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THE SMALLEST POSITIVE EIGENVALUE OF A QUASISYMMETRIC AUTOMORPHISM OF THE UNIT CIRCLE

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Abstract. This paper provides sufficient conditions on a quasisymmetric automorphism γ of the unit circle which guarantee the existence of the smallest positive eigenvalue of γ. They are expressed by means of a regular quasiconformal Teichmüller self-mapping φ of the unit disc Δ. In particular, the norm of the generalized harmonic conjugation operator $A_\gamma : \mathbb{H} \to \mathbb{H}$ is determined by the maximal dilatation of φ. A characterization of all eigenvalues of a quasisymmetric automorphism γ in terms of the smallest positive eigenvalue of some other quasisymmetric automorphism σ is given.

1. Introduction. Let us denote by $\mathbb{Q}_{\mathbf{T}}(K)$, $1 \le K < \infty$, the class of all homeomorphic self-mappings of the unit circle $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ which admit a K-quasiconformal extension to the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathbb{Q}_{\mathbf{T}} = \bigcup_{1 \le K < \infty} \mathbb{Q}_{\mathbf{T}}(K)$. For any homeomorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$ we set $K(\gamma) = \inf\{K : \gamma \in \mathbb{Q}_{\mathbf{T}}(K)\}$. Due to J. G. Krzyż, cf. [K], the class $\mathbb{Q}_{\mathbf{T}}$ coincides with the class of all quasisymmetric automorphisms of the unit circle $\mathbf{T}$, i.e. all sense-preserving homeomorphisms $\gamma : \mathbf{T} \to \mathbf{T}$ satisfying

$$k^{-1} \le |\gamma(I_1)|/|\gamma(I_2)| \le k$$

for each pair of adjacent closed arcs $I_1, I_2 \subset \mathbf{T}$ of equal length: $0 < |I_1| = |I_2| \le \pi$ where the constant k depends on γ only. Let us denote by $L^p_{\mathbf{T}}$, $1 \le p < \infty$, the space of all functions $f : \mathbf{T} \to \mathbb{R}$, p-integrable on $\mathbf{T}$, i.e. $\|f\|_p =$

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$(\int_{\mathbf{T}} |f(z)|^p |dz|)^{1/p} < \infty$ and let $L^\infty_{\mathbf{T}} = \{f \in L^1_{\mathbf{T}} : \|f\|_\infty = \operatorname{sup\,ess}_{z\in\mathbf{T}} |f(z)| < \infty\}$. The space $L^2_{\mathbf{T}}$ is a real Hilbert space with the inner product

$$(f,g) = \int_{\mathbf{T}} f(z)g(z)|dz|, \qquad f,g \in L^2_{\mathbf{T}}.$$

With any function $f \in L^1_{\mathbf{T}}$ we can associate an analytic function $f_\Delta : \Delta \to \mathbb{C}$ given by the formula

$$\begin{aligned} f_\Delta(z) &= \frac{1}{2\pi} \int_{\mathbf{T}} f(u) \frac{u+z}{u-z} |du| \\ &= \frac{1}{2\pi} \int_{\mathbf{T}} f(u)|du| + \frac{1}{\pi} \sum_{n=1}^{\infty} \Big(\int_{\mathbf{T}} f(u)\overline{u}^n |du| \Big) z^n, \qquad z \in \Delta. \end{aligned}$$

The space $\mathbb{H} = \{f \in L^1_{\mathbf{T}} : \int_\Delta |f'_\Delta|^2 dS < \infty$ and $f_\Delta(0) = 0\}$, where $f'_\Delta = (f_\Delta)'$, equipped with the inner product $(f,g)_{\mathbb{H}} = \operatorname{Re} \int_\Delta f'_\Delta \overline{g'_\Delta} dS$, $f,g \in \mathbb{H}$, is a real Hilbert space, isometric with the space $\widetilde{L}^2_{\mathbf{T}} = \{f \in L^2_{\mathbf{T}} : f_\Delta(0) = 0\}$, cf. [P1, Theorem 1.2]. In the paper [P2] a linear homeomorphism A_γ of the Hilbert space $\mathbb{H}$ onto itself was associated with every quasisymmetric automorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$. If $\gamma \in \mathbb{Q}_{\mathbf{T}}$ is sufficiently regular then the operator A_γ has a nice form, cf. [P2, Theorem 1.4], given by means of a singular integral

$$A_\gamma(f)(z) = \frac{1}{\pi} \operatorname{Re} \mathrm{P.V.} \int_{\mathbf{T}} \frac{f(u)}{\gamma(z) - \gamma(u)} d\gamma(u) - a_\gamma(f)$$

