TOPICS IN COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 31
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

ON $\overline{\partial}$-PROBLEMS ON (PSEUDO)-CONVEX DOMAINS

R. MICHAEL RANGE

Department of Mathematics, State University of New York at Albany
Albany, New York 12222, U.S.A.
E-mail: range@math.albany.edu

In this survey we shall tour the area of multidimensional complex analysis which centers around $\overline{\partial}$-problems (i.e., the Cauchy-Riemann equations) on pseudoconvex domains. Along the way we shall highlight some of the classical milestones as well as more recent landmarks, and we shall discuss some of the major open problems and conjectures. For the sake of simplicity we will only consider domains in $\mathbb{C}^n$; intriguing phenomena occur already in the simple setting of (Euclidean) convex domains. We will not discuss at all the closely related theory of the induced Cauchy-Riemann equations on boundaries of domains or on submanifolds of higher codimension. The reader interested in such $\overline{\partial}_b$-problems may consult the recent monograph of Boggess [Bo].

1. General domains. We consider the problem of solving the equation

$$\overline{\partial}u = f \tag{1.1}$$

for a given $\overline{\partial}$-closed $(0,1)$ form $f = \sum f_j d\overline{z}_j$ on the domain $D \subset \mathbb{C}^n$. The following are two classical results which require minimal regularity hypothesis.

THEOREM 1 (Hörmander [Hö]). *Suppose that D is bounded and pseudoconvex, and that φ is plurisubharmonic on D. There exists a constant $C < \infty$ which depends only on n and on the diameter of D, such that if $f \in L^2_{0,1}(D, e^{-\varphi})$ is $\overline{\partial}$-closed then there exists a solution u of* (1.1) *which satisfies*

$$\int_D |u|^2 e^{-\varphi} \le C \int_D |f|^2 e^{-\varphi}. \tag{1.2}$$

THEOREM 2 (Kohn [Ko2]). *Suppose that D is a bounded pseudoconvex domain*

1991 *Mathematics Subject Classification*: 32F20, 32F15.
The paper is in final form and no version of it will be published elsewhere.

with C^∞ boundary bD. If $f \in C^\infty_{0,1}(\overline{D})$ is $\overline{\partial}$-closed, then there exists a solution u of (1.1) *which is in $C^\infty(\overline{D})$.*

There is a major difference between the solutions u in Theorem 1 and in Theorem 2. In Theorem 1, there is no loss of generality in assuming that u is orthogonal to the holomorphic functions in $L^2(D, e^\varphi)$ — just replace u by $u - Pu$, where $P : L^2 \to \mathcal{O}L^2$ is the orthogonal projection. The (unique) solution orthogonal to $\mathcal{O}L^2$ is also given in terms of the complex Neumann operator N on D by the formula $u = \overline{\partial}^* Nf$. For $D \subset \mathbb{C}^n$, N is just the inverse of the complex Laplacian

$$\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : L^2_{0,1}(D, e^{-\varphi}) \to L^2_{0,1}(D, e^{-\varphi}).$$

In contrast, in Theorem 2, the solution u cannot be described so easily. In fact, the following is one of the major open problems in the area.

Problem 1. *Under the hypothesis of Theorem* 2, *is $u = \overline{\partial}^* Nf \in C^\infty(\overline{D})$* (*i.e. smooth up to the boundary*) *whenever $f \in C^\infty_{0,1}(\overline{D})$?*

Via the formula $P = Id - \overline{\partial}^* N \overline{\partial}$ for the Bergman projection P (here we take L^2 with respect to Lebesgue measure), an affirmative answer to Problem 1 is equivalent to the so-called regularity condition R for D: *The domain D satisfies condition R if $u \in C^\infty(\overline{D})$ implies $Pu \in C^\infty(\overline{D})$.*

Condition R has far reaching applications in the theory of boundary regularity for holomorphic maps (Bell and Ligocka [BL]; see [Ra2] for other references).

