PS^M(p)$ of the space $NPS(M)$ of all negative plurisubharmonic functions on M as follows:

$$PS^M(p) = \{f \in NPS(M); f(q) - \log \|z(q) - z(p)\| \le O(1) \text{ as } q \to p\},$$

where z is a holomorphic coordinate around p and $\| \ \|$ means the complex euclidian norm on $\mathbb{C}^m, m = \dim M$. Here, we assume the function $-\infty$ identically belongs to $NPS(M)$. The definition of $PS^M(p)$ does not depend on the choice of the coordinate z. For $q \in M$, let

$$u_p^M(q) = u^M(q,p) = \sup\{f(q); f \in PS^M(p)\}.$$

The function u_p^M is called the *pluri-complex Green function* with pole at p (cf. [14], [9], [1], [2], [3], [6], [10], [8], [11]).

Let $X \in T_pM$ be a holomorphic tangent vector at $p \in M$. Let $E = \{\lambda \in \mathbb{C}; |\lambda| < 1\}$ be the unit disk in $\mathbb{C}$. Taking a holomorphic function φ from an ε-neighborhood εE of 0 in $\mathbb{C}$ to M with $\varphi(0) = p$ and $\varphi'(0) = X$, we define

$$P^M(X) = \limsup_{\lambda \to 0, \lambda \neq 0} \frac{\exp \circ u_p^M \circ \varphi(\lambda)}{|\lambda|}$$

(cf. [1], [2], [6], [10], [11]). The definition of $P^M(X)$ does not depend on the choice of φ (cf. [2], [6]), and the function P^M is a pseudo-metric on M, that is, P^M is [0,+∞)-valued on the holomorphic tangent bundle TM satisfying $P^M(\lambda X) = |\lambda| P^M(X)$ for any $X \in TM$ and $\lambda \in \mathbb{C}$. The assignment $M \mapsto P^M$ of pseudometrics possesses the decreasing property, i.e., for a holomorphic mapping Φ from M to M', $P^{M'}(\Phi_* X) \le P^M(X)$ for all $X \in TM$ and the metric P^E for the unit disk E in $\mathbb{C}$ coincides with the Poincaré metric on E, which implies that if C^M and K^M denote the Carathéodory and the Kobayashi pseudo-metrics respectively, then $C^M \le P^M \le K^M$ for any complex manifold M (cf. [1], [2], [6]). Furthermore, if by $IS^M(p) = \{X \in T_pM; S^M(X) < 1\}$ we denote the indicatrix at $p \in M$ for a pseudo-metric S^M on M, then the following are well-known:

(1) $IC^M(p)$ is convex for all $p \in M$ ([5]).
(2) $IP^M(p)$ is pseudoconvex for all $p \in M$ ([2]).
(3) $IK^M(p)$ is not necessarily pseudoconvex ([7]).

If M is a hyperbolic Riemann surface, then the function $-u_p^M$ is the usual Green function $G^M(\cdot, p)$ of M with pole at p (cf. [9], [1]). Let z be a holomorphic coordinate around p and $\mu(d/dz)_p, \mu \in \mathbb{C}$, be a holomorphic tangent vector at p. If

$$\varphi := z^{-1} \circ (\varepsilon E \ni \lambda \mapsto z(p) + \mu\lambda \in \mathbb{C}) : \varepsilon E \to M,$$

then $\varphi(0) = p$ and $\varphi'(0) = \varphi_*((d/d\lambda)_0) = \mu(d/dz)_p$, so that

$$P^M\left(\mu\left(\frac{d}{dz}\right)_p\right) = |\mu| \left| \left. \frac{d \exp \circ u_p^M \circ z^{-1}(z(p) + \lambda)}{d\lambda} \right|_{\lambda=0} \right|. \tag{1.1}$$

It is well-known ([17], [9], [2]) that the pluri-complex Green function u^A on the annulus $A = A_q$ is given by

$$u^A_\lambda(\mu) = (1-v)\log|\mu| + \log|\Theta_\lambda(\mu)| \qquad (\lambda, \mu \in A), \tag{1.2}$$

where

$$\Theta_\lambda(\mu) = \frac{\prod_{n\ge1}(1-q^{2n}\mu/\lambda)(1-q^{2n-2}\lambda/\mu)}{\prod_{n\ge1}(1-q^{2n-2}\overline{\lambda}\mu)(1-q^{2n}/(\overline{\lambda}\mu))}$$

and $v = v(\lambda) \in (0,1)$ with

$$q^v = |\lambda|. \tag{1.3}$$

It follows from (1.1) that

$$P^A\left(\left(\frac{d}{d\lambda}\right)_\lambda\right) = \frac{q^{-v^2}\prod_{n\ge1}(1-q^{2n})^2}{\prod_{n\ge1}(1-q^{2n-2+2v})(1-q^{2n-2v})}. \tag{1.4}$$

We note that the Kobayashi metric K^A on A coincides with the usual Poincaré metric on A by virtue of the following fact ([6], [13]): If $\pi : N \to M$ is a (not necessarily universal) covering of a complex manifold M, then $K^M(\pi_* X) = K^N(X)$ for all $X \in TM$. Let $H = \{\eta \in \mathbb{C}; \operatorname{Im}\eta > 0\}$ be the upper half plane in $\mathbb{C}$. Since the mapping $H \ni \eta \mapsto e^{\tau\log\eta} \in A$ with

$$\tau = \frac{\log q}{\pi i}$$

is a covering on A ([2]), and since $|d\eta|/(2\operatorname{Im}\eta)$ is the Poincaré metric on H, we see

$$K^A\left(\left(\frac{d}{d\lambda}\right)_\lambda\right) = \frac{\pi}{(-2\log q)q^v\sin\pi v} \tag{1.5}$$

for $\lambda \in A$ with v as in (1.3).