for a.e. $z \in \mathbf{T}$ and $f \in \mathbb{H}$ where

$$a_\gamma(f) = \frac{1}{2\pi} \int_{\mathbf{T}} \Big(\frac{1}{\pi} \operatorname{Re}\ \mathrm{P.V.} \int_{\mathbf{T}} \frac{f(u)}{\gamma(z) - \gamma(u)} d\gamma(u) \Big) |dz|$$

is a normalization constant. If $\gamma(z) = z$, $z \in \mathbf{T}$, then A_γ becomes the usual harmonic conjugation operator A, see [G]. Moreover, $A^2_\gamma = -I$ for any $\gamma \in \mathbb{Q}_{\mathbf{T}}$, where I is the identity operator. Thus A_γ may be called a generalized harmonic conjugation operator. For basic properties of the operator A_γ we refer to [P2]. Now we quote two properties essential for our considerations. Namely, $AA_\gamma : \mathbb{H} \to \mathbb{H}$ is a symmetric operator and

$$A_\gamma = B_\gamma A B_{\gamma^{-1}} \tag{1.1}$$

where B_γ is a linear homeomorphism of the space $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ onto itself such that

$$B_\gamma(f) = f \circ \gamma - (f \circ \gamma)_\Delta(0) \quad \text{on } \mathbf{T} \tag{1.2}$$

for every continuous function $f \in \mathbb{H}$. This shows that the operator A_γ is related to the Neumann–Poincaré integral operator of a Jordan curve Γ. More precisely, eigenvalues of the Neumann–Poincaré kernel k, cf. [BS], [S], correspond to eigenvalues of the symmetric operator $AA_\gamma : \mathbb{H} \to \mathbb{H}$ where γ is a welding homeomorphism of a sufficiently smooth Jordan curve Γ, cf. [P1], [KP]. This justifies introducing the notion of an eigenvalue and a spectral value of a quasisymmetric automorphism of the unit circle, by means of the spectrum of the operator AA_γ, cf. [P3], or equivalently, by means of the spectrum of the operator R_γ, cf. [P1], [KP], because $R_\gamma = I + AA_\gamma$, cf. [P2, (2.4)]. For the reader's convenience we quote

Definition 1.1. A real number λ is said to be an *eigenvalue* of a quasisymmetric automorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$ if there exists a function $f \in \mathbb{H}$ with the norm $\|f\|_{\mathbb{H}} = 1$ such that

$$(\lambda + 1)A(f) = (\lambda - 1)A_\gamma(f). \tag{1.3}$$

The function f is said to be an *eigenfunction* of γ associated with the eigenvalue λ.

The set of all eigenvalues of $\gamma \in \mathbb{Q}_{\mathbf{T}}$ is denoted by Λ^*_γ.

Definition 1.2. A real number λ is said to be a *spectral value* or an *approximate eigenvalue* of a quasisymmetric automorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$ if there exist functions $f_n \in \mathbb{H}$, $\|f_n\|_{\mathbb{H}} = 1$, $n = 1, 2, \dots$ such that

$$\|(\lambda + 1)A(f_n) - (\lambda - 1)A_\gamma(f_n)\|_{\mathbb{H}} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.4}$$

The set of all spectral values of $\gamma \in \mathbb{Q}_{\mathbf{T}}$ is denoted by Λ_γ. From [P2, Theorem 2.2] we are able to infer the following basic properties of the spectra Λ^*_γ and Λ_γ:

(i) $\Lambda_\gamma = \emptyset$ iff $\gamma = \mathbb{Q}_{\mathbf{T}}(1)$;
(ii) $\Lambda^*_\gamma \subset \Lambda_\gamma$;
(iii) if $\lambda \in \Lambda_\gamma$ then $|\lambda| \ge \frac{K(\gamma)+1}{K(\gamma)-1}$;
(iv) for every $\nu, \eta \in \mathbb{Q}_{\mathbf{T}}(1)$ $\Lambda_\gamma = \Lambda_{\nu\circ\gamma\circ\eta}$ and $\Lambda^*_\gamma = \Lambda^*_{\nu\circ\gamma\circ\eta}$;
(v) $\Lambda_\gamma = \Lambda_{\gamma^{-1}} = \Lambda_{\bar{i}_{\mathbf{T}}\circ\gamma\circ\bar{i}_{\mathbf{T}}}$ and $\Lambda^*_\gamma = \Lambda^*_{\gamma^{-1}} = \Lambda^*_{\bar{i}_{\mathbf{T}}\circ\gamma\circ\bar{i}_{\mathbf{T}}}$;
(vi) if $\lambda \in \Lambda_\gamma$ then $-\lambda \in \Lambda_\gamma$ and if $\lambda \in \Lambda^*_\gamma$ then $-\lambda \in \Lambda^*_\gamma$

where $i_{\mathbf{T}}(z) = z$, $z \in \mathbf{T}$. For the proof of these properties cf. [P3]. A natural question appears when is the inequality (iii) sharp, i.e. when

$$\inf\{|\lambda| : \lambda \in \Lambda_\gamma\} = \frac{K(\gamma) + 1}{K(\gamma) - 1}. \tag{1.5}$$

