TOPICS IN COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 31
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

FOLIATIONS WITH COMPLEX LEAVES

GIUSEPPE TOMASSINI
Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy

1. Preliminaries

1. In the following new results on foliations with complex leaves are announced. Complete proofs will appear elsewhere.

A *foliation with complex leaves* is a (smooth) foliation X of dimension $2n+k$ whose local models are domains $U = V \times B$ of $\mathbb{C}^n \times \mathbb{R}^k$, $V \subset \mathbb{C}^n$, $B \subset \mathbb{R}^k$ and whose local transformations are of the form

$$(*) \qquad \begin{cases} z' = f(z,t), \\ t' = h(t), \end{cases}$$

where f is holomorphic with respect to z. A domain U as above is said to be a *distinguished coordinate domain* of X and $z = (z_1, \ldots, z_n)$, $t = (t_1, \ldots, t_k)$ are said to be *distinguished local coordinates*. k is called the real codimension of X.

As an example of such foliations we have the Levi flat hypersurfaces of $\mathbb{C}^n$ ([13], [4], [11]).

If X is a smooth foliation as above, then the leaves are complex manifolds of dimension n. Let $\mathcal{D}$ be the sheaf of germs of smooth functions, holomorphic along the leaves (namely the germs of CR-functions on X). $\mathcal{D}$ is a Fréchet sheaf and we denote by $\mathcal{D}(X)$ the Fréchet algebra $\Gamma(X, \mathcal{D})$.

It is natural to study foliations with complex leaves in the spirit of the theory of complex spaces, in particular, the convexity with respect to the algebra $\mathcal{D}(X)$ and the cohomology of X with values in $\mathcal{D}$. In this talk I will discuss some recent results obtained in a joint paper with G. Gigante.

2. Let X be a smooth foliation with complex leaves. X is said to be a q-*complete foliation* if there is an exhaustive, smooth function $\Phi : X \to \mathbb{R}$ which is strictly q-pseudoconvex along the leaves. X is a *Stein foliation* if

(a) $\mathcal{D}X$ separates points of X,

1991 *Mathematics Subject Classification*: 32F, 53C.
The paper is in final form and no version of it will be published elsewhere.

(b) X is $\mathcal{D}$-convex,

(c) for every $x \in X$ there exist $f_1, \ldots, f_n, h_1, \ldots, h_k \in \mathcal{D}(X)$ such that

$$\operatorname{rank} \frac{\partial(f_1, \ldots, f_n, h_1, \ldots, h_k)}{\partial(z_1, \ldots, z_n, t_1, \ldots, t_k)} = n + k$$

($z_1, \ldots, z_n, t_1, \ldots, t_k$ distinguished local coordinates at x).

One can prove that a Stein foliation is 1-complete.

Remark. If we replace $\mathbb{R}^k$ by $\mathbb{C}^k$ and in $(*)$ we assume $t \in \mathbb{C}^k$ and that f, h are holomorphic with respect to z, t then we obtain the notion of *complex foliation* of (*complex*) *codimension* k.

3. Every real analytic foliation can be complexified. Precisely, we have the following

THEOREM 1. *Let X be a real analytic foliation with complex leaves, of codimension k. Then there exists a complex foliation $\widetilde{X}$ of codimension k such that*:

(1) *$X \hookrightarrow \widetilde{X}$ by a closed real analytic embedding which is holomorphic along the leaves*;

(2) *every real analytic CR-function $f : X \to \mathbb{R}$ extends holomorphically to a neighbourhood of X*;

(3) *if X is a q-complete foliation with exhaustive function Φ then for every $c \in \mathbb{R}$, $\overline{X}_c = \{\Phi \le c\}$ has a fundamental system of neighbourhoods which are q-complete manifolds.*

Remark. $\widetilde{X}$ with the properties (1)–(3) is essentially unique.

As a corollary, using the approximation theorem of M. Freeman ([5]) we prove the following

THEOREM 2. *Under the assumptions of Theorem* 1, *if X is* 1-*complete*, *a smooth CR-function on a neighbourhood of $\overline{X}_c$ can be approximated by smooth global CR-functions.*

Remark. A similar argument can be applied to prove that in the previous statement $\overline{X}_c$ can be replaced by an arbitrary $\mathcal{D}$-convex compact K (i.e. $\widehat{K} = K$).