Condition R holds for many important classes of domains, including pseudoconvex domains of finite type (see below), and convex domains [BS]. (Chen [Ch] in dimension 2, Boas and Straube [BS] for the general case.) It is known to fail, in general, on non pseudoconvex domains with smooth boundary (Barrett [Ba1]). Barrett also showed recently that on certain smoothly bounded *pseudoconvex* domains D, P does not preserve Sobolev spaces H_s [Ba2]. This does not disprove condition R for the given domain D, though in all cases where condition R is known to hold, P does indeed preserve the Sobolev spaces.

2. Strictly pseudoconvex domains. This is the class of domains which has been investigated most thoroughly. In geometric terms, a domain D is strictly pseudoconvex if *locally* near a point $P \in bD$, it is biholomorphically equivalent to a strictly convex domain, i.e., the boundary can locally be described as the graph of a function with a positive definite Hessian. A very useful property of the class of strictly pseudoconvex domains is its stability under small smooth perturbations of the boundary.

2A. *L^2-Sobolev theory*

Theorem 3 (Kohn [Ko1]). *If D is strictly pseudoconvex with C^∞ boundary, then for each $s = 0, 1, 2, \ldots$ there exist $C_s < \infty$, such that*

$$\|\overline{\partial}^* Nf\|_{s+1/2} \le C_s \|f\|_s$$

for all $f \in H_s$ with $\overline{\partial} f = 0$.

Here H_s is the L^2 Sobolev space of order s, and $\|\cdot\|_s$ is the corresponding norm.

Kohn, in fact, proved much more. In particular, the result can be localized: If U is a neighborhood of a point $p \in \overline{D}$ and $f \in L^2_{0,1}(D) \cap H_s(\overline{D} \cap U)$ is $\overline{\partial}$-closed, then for $V \subset\subset U$ one has $\overline{\partial}^* N f \in H_{s+1/2}(\overline{D} \cap V)$. A self contained exposition of Kohn's results may be found in [FoK].

2B. *Integral solution operators.* Beginning in 1969, integral solution operators for $\overline{\partial}$ on strictly pseudoconvex domains were introduced, independently, by Henkin [He] and Grauert-Lieb [GL]. These operators have been investigated thoroughly by numerous authors, resulting in estimates for $\overline{\partial}$ in many of the classical function spaces, like L^p, $1 \le p \le \infty$, C^k, Λ_α, etc. These operators are now one of the standard tools in complex analysis on strictly pseudoconvex domains. Basic references include the monographs by Henkin and Leiterer [HL] and by Range [Ra2]. In the mid 1980's, these techniques led to an explicit description via integral kernels of the principal parts of abstract L^2 operators like $\overline{\partial}^* N$ and N (see Lieb and Range [LR1,2], and Phong and Stein [PS]).

3. Finite type in $\mathbb{C}^2$. Domains of finite type in $\mathbb{C}^2$ were introduced by Kohn in 1972 ([Ko3]), as a natural generalization of strict pseudoconvexity, as follows. Suppose $p \in bD \subset \mathbb{C}^2$, and let L be a nonzero tangent vector field to bD near p of type $(1,0)$. Then the complexified tangent space $\mathbb{C}TbD$ is spanned near p by L, $\overline{L}$, and T, for a suitably chosen vector field T. If D is pseudoconvex near p, it is well known that D is strictly pseudoconvex at p if and only if $\mathbb{C}T_p bD$ is spanned by L_p, $\overline{L}_p$ and $[L, \overline{L}]_p$, i.e., if T_p is a linear combination of these three vectors. The following definition thus introduces a natural higher order analogon of strict pseudoconvexity.

