1. Preliminaries

1. In the following new results on foliations with complex leaves are announced. Complete proofs will appear elsewhere.

A foliation with complex leaves is a (smooth) foliation $X$ of dimension $2n+k$ whose local models are domains $U = V \times B$ of $\mathbb{C}^n \times \mathbb{R}^k$, $V \subset \mathbb{C}^n$, $B \subset \mathbb{R}^k$ and whose local transformations are of the form

$$\begin{cases}
z' = f(z,t), \\
t' = h(t),
\end{cases}$$

where $f$ is holomorphic with respect to $z$. A domain $U$ as above is said to be a distinguished coordinate domain of $X$ and $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_k)$ are said to be distinguished local coordinates. $k$ is called the real codimension of $X$.

As an example of such foliations we have the Levi flat hypersurfaces of $\mathbb{C}^n$ ([13], [4], [11]).

If $X$ is a smooth foliation as above, then the leaves are complex manifolds of dimension $n$. Let $\mathcal{D}$ be the sheaf of germs of smooth functions, holomorphic along the leaves (namely the germs of CR-functions on $X$). $\mathcal{D}$ is a Fréchet sheaf and we denote by $\mathcal{D}(X)$ the Fréchet algebra $\Gamma(X, \mathcal{D})$.

It is natural to study foliations with complex leaves in the spirit of the theory of complex spaces, in particular, the convexity with respect to the algebra $\mathcal{D}(X)$ and the cohomology of $X$ with values in $\mathcal{D}$. In this talk I will discuss some recent results obtained in a joint paper with G. Gigante.

2. Let $X$ be a smooth foliation with complex leaves. $X$ is said to be a $q$-complete foliation if there is an exhaustive, smooth function $\Phi : X \to \mathbb{R}$ which is strictly $q$-pseudoconvex along the leaves. $X$ is a Stein foliation if

(a) $\mathcal{D}X$ separates points of $X$.

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The paper is in final form and no version of it will be published elsewhere.
(b) $X$ is $\mathcal{D}$-convex,
(c) for every $x \in X$ there exist $f_1, \ldots, f_n, h_1, \ldots, h_k \in \mathcal{D}(X)$ such that
\[ \text{rank} \frac{\partial (f_1, \ldots, f_n, h_1, \ldots, h_k)}{\partial (z_1, \ldots, z_n, t_1, \ldots, t_k)} = n + k \]
($z_1, \ldots, z_n, t_1, \ldots, t_k$ distinguished local coordinates at $x$).

One can prove that a Stein foliation is 1-complete.

Remark. If we replace $\mathbb{R}^k$ by $\mathbb{C}^k$ and in ($*$) we assume $t \in \mathbb{C}^k$ and that $f$, $h$ are holomorphic with respect to $z, t$ then we obtain the notion of complex foliation of (complex) codimension $k$.

3. Every real analytic foliation can be complexified. Precisely, we have the following

**Theorem 1.** Let $X$ be a real analytic foliation with complex leaves, of codimension $k$. Then there exists a complex foliation $\tilde{X}$ of codimension $k$ such that:

1. $X \hookrightarrow \tilde{X}$ by a closed real analytic embedding which is holomorphic along the leaves;
2. every real analytic CR-function $f : X \to \mathbb{R}$ extends holomorphically to a neighbourhood of $X$;
3. if $X$ is a $q$-complete foliation with exhaustive function $\Phi$ then for every $c \in \mathbb{R}$, $\overline{X}_c = \{ \Phi \leq c \}$ has a fundamental system of neighbourhoods which are $q$-complete manifolds.

Remark. $\tilde{X}$ with the properties (1)–(3) is essentially unique.

As a corollary, using the approximation theorem of M. Freeman ([5]) we prove the following

**Theorem 2.** Under the assumptions of Theorem 1, if $X$ is 1-complete, a smooth CR-function on a neighbourhood of $\overline{X}_c$ can be approximated by smooth global CR-functions.