2. Applications

1. The approximation theorem allows us to prove an embedding theorem for real analytic Stein foliations ([7]).

Let X be a smooth foliation with complex leaves of dimension n and of codimension k. Let us denote by $\mathcal{A}(X; \mathbb{C}^N)$ the set of smooth CR-maps $X \to \mathbb{C}^N$. Then $\mathcal{A}(X; \mathbb{C}^N)$ is Fréchet. We have the following

THEOREM 3. *Assume X is a real analytic Stein foliation. Then there exists a smooth CR-map $X \to \mathbb{C}^N$, $N = 2n + k + 1$, which is one-to-one*, *proper and regular.*

2. We apply the above theorem to obtain information about the topology of X.

THEOREM 4. *Let X be a real analytic Stein foliation. Then $H_j(X,\mathbb{Z})=0$ for $j\geq n+k+1$ and $H_{n+k}(X,\mathbb{Z})$ has no torsion.*

SKETCH OF PROOF. Embed X in $\mathbb{C}^N$ and consider on X the distance function ϱ from a point $z^\circ\in\mathbb{C}^N\setminus X$. z° can be chosen in such a way that ϱ is a Morse function. Next we show that ϱ has no critical point of index $j\geq n+k+1$ ([14]).

COROLLARY 5. *Let $X\subset\mathbb{P}^N(\mathbb{C})$ be a closed oriented real analytic foliation and let W be a smooth algebraic hypersurface which does not contain X. Then the homomorphism*

$$H^j(X,\mathbb{Z})\to H^j(X\cap W,\mathbb{Z})$$

induced by $X\cap W\to X$ is bijective for $j<n-1$ and injective for $j=n-1$. Moreover, the quotient group $H^{n-1}(X\cap W,\mathbb{Z})/H^{n-1}(X,\mathbb{Z})$ has no torsion.

3. Cohomology

1. Given a q-complete smooth foliation X, according to the Andreotti and Grauert theory for complex spaces it is natural to expect that the cohomology groups $H^j(X,\mathcal{D})$ vanish for $j\geq q$. This is actually true for domains in $\mathbb{C}^n\times\mathbb{R}^k$ ([1]). More generally, we prove the following:

THEOREM 6. *Let X be a 1-complete real analytic foliation. Then $H^j(X,\mathcal{D})=0$ for $j\geq 1$.*

SKETCH OF PROOF. Assume $k=1$ and let Φ be an exhaustive function for X. Then the vanishing theorem for domains in $\mathbb{C}^n\times\mathbb{R}^k$, the bumps lemma and the Mayer–Vietoris sequence ([1]) yield the following: for every $c>0$ there is $\varepsilon>0$ such that

$$H^j(X_{c+\varepsilon},\mathcal{D})\to H^j(X_c,\mathcal{D}) \tag{1}$$

is onto for $j\geq 1$ (and this holds true for $j\geq q$ whenever X is a q-complete smooth foliation).

Now let $\widetilde{X}$ be the complexification of X and consider the compact $\overline{X}_c=\{\Phi\leq c\}$. In view of Theorem 1, $\overline{X}_c$ has a fundamental system of Stein neighbourhoods U in $\widetilde{X}$. X is oriented around $\overline{X}_c$ and consequently $U\setminus X$ has two connected components U_+, U_- (U is connected).

Denote by $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) the sheaf of germs of holomorphic functions on U_+ (resp. U_-) that are smooth on $U_+\cup(U_+\cap X)$ (resp. $U_-\cup(U_-\cap X)$. Then we have the exact sequence

$$0\to\mathcal{O}\to\mathcal{O}_+\oplus\mathcal{O}_-\stackrel{\mathrm{re}}{\to}\mathcal{D}\to 0 \tag{2}$$

([2]) (here $\mathcal{O}_+$ (resp. $\mathcal{O}_-$) is a sheaf on $\overline{U}_+$ (resp. $\overline{U}_-$) extended by 0 on all U and $\mathrm{re}\,(f\oplus g)=f_{|x}-g_{|x}$). Since U is Stein we derive from (2) that

$$H^j(\overline{U}_+,\mathcal{O}_+)\oplus H^j(\overline{U}_-,\mathcal{O})\stackrel{\sim}{\to} H^j(U\cap X,\mathcal{D}) \tag{3}$$

for $j \geq 1$ (and this holds true for $j \geq q$ whenever X is a q-complete real-analytic foliation of codimension 1).