Definition. D is of *finite type* m at $p \in bD$ if L_p, $\overline{L}_p$ and $[X_1, [X_2, [\ldots, X_m] \ldots]_p$ span $\mathbb{C}T_p bD$, for some sequence $X_1, \ldots, X_m$, where $X_j = L$ or $\overline{L}$, while $L_p, \overline{L}_p$, and $[X_1, [\ldots, X_{m-1}] \ldots]_p$ do *not* span $\mathbb{C}T_p bD$ for all choices of $X_1, \ldots, X_{m-1}$.

In order to formulate the next results precisely, we need the following condition.

Definition. The $\overline{\partial}$-Neumann problem is *subelliptic* at $p \in bD$ (of order ε) if there exist a neighborhood U of p, a constant C, and $\varepsilon > 0$, such that

$$\|\varphi\|_\varepsilon \le C[\|\overline{\partial}\varphi\|_0 + \|\overline{\partial}^*\varphi\|_0 + \|\varphi\|_0] \tag{3.1}$$

for all smooth $(0,1)$ forms $\varphi \in \operatorname{dom} \overline{\partial}^*$ with support in U.

Subellipticity has numerous important consequences. For example, if it holds at every boundary point of D, then D satisfies condition R, and the solution operator $\overline{\partial}^* N$ for $\overline{\partial}$ satisfies

$$\|\overline{\partial}^* N f\|_{s+\varepsilon} \le C_s \|f\|_s \qquad \text{for } \overline{\partial}\text{-closed } f,$$

and all $s \ge 0$. In the proof of Theorem 3, Kohn proved that on strictly pseudoconvex domains the $\overline{\partial}$-Neumann problem is subelliptic of order $\frac{1}{2}$ at every boundary point.

Kohn's 1972 generalization [Ko3], as improved later by Hörmander, states:

THEOREM 4. *If $D \subset\subset \mathbb{C}^2$ is pseudoconvex and of finite type m at $p \in bD$, then the $\overline{\partial}$-Neumann problem is subelliptic of order $1/m$ at p.*

Greiner [Gr] subsequently showed that finite type m is indeed necessary for subellipticity of order $1/m$.

These results were within the frame work of L^2 Sobolev spaces, and no concrete information in other norms became available until much later.

4. Refined geometry of finite type domains in $\mathbb{C}^2$. Beginning in the mid 1980's, E.M. Stein and his collaborators (see [NRWS]), and, independently, Catlin [Ca3], introduced certain analytic/geometric invariants near points of finite type in $\mathbb{C}^2$, which provided the basic tools for obtaining much detailed information for domains of finite type.

Following Catlin's approach, the key invariant can be described, geometrically, as follows.

Let $p \in bD$ be a point of finite type m. Then, on a suitable neighborhood U of p, one can introduce, analytically, a function $\tau : U \times \mathbb{R}^+ \to \mathbb{R}^+$, such that for $q \in U \cap D$, and for $\delta = \delta(q) = \mathrm{dist}(q, bD)$, $\tau(q,\delta)$ measures the radius of the "maximal analytic disc centered at q" which is still contained in D. It is easy to see that if D is strictly pseudoconvex at p (i.e., $m = 2$), then $\tau(q,\delta) \approx \delta^{1/2}$. In general, one has

$$c_1 \delta^{1/2} \le \tau(q,\delta) \le c_2 \delta^{1/m}$$

for suitable constants c_1, c_2, independent of q. For the precise definition of τ the reader is referred to [Ca3]. It turns out that the boundary behavior of basic analytic objects can be described precisely in terms of τ.

We now list several major results in the theory of finite type domains in $\mathbb{C}^2$ which have been obtained in recent years building upon the foundations laid by Stein et al. and Catlin.

4.1. Precise estimates for the Szegö and Bergman kernels and their derivatives (Nagel et al. [NRSW], McNeal [Mc1]). In particular, these operators — which, by definition, are bounded on L^2 — are also bounded on L^p for $1 < p < \infty$.

