AN ESTIMATE FROM BELOW FOR THE MARKOV CONSTANT OF A CANTOR REPELLER

ALEXANDER VOLBERG

Department of Mathematics, Michigan State University
East Lansing, Michigan 48824, U.S.A.

1. Here we will be concerned with the following version of the classical Markov inequality:

$$(1) \qquad \|p'\|_J \le M(\deg p)^r \|p\|_J,$$

where $\| \; \|_J$ stands for the supremum norm on compact J, $J \subset \mathbb{C}$, and p denotes complex polynomials.

So we deal with a particular case of the inequality

$$(2) \qquad \|D^\alpha p\|_E \le M(\deg p)^{r|\alpha|} \|p\|_E,$$

E is a compact in $\mathbb{R}^n$, p is a polynomial of n real variables.

When J lies on a line in $\mathbb{C}$ (1) becomes exactly the one-dimensional version of (2).

Let us start with mentioning that the description of compacts E (or J) satisfying (2) is not known.

Some partial results are known. W. Pawłucki and W. Pleśniak [1], [2] showed that (2) holds whenever E is uniformly polynomially cuspidal (UPC). In particular, every subanalytic compact set E with $E = \overline{\operatorname{int} E}$ being UPC satisfies (2).

J. Siciak gave an important sufficient condition for (2). Let us consider $E \subset \mathbb{R}^n \subset \mathbb{C}^n$. For (2) to hold, it is sufficient that the function

$$G_E(a) = \sup \left\{ \frac{1}{\deg p} \log |p(z)| : p \text{ is a polynomial on } \mathbb{C}^n, \; \deg p \ge 1, \; \|p\|_E \le 1 \right\}$$

satisfies the following estimate

$$(3) \qquad G_E(z) \le M d(z)^m, \quad d(z) \stackrel{\text{def}}{=} \operatorname{dist}(z, E).$$

Moreover, the constants in (2) and (3) are connected by an inequality

$$(4) \qquad\qquad\qquad r \le m^{-1}.$$

Siciak constructed a Cantor type set on $\mathbb{R}$ which satisfies (3). This shows that the class of sets for which (2) holds is strictly larger than UPC.

Using Siciak's implication (3)$\Rightarrow$(2) L. Białas and the author showed in [4] that standard Cantor sets on $\mathbb{R}$ have property (3), so (1). It is not clear how necessary Siciak's condition is for Markov's inequality.

Let us remind how the inequalities (2), (1) can be applied. In [1], [2] it was proved that Markov's inequality is equivalent to the existence on linear continuous right inverse to the restriction operator $C^\infty(\mathbb{R}^n) \ni f \to f|E$. Another way of expressing this is as follows. Let $\mathcal{P}_\ell$ denote all polynomials of degree ℓ. If $\mathrm{dist}_E(f, \mathcal{P}_\ell)$ tends to zero faster than any ℓ^{-n} then f is a restriction on E of a function from $C^\infty(\mathbb{R}^n)$. This form of Bernstein's theorem is also proved by W. Pleśniak.

The goal of this paper is to get some estimates on m and r when J is a Cantor repeller in $\mathbb{C}$.

2. Cantor repellers. Let $U, U_1, \ldots, U_d$ be topological discs with real analytic boundaries such that $\bar{U}_i \subset U$, $i = 1, \ldots, d$. Consider a map $f : \bigcup_{i=1}^d U_i \to U$ which is univalent on $\bar{U}_i$, $i = 1, \ldots, d$, and is a conformal isomorphism $f_i : U_i \to U$ on each U_i. By Cantor repeller we mean the set

$$J = J(f) = \{x \in \mathbb{C} : f^n x \in U, \ n = 0, 1, \ldots\}.$$

When $f = p = $ polynomial this is a Julia set of p.

We will also need the following notations. Let G denote Green's function of $\Omega = \tilde{\mathbb{C}} \backslash J$ with pole at infinity, and let $\omega = \Delta G$ denote the harmonic measure of J. Let $\partial = \dim_H J$,

$$\partial_0 = \inf\{\dim_H J', \ J' \text{ is a Borel subset of } J, \ \omega(J') = 1\}.$$

The symbol $\dim_H$ denotes the Hausdorff dimension. There is a vast literature concerning the estimates of ∂_0. In particular, in [5] it is shown that always $\partial_0 \le 1$. And in [6] the estimate

$$(5) \qquad\qquad\qquad \partial_0 < \partial$$

is proved for the case $f_i(\bar{z}) = \overline{f_i(z)}$, $U_i \cap \mathbb{R} \ne \phi$. Also (5) holds when $f = p = $ polynomial, see [7].