Concerning the Carathéodory metric C^A on A, the following is well-known ([17], [2]): For $\lambda \in A$ with v in (1.3),

$$C^A\left(\left(\frac{d}{d\lambda}\right)_\lambda\right) = \frac{\prod_{n\ge1}(1-q^{2n})^2(1+q^{2n-1+2v})(1+q^{2n-1-2v})}{\prod_{n\ge1}(1+q^{2n-1})^2(1-q^{2n-2+2v})(1-q^{2n-2v})}. \tag{1.6}$$

2. Theta functions and their transformation formulas. By $\mathbb{T}(\omega_1,\omega_2)$ we denote the torus $\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ with basic periods (ω_1,ω_2) satisfying $\omega_2/\omega_1 \in H$; the number ω_2/ω_1 is called the modulus of the torus $\mathbb{T}(\omega_1,\omega_2)$. For $\tau \in H$ and $v \in \mathbb{C}$, let

$$\theta_0^*(v,\tau) = 2e^{\frac{\pi i\tau}{4}}\prod_{n\ge1}(1-e^{2n\pi i\tau})(1-e^{2\pi i(n\tau+v)})(1-e^{2\pi i(n\tau-v)}), \tag{2.1}$$

$$\theta_0(v,\tau) = (\sin\pi v)\theta_0^*(v,\tau),$$

$$\theta_3(v,\tau) = \prod_{n\ge1}(1-e^{2n\pi i\tau})(1+e^{2\pi i((n-1/2)\tau+v)})(1+e^{2\pi i((n-1/2)\tau-v)}) \tag{2.2}$$

(cf. [4, p. 69]). Then, the functions $\theta_j(\cdot,\tau)$ $(j=0,3)$ are two of four theta functions attached to the torus $\mathbb{T}(1,\tau)$ and satisfy

$$\theta_0(v+1,\tau) = -\theta_0(v,\tau), \quad \theta_0(v+\tau,\tau) = -q^{-1}e^{-2\pi i v}\theta_0(v,\tau),$$
$$\theta_3(v+1,\tau) = \theta_3(v,\tau), \quad \theta_3(v+\tau,\tau) = q^{-1}e^{-2\pi i v}\theta_3(v,\tau)$$

(cf. [4, pp. 58, 64]).

Since we have holomorphic isomorphisms

$$\mathbb{T}(1,\omega_2/\omega_1) \cong \mathbb{T}(\omega_1,\omega_2) \cong \mathbb{T}(-\omega_2,\omega_1) \cong \mathbb{T}(1,-\omega_1/\omega_2),$$

the second one of which comes from the mapping $\mathbb{C} \ni \lambda \mapsto \lambda - \omega_1 \in \mathbb{C}$, if $\tau \in H$, then $\mathbb{T}(1,\tau) \cong \mathbb{T}(1,-1/\tau)$. We need the transformation formulas connecting $\theta_j(\cdot,\tau)$ and $\theta_j(\cdot,-1/\tau)$ for $j=0,3$. If $v \in \mathbb{C}$ and $\tau \in H$, then

$$\theta_0(v,-1/\tau) = -ie^{\pi i\tau v^2}\sqrt{\tau/i}\,\theta_0(\tau v,\tau), \tag{2.3}$$

$$\theta_3(v,-1/\tau) = e^{\pi i\tau v^2}\sqrt{\tau/i}\,\theta_3(\tau v,\tau), \tag{2.4}$$

where the square root is taken so that $\sqrt{\tau/i} = 1$ for $\tau = i$ (cf. [4, pp. 73, 75]).

3. Proof of Theorem A and Theorem B.

3. Proof of Theorem A and Theorem B. We first show the following.

THEOREM C. *Let $\tau \in H$ be defined by*

$$q = e^{\pi i\tau}. \tag{3.1}$$

For $\lambda \in A$ with $v = v(\lambda) \in (0,1)$ such that

$$|\lambda| = q^v, \tag{3.2}$$

we have

$$CP^A(\lambda) = \frac{\theta_3(v,-1/\tau)}{\theta_3(0,-1/\tau)}, \tag{3.3}$$

$$KP^A(\lambda) = \frac{\theta_0^*(v,-1/\tau)}{\theta_0^*(0,-1/\tau)}. \tag{3.4}$$

We note that (3.1) is equivalent to (0.5).