4.2. Sharp Hölder estimates for solutions of $\overline{\partial}$ (Fefferman and Kohn [FeK]).

4.3. Precise description of the boundary behavior of the Bergman, Carathéodory, and Kobayashi metrics (Catlin [Ca3]).

4.4. Construction of integral solution operators for $\overline{\partial}$ which satisfy close to optimal Hölder estimates (Range [Ra3]).

4.5. Identification of the Bergman projection of $L^\infty(D)$ with the Bloch space $\mathcal{B}$ (Burke [Bu]).

4.6. Complete analysis of N and $\overline{\partial}^* N$ (Chang, Nagel, and Stein [CNS]).

5. Finite type in dimension greater than 2. Ever since Kohn proved his basic result on domains of finite type in $\mathbb{C}^2$, there have been attempts to generalize this work to higher dimension. Motivated by the results in dimension two, Kohn was led to the following.

CONJECTURE 1. *If D is pseudoconvex near $p \in bD$, then the $\overline{\partial}$-Neumann problem is subelliptic at p if and only if "bD is of finite type at p".*

The first major progress was made by Kohn in 1977 ([Ko4]). By introducing a new technique of "subelliptic multipliers", he proved a sufficient local condition for subellipticity which, in principle, could be verified in many cases. In particular, Diederich and Fornaess [DF] showed that Kohn's condition was satisfied at every boundary point of a bounded pseudoconvex domain with real analytic boundary. Consequently, the $\overline{\partial}$-Neumann problem is subelliptic on such domains.

A major difficulty with Conjecture 1 involved finding the right geometric/analytic notion of finite type. Early examples showed that the approach via higher order commutators (see section 3) did not work in dimension greater than 2. It turns out that the right notion involves the order of contact of bD with complex analytic varieties. More precisely, suppose bD is defined near $p \in bD$ by the vanishing of the function r, with $dr \neq 0$. Consider holomorphic maps $f : \varDelta \to \mathbb{C}^n$, with $f(0) = p$, where $\varDelta$ is open unit disc in $\mathbb{C}$. Denote by $\nu_0(f)$ the order of f at 0. If f is non constant, $\nu_0(f) < \infty$, and f parametrizes the germ of a one dimensional analytic variety through p.

DEFINITION. $t(p) = \sup\{\nu_0(r \circ f)/\nu_0(f) : f \in \mathcal{O}(\varDelta, \mathbb{C}^n),\ f(0) = p\}$.

It is clear that if there is an analytic variety A through p of positive dimension, which is contained in bD, then $t(p) = \infty$, but the converse is of course not true for C^∞ boundaries bD.

DEFINITION. (D'Angelo [Da1]). bD is said to be of *finite type* at p if $t(p) < \infty$.

The proof of the following "natural" result, which obviously must hold for any notion of finite type for which Conjecture 1 is true, is quite non-trivial:

THEOREM 5 ([Da1]). *Suppose D is pseudoconvex with C^∞ boundary. Then $\{p \in bD : t(p) < \infty\}$ is open in bD.*

We also note that if D is in $\mathbb{C}^2$ and $p \in bD$ is of finite type m, as defined in section 3, then $t(p) = m$. The reader may find an extensive, up-to-date treatment of finite type and related concepts in the recent monograph by D'Angelo [Da2].

With D'Angelo's notion of finite type, Kohn's conjecture was eventually proved by Catlin [Ca1,2].

THEOREM 6. *Suppose $D \subset \mathbb{C}^n$ is pseudoconvex with smooth boundary. Let $p \in bD$. Then the $\overline{\partial}$-Neumann problem is subelliptic at p if and only if $t(p) < \infty$.*

In contrast to the situation in dimension 2, there is no simple relationship between $t(p)$ and the maximal ε describing the order of subellipticity.