3. Main results. We are going to give some estimates of m from above and (in the polynomial case) for r from below.

THEOREM 3.1. *Let $(U, U_1, \ldots, U_d, f)$ define a Cantor repeller $J = \bigcap_{n \ge 0} f^{-n}(U)$. Let m satisfy (3) with $E = J$. Then the following assertions are equivalent*:

1) *any such m is strictly less than ∂;*

2) *any such m is strictly less than ∂_0;*
3) $\partial_0 < \partial$. ■

R e m a r k. The author believes that $\partial_0 < \partial$ for any Cantor repeller. However, let us remind that (5) is still proved only for f = polynomial [7] and for symmetric f [8], or "linear" f [11].

For polynomial case one can give the estimate not only for m but also for r.

THEOREM 3.2. *Let J be a Julia set of a polynomial p. Then the Markov's inequality* (1) *holds only with*

$$(6) \qquad r \geq \frac{1}{\partial_0}. \ ■$$

R e m a r k. Let us note that J is not assumed to be a Cantor repeller.

Next is considered a "standard Cantor set" situation. And we illustrate how one can easily see that $m < \partial_0$.

THEOREM 3.3. *Let $J = J(f)$, $f = (f_1, \ldots, f_d)$; $|f_i'(z)| \equiv l$, $z \in U_i$, $i = 1, \ldots, d$. Then*

$$(7) \qquad m < \partial_0. \ ■$$

4. Gibbs measures. It will be convenient to use the notion of special class of ergodic measures, namely the class of Gibbs measures in what follows. The reader may refer to [8] for the theory of these measures.

Let $J = J(f)$, $f = (f_1, \ldots, f_d)$ be a Cantor repeller and let f^* denote the pushing forward of the measures on J. What we mean is the following. Let ν be a measure on J. Then $f^*\nu$ is defined on $J \cap U_i$ as $(f^*\nu)(E) = \nu(f_i E)$. A measure ν will be called quasi-invariant if ν is mutually absolutely continuous with respect to $f^*\nu$. For any quasi-invariant measure ν on J let us consider its Jacobian

$$\mathcal{G}_\nu(x) = \frac{df^*\nu}{d\nu}(x), \quad \text{for } \nu\text{-a.e. } x \in J,$$

which is the derivative of f with respect to ν. The function $\varphi_\nu(x) \stackrel{\text{def}}{=} -\log \mathcal{G}_\nu(x)$ will be called the potential of ν. Generally speaking Gibbs measures are those having Hölder potentials. Let us state the definition and some properties of Gibbs measures.

1) By a Gibbs measure on $J(f)$ we mean an f-invariant μ with Hölder potential $\varphi_\mu = -\log \mathcal{G}_\mu$.

2) If ν is an f-quasi-invariant measure with Hölder potential, then there exists a unique f-invariant μ absolutely continuous with respect to ν. Also μ is a Gibbs measure and $\log \frac{d\nu}{d\mu}$ is Hölder continuous.

3) For any Hölder continuous φ there exists a unique measure μ which provides the maximum for the functional

$$(8) \qquad L_\varphi(\nu) = h_\nu + \int_J \varphi d\nu, \ \nu \text{ is } f\text{-invariant, probabilistic}$$

(here h_ν denotes the entropy). This measure μ is Gibbs. Any other Hölder function ψ defines the same μ if and only if

$$(9) \qquad \varphi - \psi = \gamma \circ f - \gamma + C,$$

where C is a constant and γ is Hölder continuous.

Equation (9) is called the homologous equation. If we define $P(\varphi) = \max L_\varphi(\nu)$ (called pressure of φ) then the constant C in (9) is $P(\varphi) - P(\psi)$.

4) If μ is a Gibbs measure on J, then $X_n(x) = \{i : f^n x \in U_i\}$ form the sequence of exponentially independent random variables on (J, μ), in particular

$$(10) \qquad |\mathbb{E}X_n X_{n+k} - \mathbb{E}X_n \mathbb{E}X_{n+k}| \leq Cq^k, \qquad q \in (0,1).$$

The property 4) will be used soon. We need the following corollary of (1).