Let be a j-cocycle of $\mathcal{D}$ on a neighbourhood of $\overline{X}_c$. In view of (2) we have $\xi = \xi_+ - \xi_-$ where ξ_+ and ξ_- are represented by two $(0, j)$-forms ω_+, ω_- on U_+, U_- respectively which are smooth up to X.

Moreover, according to [6] it is possible to construct pseudoconvex domains U'_+ and U'_- satisfying the following conditions: $U'_+ \subset U_+$, $U'_- \subset U'_-$, $\partial U'_+$, $\partial U'_-$ are smooth and $\partial U'_+ \cap X$, $\partial U'_- \cap X$ contain a neighbourhood of $\overline{X}_c$.

Then Kohn's theorem ([10]) implies that on U'_+ and U'_- respectively we have $\omega_+ = \overline{\partial} v_+$, $\omega_- = \partial v_-$ where $v_+ \in C^\infty(\overline{U}'_+)$, $v_- \in C^\infty(\overline{U}'_-)$. It follows that $H^j(\overline{X}_c, \mathcal{D}) = 0$ for $j \geq 1$ and from (1) we deduce that $H_j(x_c, \mathcal{D}) = 0$ for every $c \in \mathbb{R}$ and $j \geq 1$.

At this point, in order to conclude our proof we can repeat step by step the proof of the Andreotti–Grauert vanishing theorem for q-complex spaces ([1]).

If $k \geq 2$ the situation is much more involed. Using the Nirenberg Extension Lemma ([10]) it is possible to reduce the cohomology $H^*(x, \mathcal{D})$ to the $\overline{\partial}$-cohomology of $\widetilde{X}$ with respect to the differential forms on $\widetilde{X}$ which are flat on X and to conclude invoking a theorem of existence proved by J. Chaumat and A. M. Chollet ([3]).

Assume that X is real analytic and let $\mathcal{O}'$ be the sheaf of germs of real analytic CR-functions. Then an analogous statement for $\mathcal{O}'$ is not true. Andreotti and Nacinovich ([2]) showed that $H^1(X, \mathcal{O}')$ is never zero. However by Theorem 1 we have for arbitrary k, $H^j(\overline{X}_c, \mathcal{O}') = 0$ for $j > 0$ whenever X is q-complete.

2. Using the same method of proof, under the hypothesis of Theorem 6, we have the following

THEOREM 7. *Let $A = \{x_\nu\}$ be a discrete subset of X and let $\{c_\nu\}$ be a sequence of complex numbers. Then there exists $f \in \mathcal{D}(X)$ such that $f(x_\nu) = c_\nu$, $\nu = 1, 2, \ldots$ In particular, X is $\mathcal{D}$-convex and $\mathcal{D}(X)$ separates points of X.*

Remark. A vanishing theorem can be also proved for the sheaf of germs of "CR-sections" of $E \to X$ where E is a fibre vector bundle with fibre $\mathbb{C}^m \times \mathbb{R}^h$.

4. The Kobayashi metric

1. Let X be a foliation with complex leaves of codimension k, and let $T(X) \xrightarrow{\pi} X$ be the tangent bundle of X. The collection of all tangent spaces to the leaves of X forms a complex subbundle $T_H(X)$ of $T(X)$. Let D be the unit disc in $\mathbb{C}$ and denote by $\mathrm{CR}(D, X)$ the set of all CR-maps $D \to X$.

Given $\zeta \in T_H(X)$ with $x = \pi(\zeta)$ we define the function $F = F_X$ on $X \times T_H(X)$ by

$$F(x, \zeta) = \inf\{s \in \mathbb{R} : s \geq 0, s\varphi'(0) = \zeta\}$$

where $\varphi \in \mathrm{CR}(D, X)$ and $\varphi(0) = x$.

When $k = 0$, F reduces to the Kobayashi "infinitesimal metric" of the complex manifold X ([8]). In particular, if $X = \mathbb{C}^n \times \mathbb{R}^k$, then $F = 0$.

If X' is another foliation as above and $\phi : X \to X'$ is a CR-map then $d\phi : T_H(X) \to T_H(X')$ and

$$F_X(\phi(x), d\phi\zeta) \leq F_X(x, \zeta).$$

TEOREM 7. *F_X is upper semicontinuous.*

According to the complex case [8], X is said to be *hyperbolic* if $F(x, \zeta) > 0$ for every $x \in X$ and $\zeta \in T_H(X)$, $\zeta \neq 0$.