P r o o f o f T h e o r e m C. Items (1.4) and (1.6) imply that

$$CP^A(\lambda) = \frac{q^{v^2}\prod_{n\ge1}(1+q^{2n-1-2v})(1+q^{2n-1+2v})}{\prod_{n\ge1}(1+q^{2n-1})^2}. \tag{3.5}$$

Using (2.1) to get

$$\theta_3(\tau v,\tau) = \prod_{n\ge1}(1-q^{2n})(1+q^{2n-1+2v})(1+q^{2n-1-2v}),$$

we have

$$CP^A(\lambda) = \frac{q^{v^2}\theta_3(\tau v,\tau)}{\theta_3(0,\tau)}. \tag{3.6}$$

By the transformation formula (2.4) we have

$$\theta_3(v,-1/\tau) = q^{v^2}\sqrt{\tau/i}\,\theta_3(\tau v,\tau);$$

therefore, assertion (3.3) follows.

Similarly, items (1.4) and (1.5) imply that

$$KP^A(\lambda) = \frac{\pi}{-\log q}\frac{q^{v^2}}{2q^v\sin\pi v}\frac{\prod_{n\geq 1}(1-q^{2n-2+2v})(1-q^{2n-2v})}{\prod_{n\geq 1}(1-q^{2n})^2}.$$

Since $\sin\pi\tau v = (1-q^{2v})/(2iq^v)$, it follows that

$$KP^A(\lambda) = \frac{q^{v^2}}{\tau}\frac{\sin\pi\tau v}{\sin\pi v}\frac{\prod_{n\geq 1}(1-q^{2n+2v})(1-q^{2n-2v})}{\prod_{n\geq 1}(1-q^{2n})^2}.$$

Using (2.1) to get

$$\theta_0^*(\tau v,\tau) = 2q^{1/4}\prod_{n\geq 1}(1-q^{2n})(1-q^{2n+2v})(1-q^{2n-2v}),$$

we have

$$KP^A(\lambda) = \frac{q^{v^2}}{\tau}\frac{\sin\pi\tau v}{\sin\pi v}\frac{\theta_0^*(\tau v,\tau)}{\theta_0^*(0,\tau)}. \tag{3.7}$$

By the transformation formula (2.3) we have

$$\theta_0^*(v,-1/\tau)\sin\pi v = -iq^{v^2}\sqrt{\tau/i}\,(\sin\pi\tau v)\theta_0^*(\tau v,\tau).$$

Dividing both sides by $\sin\pi v$ and taking the limit as $v\to 0$, we see

$$\theta_0^*(0,-1/\tau) = -i\sqrt{\tau/i}\,\tau\theta_0^*(0,\tau),$$

so that we get

$$\frac{\theta_0^*(v,-1/\tau)}{\theta_0^*(0,-1/\tau)} = \frac{\sin\pi\tau v}{\sin\pi v}\frac{q^{v^2}}{\tau}\frac{\theta_0^*(\tau v,\tau)}{\theta_0^*(0,\tau)}.$$

Combining this with (3.7) we obtain formula (3.4) and complete the proof of Theorem C.

We shall show Theorem A stated in Introduction.

P r o o f o f T h e o r e m A. By virtue of (3.2), using the definition (2.2) of $\theta_3(\cdot,-1/\tau)$, noticing the fact

$$r = e^{-2\pi i/\tau}$$

(see (0.1)), we have

$$CP^A(\lambda) = \frac{\prod_{n\geq 1}(1+r^{2n-1}e^{2\pi iv})(1+r^{2n-1}e^{-2\pi iv})}{\prod_{n\geq 1}(1+r^{2n-1})^2}.$$

Since $\overline{e^{2\pi iv}} = e^{-2\pi iv}$ because v is real, we have obtained assertion (0.3) in Theorem A.

Similarly, by virtue of (3.3), using (2.1) we have

$$KP^A(\lambda) = \frac{\prod_{n\ge1}(1 - r^{2n}e^{2\pi iv})(1 - r^{2n}e^{-2\pi iv})}{\prod_{n\ge1}(1 - r^{2n})^2},$$

and assertion (0.4) in Theorem A. The proof is complete.

Proof of Theorem B. We first note that the Green function $G^A(\cdot, \lambda)$ of A with pole at $\lambda \in A$ coincides with $-u^A_\lambda$ (see Section 1). It follows from (1.2) that

$$\begin{aligned}\exp(-G^A(-q/\overline{\lambda}, \lambda)) &= \exp u^A(-q/\overline{\lambda}, \lambda)\\ &= \frac{q^{(1-v)^2}\prod_{n\ge1}(1 + q^{2n}q/|\lambda|^2)(1 + q^{2n-2}|\lambda|^2/q)}{\prod_{n\ge1}(1 + q^{2n-1})^2}\\ &= \frac{q^{v^2}\prod_{n\ge1}(1 + q^{2n-1-2v})(1 + q^{2n-1+2v})}{\prod_{n\ge1}(1 + q^{2n-1})^2}.\end{aligned}$$

By virtue of (3.5) we have proved the desired assertion of Theorem B.

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