THEOREM 4.1. *Let ξ be a Hölder continuous function on J. Let μ be a Gibbs measure $\int \xi d\mu = 0$ and let $\{Y_n\}$ denote the sequence of random variables on (J, μ) defined as follows*

$$Y_n = \xi(f^n x), \qquad x \in J, \ n = 0, 1, \dots$$

Then $\{Y_n\}$ are exponentially independent. In particular the limit

$$(11) \qquad \sigma^2 = \lim \frac{1}{n} \mathbb{E}(Y_1 + \dots + Y_n)^2$$

exists and if $\sigma > 0$ the law of iterated logarithm (LIL) *holds for $\{Y_n\}$, namely*

$$(12) \qquad \begin{aligned} &\mu\{x : Y_1 + \dots + Y_n < -\sqrt{2\sigma^2 n \log\log n} \ \text{for infinitely many } n\} = 1, \\ &\mu\{x : Y_1 + \dots + Y_n > +\sqrt{2\sigma^2 n \log\log n} \ \text{for infinitely many } n\} = 1. \ \blacksquare \end{aligned}$$

Theorem 4.1 can be found in [9].

At the same time a result of Ibragimov [10] describes the cases when $\sigma = 0$.

THEOREM 4.2. *Let $\sigma = 0$. Then*

$$Y = u \circ f - u,$$

where $u \in L^2(J, d\mu)$. $\blacksquare$

5. Proof of Theorem 3.1. Let G be Green's function of $\mathbb{C}\backslash J$ (that trivially coincides with Siciak's function defined before (3)). It is proved in [6] that the harmonic measure $\omega = \Delta G$ of J is f quasi-invariant and has Hölder potential $\varphi_\omega = -\log \mathcal{G}_\omega$. According to Section 4 there exists a Gibbs measure μ_0 equivalent with ω.

Let φ denote the Hölder function $\log \mathcal{G}_{\mu_0}$, $\psi \stackrel{\text{def}}{=} \log |f'|$ which is certainly Hölder. Finally

$$\xi \stackrel{\text{def}}{=} \varphi - m\psi, \qquad Y_k = \xi \circ f^k.$$

LEMMA 5.1. *Let $U_n(x)$ denote the component of $f^{-n}U$ containing $x \in J$. Then*

$$\operatorname{diam} U_n(x) \asymp |(f^n)'(x)|^{-1}; \quad d(z, J) \asymp \operatorname{diam} U_n(x), \quad z \in \partial U_n(x),$$
$$\mu_0(U_n(x)) \asymp \sup_{z \in U_n(x)} G(z). \quad \blacksquare$$

This is proved in [6].

LEMMA 5.2. $|\log \mu_0(U_n(x)) + \sum_{k=1}^n \varphi(f^k x)| \leq C.$ $\blacksquare$

This follows immediately from the fact that φ is the potential of μ_0 and from the Hölder continuity of φ. On the other hand, one can derive Lemma 5.2 from the alternative description of Gibbs measures in [8].

Now we may rewrite (3) as

$$(13) \qquad \sum_1^n Y_k = \sum_1^n \varphi(f^k x) - m \sum_1^n \log |f'(f^k x)| \geq C_1.$$

We use Lemmas 5.1, 5.2 to obtain (13) from (3).

On the other hand $\int_J (\varphi - \partial_0 \psi) d\mu_0 = 0$. This is just the form of writing Manning's formula for the dimension of the f-invariant measure μ_0:

$$(14) \qquad \partial_0 = \dim \mu_0 = \frac{h\mu_0}{\int_J \log |f'| d\mu_0} = \frac{\int \varphi d\mu_0}{\int \log |f'| d\mu_0}.$$

Thus if ∂_0 were equal to m we would have

$$\mathbb{E} Y_k = \int \xi d\mu = \int (\varphi - m\psi) d\mu_0 = \int (\varphi - \partial_0 \psi) d\mu_0 = 0.$$

this would bring us under the conditions of Theorem 4.1, 4.2. But (13) contradicts LIL of Theorem 4.1. We conclude that $\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(Y_1 + \ldots + Y_n)^2 = 0$ which means that

$$(15) \qquad \log \mathcal{G}_{\mu_0} - \partial_0 \log |f'| = \varphi - m\psi = u \circ f - u.$$

In particular, using (9) we obtain

$$P(-\partial_0 \log |f'|) = 0.$$

But by Bowen's theorem the unique ∂ such that $P(-\partial \log |f'|) = 0$ is the Hausdorff dimension of J. Thus $\partial_0 = \partial$. The implication 3)$\Rightarrow$2) is proved.