PROBLEM 2. *Describe the detailed geometric/analytic properties of finite type in dimension > 2.*

Very little is known, so far, regarding this problem. It appears to present formidable difficulties. Some results have been obtained by Fefferman, Kohn and Machedon ([FKM]) in the very restrictive case when the Levi form is diagonalizable. Most recently there has been a resurgence of interest in (Euclidean) convex domains, prompted — in part — by the fact that for convex domains the invariant $t(p)$ can be calculated by just considering the order of contact of bD at p with *complex lines*. McNeal [Mc2] obtained estimates for the Bergman kernel on convex domains of finite type, which imply, in particular, that on such domains the Bergman projection is bounded on L^p for $1 < p < \infty$. But the following question is still open.

CONJECTURE 2. *Suppose $D \subset\subset \mathbb{C}^n$ is convex with real analytic boundary (and hence, of finite type). Then there exist $\alpha > 0$ and a bounded solution operator for $\overline{\partial}$ from L^∞ into the Lipschitz space Λ_α.*

The conjecture was proved in case $n = 2$ by Range ([Ra1]) in 1976. In spite of the fact that on convex domains there exists a simple explicit integral solution operator for $\overline{\partial}$ (see [Ra2], for example), Conjecture 2 has remained open in higher dimensions for a long time. Partial results are known in very special cases ([Ra1], [DFW]).

6. General domains—revisited. In section 1 we stated two classical results which did not require any finite type hypothesis. In recent years there has been much interest in studying the $\overline{\partial}$ problem on such general domains in function spaces other than the classical L^2 or $C^\infty(\overline{D})$ settings. Most of the results obtained so far are negative.

Already in 1980, Sibony [Si] found a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$, on which it is not possible to solve $\overline{\partial}$ with L^∞ estimates. Such counterexamples have recently (in 1991) been found also in $\mathbb{C}^2$ (Berndtsson [Be2]). Around 1989, Fornaess and Sibony began investigating versions of Hörmander's Theorem 1 in weighted L^p norm, for $p \neq 2$ [FS1]. Quite surprisingly, they discovered that when $p > 2$, the weighted estimate (1.2), with 2 replaced by p, already fails on the unit disc $\Delta \subset \mathbb{C}$ for a general subharmonic function φ. On the other hand, they showed that Theorem 1 remains true for $D \subset\subset C^1$ and $1 < p \leq 2$; later Berndtsson [Be1] proved Theorem 1 for $D \subset\subset \mathbb{C}^1$ in case $p = 1$ as well. Fornaess and Sibony used the failure of the weighted estimate (1.2) for solutions of $\overline{\partial}$ on Δ in $\mathbb{C}^1$ when $p > 2$, to construct smoothly bounded (Hartogs) pseudoconvex domains D in $\mathbb{C}^3$, on which there is *no* L^p estimates with respect to Lebesgue measure (i.e. $\varphi = 0$) for solutions of $\overline{\partial}$ for $p > 2$. In fact, they showed that there is a definite loss of regularity, as follows: Given $p > 2$, there exist p', $2 < p' < p$, and a $\overline{\partial}$-closed (0,1) form $f \in L^p_{0,1}(D)$, such that there is no solution u of $\overline{\partial} u = f$ with $u \in L^{p'}(D)$. Using the technical improvements introduced by Berndtsson [Be2], Fornaess and Sibony [FS2] recently constructed smoothly bounded pseudoconvex domains in dimension *two*, for which L^p estimates for $\overline{\partial}$ do not hold for $2 < p \le \infty$. However, it is not known whether in $\mathbb{C}^2$ there is a definite loss of regularity analogous to the one in $\mathbb{C}^3$ discussed above. (See also Conjecture 3 below.) For the reader familiar with the so called "Corona Problem", we mention in passing that in [FS2] Fornaess and Sibony also constructed smoothly bounded pseudoconvex domains in $\mathbb{C}^2$ for which the Corona Theorem fails.

Next, we mention the main positive results in this area.