Remark. A similar argument can be applied to prove that in the previous statement $\overline{X}_c$ can be replaced by an arbitrary $\mathcal{D}$-convex compact $K$ (i.e. $\hat{K} = K$).

2. Applications

1. The approximation theorem allows us to prove an embedding theorem for real analytic Stein foliations ([7]).

Let $X$ be a smooth foliation with complex leaves of dimension $n$ and of codimension $k$. Let us denote by $\mathcal{A}(X; \mathbb{C}^N)$ the set of smooth CR-maps $X \to \mathbb{C}^N$. Then $\mathcal{A}(X; \mathbb{C}^N)$ is Fréchet. We have the following

**Theorem 3.** Assume $X$ is a real analytic Stein foliation. Then there exists a smooth CR-map $X \to \mathbb{C}^N$, $N = 2n + k + 1$, which is one-to-one, proper and regular.
2. We apply the above theorem to obtain information about the topology of $X$.

**Theorem 4.** Let $X$ be a real analytic Stein foliation. Then $H_j(X, \mathbb{Z}) = 0$ for $j \geq n + k + 1$ and $H_{n+k}(X, \mathbb{Z})$ has no torsion.

**Sketch of proof.** Embed $X$ in $\mathbb{C}^N$ and consider on $X$ the distance function $\varrho$ from a point $z^0 \in \mathbb{C}^N \setminus X$. $z^0$ can be chosen in such a way that $\varrho$ is a Morse function. Next we show that $\varrho$ has no critical point of index $j \geq n + k + 1$ ([14]).

**Corollary 5.** Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a closed oriented real analytic foliation and let $W$ be a smooth algebraic hypersurface which does not contain $X$. Then the homomorphism $H_j(X, \mathbb{Z}) \to H_j(X \cap W, \mathbb{Z})$ induced by $X \cap W \to X$ is bijective for $j < n - 1$ and injective for $j = n - 1$. Moreover, the quotient group $H^{n-1}(X \cap W, \mathbb{Z})/H^{n-1}(X, \mathbb{Z})$ has no torsion.

3. Cohomology

1. Given a $q$-complete smooth foliation $X$, according to the Andreotti and Grauert theory for complex spaces it is natural to expect that the cohomology groups $H^j(X, \mathcal{D})$ vanish for $j \geq q$. This is actually true for domains in $\mathbb{C}^n \times \mathbb{R}^k$ ([1]). More generally, we prove the following:

**Theorem 6.** Let $X$ be a $1$-complete real analytic foliation. Then $H^j(X, \mathcal{D}) = 0$ for $j \geq 1$.

**Sketch of proof.** Assume $k = 1$ and let $\Phi$ be an exhaustive function for $X$. Then the vanishing theorem for domains in $\mathbb{C}^n \times \mathbb{R}^k$, the bumps lemma and the Mayer–Vietoris sequence ([1]) yield the following: for every $c > 0$ there is $\varepsilon > 0$ such that

\[
H^j(X_{c+\varepsilon}, \mathcal{D}) \to H^j(X_c, \mathcal{D})
\]

is onto for $j \geq 1$ (and this holds true for $j \geq q$ whenever $X$ is a $q$-complete smooth foliation).

Now let $\tilde{X}$ be the complexification of $X$ and consider the compact $\overline{X}_c = \{ \Phi \leq c \}$. In view of Theorem 1, $\overline{X}_c$ has a fundamental system of Stein neighbourhoods $U$ in $\tilde{X}$. $X$ is oriented around $\overline{X}_c$ and consequently $U \setminus X$ has two connected components $U_+, U_-$ ($U$ is connected).