This was the most difficult implication. Clearly $m \leq \partial_0 \leq \partial$ ((3) shows that $H_{m-\varepsilon}(E) = \infty$ as soon as $\omega(E) > 0 \Rightarrow \partial_0 = \dim \omega \geq m - \varepsilon$). So 2)$\Rightarrow$1) is trivial.

The implication 1)$\Rightarrow$3). This repeats the consideration in [6]. If $\partial_0 = \partial$ then Manning's formula (14) gives

$$h_{\mu_0} - \partial \int \log |f'| d\mu_0 = 0.$$

So μ_0 is the measure giving the maximum $P(-\partial \log |f'|) = 0$ to the functional $L_{\partial \log |f'|}(\nu) = h_\nu - \partial \int \log |f'| d\nu$. We see that μ_0 is the Gibbs measure constructed by means of $-\partial \log |f'|$. Now using (9) we come to the homologous equation (15) with Hölder u. Lemma 5.1 shows that $G(z) \leq Md(z)^\partial$. This contradicts 1) and 1)$\Rightarrow$3) is proved together with Theorem 3.1. ∎

Let $m_0 = \sup\{m : \ (3)$ is satisfied with a certain $M = M(m) < \infty\}$. Then clearly

$$m_0 = \lim_{z \to J} \frac{\log G(z)}{\log d(z)}.$$

QUESTION 1. Is always $m_0 < \partial_0$?

6. A simple estimate for m_0 from above. Let us consider the following expression in which supremum is taken over all ergodic measures on J:

$$r_1 \stackrel{\text{def}}{=} \sup_\mu \frac{\int \log |f'| d\mu}{\int \log \mathcal{G}_\omega d\mu}.$$

Let ν provide "almost" supremum to this expression. Then by the ergodic theorem we get that

$$\omega(U_n(x)) \geq C(\operatorname{diam} U_n(x))^{\frac{1}{r_1} + \varepsilon}, \quad \text{for } \nu\text{-a.e. } x.$$

Using Lemma 5.1 we derive immediately the estimate

$$(16) \qquad\qquad\qquad \frac{1}{m_0} \geq r_1.$$

7. Polynomial case. For $f = p = $ polynomial we can write r_1 as follows:

$$r_1 = \frac{1}{\log d} \sup_\mu \ \int \ \log |p'| d\mu \stackrel{\text{def}}{=} \frac{\sup \chi_\mu}{\log d}.$$

Let us introduce $r_0 = \inf\{r : (1)$ is satisfied$\}$.

LEMMA 7.1. $\frac{1}{m_0} \geq r_0 \geq r_1$. ∎

Proof. The first inequality is a trivial consequence of (4). To prove the second let us introduce

$$\chi_p(x) = \overline{\lim_{n \to \infty}} \frac{1}{n} \log |(p^n)'(x)| \quad \text{and} \quad r_s = \frac{\sup_{x \in J} \chi_p(x)}{\log d}.$$

The functions $f_n = \frac{1}{n} \log |(p^n)'(x)|$ are bounded from above (but maybe not from below if p' vanishes on J) and so for them $\overline{\lim} \int f_n \leq \int \overline{\lim} f_n$. So

$$\int \ \log |p'| d\mu = \frac{1}{n} \int \ \log |(p^n)'| d\mu = \overline{\lim} \ \int \frac{1}{n} \log |(p^n)'| d\mu$$

$$\leq \ \int \ \overline{\lim} \frac{1}{n} \log |(p^n)'| d\mu \leq \sup_{x \in J} \chi_p(x).$$

Thus

(17) $$r_1 \leq r_s.$$

On the other hand (1) shows that

$$|(p^n)'(x)| \leq M(d^n)^{r_0+\varepsilon}$$

and so for each $x \in J$

$$\chi_p(x) \leq (r_0 + \varepsilon)\log d$$

which means that $r_s \leq r_0$. Combining this with (16) we complete the proof of Lemma 7.1. ■

Let us introduce

$$\chi_p^{\mathrm{orb}} = \sup\{\chi_p(x) : x \text{ is a periodic point of } p\}.$$

LEMMA 7.2. $\sup_\mu \chi_\mu \leq \chi_p^{\mathrm{orb}} \leq \sup_{x\in J} \chi_p(x)$, *where the first supremum is taken over all ergodic measures.* ■

We will not use this refinement of (16) and we cite this fact about Lyapunov exponents only for the sake of completeness.