THEOREM 7. *Suppose* $D \subset\subset \mathbb{C}^2$ *is convex, with smooth boundary. Then there exist integral solution operators* $\widehat{T}_D$ *and* T_D *for* $\overline{\partial}$ *on* D *such that*

(a) $\widehat{T}_D$ *is bounded from* $L^p_{0,1}(D)$ *to* $L^p(D)$ *for* $1 < p < \infty$.

(b) T_D *is bounded from* $\Lambda_{\alpha,(0,1)}(D)$ *to* $\Lambda_\alpha(D)$ *for* $0 < \alpha < 2$, *and, on the subspace of* $\overline{\partial}$*-closed forms, for all* $\alpha \ge 2$ *as well.*

(c) T_D *is bounded from* $L^\infty_{0,1}(D)$ *to* $BMO(D)$ *with respect to Lebesgue measure on* D.

Part (a) was proved by Polking [Po], while parts (b) and (c) were proved by Range [Ra4]; partial results closely related to (b) were obtained earlier by Chaumat and Chollet [CC]. The operator T_D is the standard integral operator on convex domains (see [Ra2]), and $\widehat{T}_D$ is a simple modification of it. In fact, $\widehat{T}_D = T_D$ on $\overline{\partial}$-closed forms $f \in C^1_{0,1}(\overline{D})$.

While the operators T_D and $\widehat{T}_D$ exist on convex domains in arbitrary dimension, the proof of Theorem 7 breaks down when $n \ge 3$. The reason is as follows. In $\mathbb{C}^n$, the critical part of the kernel of T_D contains a factor $1/\phi^{n-1}(\zeta, z)$, where $\phi(\zeta, z)$ plays the role of $\zeta - z$ in $\mathbb{C}^1$. For fixed $z \in D$, $t = \operatorname{Im} \phi(\cdot, z)$ can be used as a (local) coordinate on bD, which allows to control the integral over bD in the definition of T_D when $n = 2$. But $\int_0^1 dt/t^2$ diverges hopelessly, and hence the analogous estimation of T_D fails already for $n = 3$.

However, the failure of the *proof* of Theorem 7 for $n \ge 3$ does not tell us much. Note, for example, that on polydiscs there are L^∞ and C^k estimates for $\overline{\partial}$ in arbitrary dimension (Bertrams [Br])! So the following question remains open.

PROBLEM 3. *Are there* L^p, $1 < p < \infty$, *and* Λ_α *estimates for* $\overline{\partial}$ *on convex domains in dimension* ≥ 3*?*

Note that the counterexamples by Fornaess and Sibony are definitely *not convex.*

PROBLEM 4. *Are there* L^1 *and* L^∞ *estimates for* $\overline{\partial}$ *on convex domains in* $\mathbb{C}^2$*?*

Lastly, there remains the question of L^p and Λ_α estimates on pseudoconvex domains in $\mathbb{C}^2$. While the *sharp* estimates are known to fail ([FS2] for L^p, and [Be3] for Λ_α), experience in dimension two with convex domains and domains of finite type, suggests that, "morally", such estimates should hold, if one allows for a "negligible" loss in regularity. So we end with the following conjecture.

CONJECTURE 3. *Let $D \subset\subset \mathbb{C}^2$ be a smoothly bounded pseudoconvex domain. Then there exists a solution operator T for $\overline{\partial}$ on D, which satisfies the estimate*

$$\|Tf\|_{L^p(D)} \leq C_p \| |\log \operatorname{dist}(\cdot, bD)|^\beta f\|_{L^p_{0,1}(D)}$$

for $1 \leq p \leq \infty$ and for some $\beta = \beta(p) > 0$.

Note that the Conjecture is true for $p = 1$ with $\beta = 1$ (Bonneau and Diederich [BD]).

References

[Ba1] D. Barrett, *Irregularity of the Bergman projection on a smooth bounded domain in $\mathbb{C}^2$*, Ann. of Math. 119 (1984), 431–436.

[Ba2] —, *Behavior of the Bergman projection on the Diederich-Fornaess Worm*, preprint.