Denote by $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) the sheaf of germs of holomorphic functions on $U_+$ (resp. $U_-$) that are smooth on $U_+ \cup (U_+ \cap X)$ (resp. $U_- \cup (U_- \cap X)$). Then we have the exact sequence

\[
0 \to \mathcal{O} \to \mathcal{O}_+ \oplus \mathcal{O}_- \xrightarrow{\text{re}} \mathcal{D} \to 0
\]

(2) (here $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) is a sheaf on $\overline{U}_+$ (resp. $\overline{U}_-$) extended by 0 on all $U$ and $\text{re}(f \oplus g) = f|_X - g|_X$). Since $U$ is Stein we derive from (2) that

\[
H^j(U_+, \mathcal{O}_+) \oplus H^j(U_-, \mathcal{O}_-) \cong H^j(U \cap X, \mathcal{D})
\]

(3)
for \( j \geq 1 \) (and this holds true for \( j \geq q \) whenever \( X \) is a \( q \)-complete real-analytic foliation of codimension 1).

Let be a \( j \)-cocycle of \( D \) on a neighbourhood of \( X \). In view of (2) we have \( \xi = \xi_+ - \xi_- \) where \( \xi_+ \) and \( \xi_- \) are represented by two \((0,j)\)-forms \( \omega_+ \), \( \omega_- \) on \( U_+ \), \( U_- \) respectively which are smooth up to \( X \).

Moreover, according to [6] it is possible to construct pseudoconvex domains \( U'_+ \) and \( U'_- \) satisfying the following conditions: \( U'_+ \subset U_+ \), \( U'_- \subset U_- \), \( \partial U'_+ \), \( \partial U'_- \) are smooth and \( \partial U'_+ \cap X \), \( \partial U'_- \cap X \) contain a neighbourhood of \( X \).

Then Kohn’s theorem ([10]) implies that on \( U'_+ \) and \( U'_- \) respectively we have \( \omega_+ = \partial v_+ \), \( \omega_- = \partial v_- \) where \( v_+ \in C^\infty(\overline{U'_+}) \), \( v_- \in C^\infty(\overline{U'_-}) \). It follows that \( H^j(\overline{X}_c, D) = 0 \) for \( j \geq 1 \) and from (1) we deduce that \( H_j(x_c, D) = 0 \) for every \( c \in \mathbb{R} \) and \( j \geq 1 \).

At this point, in order to conclude our proof we can repeat step by step the proof of the Andreotti–Grauert vanishing theorem for \( q \)-complex spaces ([1]).

If \( k \geq 2 \) the situation is much more involved. Using the Nirenberg Extension Lemma ([10]) it is possible to reduce the cohomology \( H^*(x, D) \) to the \( \mathcal{O} \)-cohomology of \( \tilde{X} \) with respect to the differential forms on \( \tilde{X} \) which are flat on \( X \) and to conclude invoking a theorem of existence proved by J. Chaumat and A. M. Chollet ([3]).

Assume that \( X \) is real analytic and let \( O' \) be the sheaf of germs of real analytic CR-functions. Then an analogous statement for \( O' \) is not true. Andreotti and Nacinovich ([2]) showed that \( H^1(X, O') \) is never zero. However by Theorem 1 we have for arbitrary \( k \), \( H^j(\overline{X}_c, O') = 0 \) for \( j > 0 \) whenever \( X \) is \( q \)-complete.

2. Using the same method of proof, under the hypothesis of Theorem 6, we have the following

**Theorem 7.** Let \( A = \{x_\nu\} \) be a discrete subset of \( X \) and let \( \{c_\nu\} \) be a sequence of complex numbers. Then there exists \( f \in D(X) \) such that \( f(x_\nu) = c_\nu \), \( \nu = 1, 2, \ldots \). In particular, \( X \) is \( D \)-convex and \( D(X) \) separates points of \( X \).

**Remark.** A vanishing theorem can be also proved for the sheaf of germs of “CR-sections” of \( E \to X \) where \( E \) is a fibre vector bundle with fibre \( \mathbb{C}^m \times \mathbb{R}^b \).