Remarks. 1) The fact that all the leaves are hyperbolic does not imply that X itself is hyperbolic.

2) Every bounded domain in $\mathbb{C}^n \times \mathbb{R}^k$ is hyperbolic.

3) Following [12] it can be proved that if X admits a continuous bounded function u, p.s.h. along the leaves and strictly p.s.h. in a neighbourhood of x, then X is hyperbolic at x.

2. Now consider a riemannian metric on X and let V be a smooth distribution of transversal tangent k-spaces. Then every $\zeta \in T(X)$ splits into $\zeta_0 + \zeta_c$ where $\zeta_0 \in V$, $\zeta_c \in T_H(X)$ and we denote by $\tau(\zeta_0)$ the length of ζ_0.

Let F be the infinitesimal Kobayashi metric on X and for $\zeta \in T_x(X)$ set $g(x, \zeta) = F(x, \zeta_c) + \tau(x, \zeta_0)$. Then g is an upper semicontinuous pseudometric.

If $\gamma = \gamma(s)$, $0 \leq s \leq 1$, is a smooth curve joining $x, y \in X$ the *pseudo-length* of γ with respect to g is

$$L(\gamma) = \int_0^1 g(\gamma(s), \dot{\gamma})\, ds$$

and the *pseudo-distance* between x, y is

$$d(x, y) = \inf_{\gamma} L(\gamma).$$

d is a real distance on X inducing the topology of X if X is hyperbolic. X is said to be *complete* if a field V can be chosen making X complete with respect to d.

For example, the unit ball in $\mathbb{C} \times \mathbb{R}$ is complete for the choice

$$V = \lambda(t)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) + (1 + t^2)^{-1}\frac{\partial}{\partial t}$$

where $\lambda(t) = 2\arctan t[(1 + t^2)^{-1}(1 - \arctan^2 t)^{-3/2}]$.

The interest of this construction is due to the following

THEOREM 8. *Let $\Omega \subset \mathbb{C}^n \times \mathbb{R}^k$ be with the riemannian structure induced by $\mathbb{C}^n \times \mathbb{R}^k$. If Ω is hyperbolic and complete then Ω is $\mathcal{D}$-convex.*

References

[1] A. Andreotti et H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France 90 (1962), 193–259.

[2] A. Andreotti and M. Nacinovich, *Analytic convexity*, Ann. Scuola Norm. Sup. Pisa 7 (1980), 287–372.

[3] J. Chaumat et A. M. Chollet, *Noyaux pour résoudre l'équation* $\overline{\partial}$ *dans des classes ultradifférentiables sur des compacts irréguliers de* $\mathbb{C}^n$, preprint.

[4] M. Freeman, *Local complex foliations of real submanifolds*, Math. Ann. 209 (1970), 1–30.

[5] —, *Tangential Cauchy–Riemann equations and uniform approximation*, Pacific J. Math. 33 (1970), 101–108.

[6] R. Gay et A. Sebbar, *Division et extension dans l'algèbre* $A^\infty(\Omega)$ *d'un ouvert pseudo-convexe à bord lisse de* $\mathbb{C}^n$, Math. Z. 189 (1985), 421–447.

[7] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, North-Holland, 1973.

[8] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Dekker, New York, 1970.

[9] J. J. Kohn, *Global regularity for* $\overline{\partial}$ *on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc. 181 (1973), 273–292.

[10] L. Nirenberg, *A proof of the Malgrange preparation theorem*, in: Liverpool Singularities 1, 97–105.

[11] C. Rea, *Levi flat submanifolds and biholomorphic extension of foliations*, Ann. Scuola Norm. Sup. Pisa 26 (1972), 664–681.

[12] N. Sibony, *A class of hyperbolic manifolds*, in: Recent Developments in Several Complex Variables, Ann. of Math. Stud. 100, Princeton Univ. Press, 1981.

[13] F. Sommer, *Komplexe analytische Blätterung reeler Mannigfaltigkeiten in* $\mathbb{C}^n$, Math. Ann. 136 (1958), 111–133.

[14] G. Tomassini, *Extension d'objets CR*, Math. Z. 194 (1987), 471–486.