[BL] S. R. Bell and E. Ligocka, *A simplification and extension of Fefferman's Theorem on biholomorphic mappings*, Invent. Math. 57 (1980), 283–289.

[Be1] B. Berndtsson, *Weighted estimates for the $\overline{\partial}$-equation in domains in $\mathbb{C}$*, Duke Math. J. 66 (1992), 239–255.

[Be2] —, *A smoothly bounded pseudoconvex domain in $\mathbb{C}^2$ where L^∞ estimates don't hold*, Ark. Mat., to appear.

[Be3] —, *Some recent results on estimates for the $\overline{\partial}$-equation*, preprint, 1992.

[Br] J. Bertrams, *Randregularität von Lösungen der $\overline{\partial}$-Gleichung auf den Polyzylinder und zweidimensionalen analytischen Polyedern*, Bonner Math. Schr. 176 (1986).

[BD] P. Bonneau and K. Diederich, *Integral solution operators for the Cauchy-Riemann equations on pseudoconvex domains*, Math. Ann. 286 (1990), 77–100.

[BS] H. P. Boas and E. J. Straube, *Sobolev estimates for the $\overline{\partial}$-Neumann operator on domains admitting a defining function that is plurisubharmonic on the boundary*, Math. Z. 206 (1991), 81–88.

[Bo] A.Boggess, *CR Manifolds and the Tangential Cauchy Riemann Complex*, CRC Press, Boca Raton, FL, 1991.

[Bu] K. Burke, *Duality of the Bergman Spaces on Some Weakly Pseudoconvex Domains*, Ph.D. Dissertation, SUNY at Albany, New York, 1991.

[Ca1] D. Catlin, *Necessary conditions for subellipticity of the $\overline{\partial}$-Neumann problem*, Ann. of Math. 117 (1983), 147–171.

[Ca2] —, *Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains*, ibid. 126 (1987), 131–191.

[Ca3] —, *Estimates of invariant metrics on pseudoconvex domains of dimension two*, Math. Z. 200 (1989), 429–466.

[CNS] D. C. Chang, A. Nagel and E. M. Stein, *Estimates for the $\overline{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$*, Acta Math. 169 (1992), 153–228.

[CC] J. Chaumat et A.-M. Chollet, *Estimations hölderiennes pour les équations de Cauchy-Riemann dans les convexes compacts de $\mathbb{C}^n$*, Math. Z. 207 (1991), 501–534.

[Ch] S. C. Chen, *Global regularity of the $\overline{\partial}$-Neumann problem in dimension two*, in: Proc. Sympos. Pure Math. 52 (1991), 55–61.

[Da1] J. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. 115 (1982), 615–637.

[Da2] —, *Several Complex Variables and the Geometry of Real Hypersurfaces*, CRC Press, Boca Raton, FL, 1993.

[DF] K. Diederich and J. E. Fornaess, *Pseudoconvex domains with real analytic boundary*, Ann. of Math. 107 (1978), 371–384.

[DFW] K. Diederich, J. E. Fornaess and J. Wiegerinck, *Sharp Hölder estimates for $\overline{\partial}$ on ellipsoids*, Manuscripta Math. 56 (1986), 399–413.

[FeK] C. L. Fefferman and J. J. Kohn, *Hölder estimates on domains in two complex dimensions and on three dimensional CR manifolds*, Adv. in Math. 69 (1988), 233–303.

[FKM] C. L. Fefferman, J. J. Kohn, and M. Machedon, *Hölder estimates on CR manifolds with a diagonalizable Levi form*, ibid. 84 (1990), 1–90.

[FoK] G. Folland and J. Kohn, *The Neumann Problem for the Cauchy-Riemann complex*, Ann. of Math. Stud., Princeton, 1972.

[FS1] J. E. Fornaess and N. Sibony, *L^p estimates for $\overline{\partial}$*, in: Proc. Sympos. Pure Math. 52 (1991), 129–163.