4. The Kobayashi metric

1. Let \( X \) be a foliation with complex leaves of codimension \( k \), and let \( T(X) \xrightarrow{\pi} X \) be the tangent bundle of \( X \). The collection of all tangent spaces to the leaves of \( X \) forms a complex subbundle \( T_H(X) \) of \( T(X) \). Let \( D \) be the unit disc in \( \mathbb{C} \) and denote by \( CR(D, X) \) the set of all CR-maps \( D \to X \).

Given \( \zeta \in T_H(X) \) with \( x = \pi(\zeta) \) we define the function \( F = F_X \) on \( X \times T_H(X) \) by

\[
F(x, \zeta) = \inf \{ s \in \mathbb{R} : s \geq 0, s \varphi'(0) = \zeta \}
\]

where \( \varphi \in CR(D, X) \) and \( \varphi(0) = x \).
When \( k = 0 \), \( F \) reduces to the Kobayashi “infinitesimal metric” of the complex manifold \( X \) ([8]). In particular, if \( X = \mathbb{C}^n \times \mathbb{R}^k \), then \( F = 0 \).

If \( X' \) is another foliation as above and \( \phi : X \to X' \) is a CR-map then \( d\phi : T_H(X) \to T_H(X') \) and

\[
F_X(\phi(x), d\phi \zeta) \leq F_X(x, \zeta).
\]

**Theorem 7.** \( F_X \) is upper semicontinuous.

According to the complex case [8], \( X \) is said to be hyperbolic if \( F(x, \zeta) > 0 \) for every \( x \in X \) and \( \zeta \in T_H(X) \), \( \zeta \neq 0 \).

**Remarks.**
1) The fact that all the leaves are hyperbolic does not imply that \( X \) itself is hyperbolic.
2) Every bounded domain in \( \mathbb{C}^n \times \mathbb{R}^k \) is hyperbolic.
3) Following [12] it can be proved that if \( X \) admits a continuous bounded function \( u \), p.s.h. along the leaves and strictly p.s.h. in a neighbourhood of \( x \), then \( X \) is hyperbolic at \( x \).

2. Now consider a riemannian metric on \( X \) and let \( V \) be a smooth distribution of transversal tangent \( k \)-spaces. Then every \( \zeta \in T(X) \) splits into \( \zeta_0 + \zeta_c \) where \( \zeta_0 \in V \), \( \zeta_c \in T_H(X) \) and we denote by \( \tau(\zeta_0) \) the length of \( \zeta_0 \).

Let \( F \) be the infinitesimal Kobayashi metric on \( X \) and for \( \zeta \in T_x(X) \) set \( g(x, \zeta) = F(x, \zeta_c) + \tau(x, \zeta_0) \). Then \( g \) is an upper semicontinuous pseudometric.

If \( \gamma = \gamma(s), 0 \leq s \leq 1 \), is a smooth curve joining \( x, y \in X \) the pseudo-length of \( \gamma \) with respect to \( g \) is

\[
L(\gamma) = \int_0^1 g(\gamma(s), \dot{\gamma}) \, ds
\]

and the pseudo-distance between \( x, y \) is

\[
d(x, y) = \inf_\gamma L(\gamma).
\]

\( d \) is a real distance on \( X \) inducing the topology of \( X \) if \( X \) is hyperbolic. \( X \) is said to be complete if a field \( V \) can be chosen making \( X \) complete with respect to \( d \).

For example, the unit ball in \( \mathbb{C} \times \mathbb{R} \) is complete for the choice

\[
V = \lambda(t) \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) + (1 + t^2)^{-1} \frac{\partial}{\partial t}
\]

where \( \lambda(t) = 2 \arctan [ (1 + t^2)^{-1} (1 - \arctan^2 t)^{-3/2} ] \).

The interest of this construction is due to the following

**Theorem 8.** Let \( \Omega \subset \mathbb{C}^n \times \mathbb{R}^k \) be with the riemannian structure induced by \( \mathbb{C}^n \times \mathbb{R}^k \). If \( \Omega \) is hyperbolic and complete then \( \Omega \) is \( \mathcal{D} \)-convex.
References