8. Proof of Theorem 3.2. Just choose $\mu = \mu_0$ to estimate r_1. Then

$$r \geq r_1 \geq \frac{\int \log|p'|d\mu_0}{\log d} = \frac{1}{\partial_0}.$$

The last equality is (14). ■

Now [7] and Theorem 3.1 show that either $\frac{1}{m} > r$ or $r > \frac{1}{\partial_0}$.

9. Proof of Theorem 3.3. Here to prove (7) we will need the following:

LEMMA 9.1. *Let φ be a Hölder function on J such that φ is not homologous to a constant. Let α_t be the Gibbs measure for $t\varphi$; $\alpha = \alpha_1$. Then*

$$\left(\int \varphi d\alpha_t - \int \varphi d\alpha\right)(1 - t) > 0. \quad ■$$

P r o o f. We use 3) of Section 4 to write

$$h_\alpha - \int \varphi d\alpha > h_{\alpha_t} - \int \varphi d\alpha_t$$
$$h_{\alpha_t} - t\int \varphi d\alpha_t > h_\alpha - \int \varphi d\alpha.$$

Subtracting these two lines we get the result. ■

Now let $\varphi = \log\mathcal{G}_{\mu_0}$ and $\alpha = \mu_0$. This φ is not homologous to a constant [6]. Now Lemma 9.1 gives us an ergodic measure α_t such that

$$\int \log\mathcal{G}_{\mu_0}d\alpha_t < \int \log\mathcal{G}_{\mu_0}d\mu_0 = h_{\mu_0}.$$

Thus

$$\frac{1}{m} \geq r_1 \geq \frac{\log l}{\int \log\mathcal{G}_{\mu_0}d\alpha_t} > \frac{\log l}{\int \log\mathcal{G}_{\mu_0}d\mu_0} = \frac{\int \log|f'|d\mu_0}{h_{\mu_0}} = \frac{1}{\partial_0}. \quad ■$$

QUESTION 2. In the polynomial case, is it true that $r > \frac{1}{\partial_0}$ for any r satisfying Markov's inequality (1) unless $p = z^k$?

QUESTION 3. In the polynomial case, is it true that $r_0 > \frac{1}{\partial_0}$ or even $r_1 > \frac{1}{\partial_0}$ unless $p = z^k$?

R e m a r k. The inequality $r_1 > \frac{1}{\partial_0}$ means that harmonic measure on $J(p)$ does not provide the maximum for the functional $\nu \to \int \log |p'| d\nu$ among ergodic measures.

References

[1]   W. Pawłucki and W. Pleśniak, *Extension of C^∞ functions from sets with polynomial cusps*, Studia Math. 88 (1988), 279–287.

[2]   —, —, *Prolongement de fonctions C^∞*, C.R. Acad. Sci. Paris Ser. I 304 (1987), 167–168.

[3]   J. Siciak, *On some extremal functions and their applications in the theory of analytic functions of several complex variables*, Trans. Amer. Math. Soc. 105 (1962), 322–357.

[4]   L. Białas and A. Volberg, *Markov's property of the Cantor ternary set*, Studia Math. 104 (1993), 259–268.

[5]   P. Jones and Th. Wolff, *Hausdorff dimension of harmonic measure in the plane I*, Acta Math. 161 (1988), 133–144.

[6]   A. Volberg, *On the dimension of harmonic measure of Cantor repellers*, Michigan Math. J. 40 (1993), 239–258.

[7]   A. Zdunik, *Parabolic orbifolds and the dimension of the maximal measure for rational maps*, Invent. Math. 99 (1990), 627–649.

[8]   R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes in Math. 470 (1975).

[9]   W. Phillipp and W. Stout, *Almost sure invariant principles for partial sums of weakly dependent random variables*, Mem. Amer. Math. Soc. 161 (1975).

[10] I. A. Ibragimov and Yu. V. Linnik, *Independent and stationary sequences of random variables*, Wolters-Noordhoff, Groningen, 1971.

[11] A. Volberg, *On the harmonic measure of self-similar sets on the plane*, in: Harmonic Analysis and Discrete Potential Theory, M. Picardello (ed.), Plenum Press, 1991, 267–281.