[FS2] —, —, *Smooth pseudoconvex domains in $\mathbb{C}^2$ for which the Corona theorem and L^p estimates for $\overline{\partial}$ fail*, preprint, 1992.

[GL] H. Grauert und I. Lieb, *Das Ramirezsche Integral und die Lösung der Gleichung $\overline{\partial}f = \alpha$ im Bereich der beschränkten Formen*, Rice Univ. Stud. 56 (1970), 29–50.

[Gr] P. Greiner, *Subelliptic estimates of $\overline{\partial}$-Neumann problem $\mathbb{C}^2$*, J. Differential Geom. 9 (1974), 239–260.

[He] G. M. Henkin, *Integral representations in strictly pseudoconvex domains and applications to the $\overline{\partial}$-problem*, Mat. Sb. 82 (1970), 300–308; Math. USSR-Sb. 11 (1970), 273–281.

[HL] G. M. Henkin and J. Leiterer, *Theory of Functions on Complex Manifolds*, Akademie-Verlag, Berlin, 1984.

[Hö] L. Hörmander, *L^2-estimates and existence theorems for the $\overline{\partial}$-operators*, Acta Math. 113 (1965), 89–152.

[Ko1] J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds I, II*, Ann. of Math. 2 (1963), 112–148; ibid. 79 (1964), 450–472.

[Ko2] —, *Global regularity for $\overline{\partial}$ on weakly pseudoconvex manifolds*, Trans. Amer. Math. Soc. 181 (1973), 273–292.

[Ko3] —, *Boundary behavior of $\overline{\partial}$ on weakly pseudoconvex manifolds of dimension two*, J. Differential Geom. 6 (1972), 523–542.

[Ko4] —, *Subellipticity of the $\overline{\partial}$-Neumann problem on pseudoconvex domains: Sufficient conditions*, Acta Math. 142 (1979), 79–122.

[LR1] I. Lieb and R. M. Range, *Integral representations and estimates in the theory of the $\overline{\partial}$-Neumann problem*, Ann. of Math. 123 (1986), 265–301.

[LR2] —, —, *The kernel of the $\overline{\partial}$-Neumann operator on strictly pseudoconvex domains*, Math. Ann. 278 (1987), 151–173.

[Mc1] J. McNeal, *Boundary behaviour of the Bergman kernel function in $\mathbb{C}^2$*, Duke Math. J. 58 (1989), 499–512.

[Mc2] —, *Estimates for the Bergman kernel on convex domains*, preprint.

[PS] D. H. Phong and E. M. Stein, *Hilbert integrals, singular integrals, and Radon transforms, I*, Acta Math. 157 (1986), 99–157; II, Invent. Math. 86 (1986), 75–113.

[Po] J. Polking, *The Cauchy-Riemann equation on convex sets*, in: Proc. Sympos. Pure Math. 52 (1993), 309–322.

[Ra1] R. M. Range, *On Hölder estimates for* $\overline{\partial} u = f$ *on weakly pseudoconvex domains*, in: Proc. Int. Conf. Cortona, 1976–77, Scuola Norm. Sup. Pisa, 1978, 247–267.

[Ra2] —, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer, New York, 1986.

[Ra3] —, *Integral kernels and Hölder estimates for* $\overline{\partial}$ *on pseudoconvex domains of finite type in* $\mathbb{C}^2$, Math. Ann. 288 (1990), 63–74.

[Ra4] —, *On Hölder and BMO estimates for* $\overline{\partial}$ *on convex domains in* $\mathbb{C}^2$, J. Geom. Anal. 2 (1992), 575–584.

[Si] N. Sibony, *Un exemple de domaine pseudoconvexe régulier où l'équation* $\overline{\partial} u = f$ *n'admet pas de solution bornée pour* f *bornée*, Invent. Math. 62 (1980), 235–242.