This is strictly related to the problem of determining the norm $\|A_\gamma\|$ which always does not exceed $K(\gamma)$, cf. [P2, Theorem 1.3]. Namely, $\|A_\gamma\| = K(\gamma)$ iff the equality (1.5) holds. In Section 2 of this paper we establish Theorem 2.2 giving a sufficient condition on a quasisymmetric automorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$ which guarantees the existence of the smallest positive eigenvalue $\lambda \in \Lambda^*_\gamma$ satisfying (1.5). In particular, this implies the equality $\|A_\gamma\| = K(\gamma)$. As a consequence we obtain Corollary 2.3 which characterizes every positive eigenvalue $\lambda \in \Lambda^*_\gamma$ as the smallest positive eigenvalue of some other quasisymmetric automorphism $\sigma \in \mathbb{Q}_{\mathbf{T}}$.

2. Main results and proofs. In what follows we need the following

LEMMA 2.1 *If* $\varphi \in \mathbb{Q}_\Delta$ *is a quasiconformal extension of a quasisymmetric automorphism* $\gamma \in \mathbb{Q}_{\mathbf{T}}$ *and there exist functions* $f, g \in \mathbb{H}$ *and a constant* $c \in \mathbb{C}$ *satisfying the equality*

$$\operatorname{Re} g_\Delta(z) = \operatorname{Re} f_\Delta \circ \varphi(z) + c, \qquad z \in \Delta, \tag{2.1}$$

then $g = B_\gamma(f)$.

Proof. By [P1, Theorem 1.2] $C_{\mathbf{T}} \cap \mathbb{H}$ is a dense subset of the space $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ where the class $C_{\mathbf{T}}$ consists of all continuous real–valued functions on the unit circle $\mathbf{T}$. Thus there exist functions $f_n, h_n \in C_{\mathbf{T}} \cap \mathbb{H}$, $n \in \mathbb{N}$, approximating the functions f, g in $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$, respectively, i.e.

$$\|f_n - f\|_{\mathbb{H}} \to 0 \quad \text{and} \quad \|h_n - g\|_{\mathbb{H}} \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$

Since a harmonic function minimizes the Dirichlet integral within the class of real continuous functions on the closed unit disc $\overline{\Delta}$ with given boundary values and absolutely continuous on a.e. chord of $\overline{\Delta}$, parallel to the coordinate axes, setting $g_n = B_\gamma(f_n) = f_n \circ \gamma - (f_n \circ \gamma)_\Delta(0)$, $n \in \mathbb{N}$, we obtain by (2.1) that for any $n \in \mathbb{N}$

$$\begin{aligned}
&\int_\Delta |(g_n)'_\Delta - g'_\Delta|^2 dS \qquad (2.3)\\
&= \lim_{m\to\infty} \int_\Delta |(g_n)'_\Delta - (h_m)'_\Delta|^2 dS = \lim_{m\to\infty} \int_\Delta |((f_n \circ \gamma)_\Delta - (h_m)_\Delta)'|^2 dS\\
&= 4 \lim_{m\to\infty} \int_\Delta |\partial \operatorname{Re}((f_n \circ \gamma)_\Delta - (h_m)_\Delta)|^2 dS\\
&\le 4 \lim_{m\to\infty} \int_\Delta |\partial \operatorname{Re}((f_n)_\Delta \circ \varphi - (h_m)_\Delta)|^2 dS = 4 \int_\Delta |\partial \operatorname{Re}((f_n)_\Delta \circ \varphi - g_\Delta)|^2 dS\\
&= 4 \int_\Delta |\partial \operatorname{Re}((f_n)_\Delta \circ \varphi - f_\Delta \circ \varphi)|^2 dS = 4 \int_\Delta |\partial \operatorname{Re}((f_n - f)_\Delta \circ \varphi)|^2 dS
\end{aligned}$$

where

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

are formal derivatives. This and the K-quasi-invariance of the Dirichlet integral, cf. [A], lead to

$$\begin{aligned}
4 \int_\Delta |\partial \operatorname{Re}((f_n - f)_\Delta \circ \varphi)|^2 dS &\le 4K(\gamma) \int_\Delta |\partial \operatorname{Re}(f_n - f)_\Delta|^2 dS\\
&= K(\gamma) \int_\Delta |(f_n - f)'_\Delta|^2 dS\\
&= K(\gamma) \|f_n - f\|^2_{\mathbb{H}} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence by (2.3)

$$\|B_\gamma(f_n) - g\|_{\mathbb{H}} \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

On the other hand, by the continuity of the operator $B_\gamma : (\mathbb{H}, \|\cdot\|_{\mathbb{H}}) \to (\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ and (2.2) we have

$$\|B_\gamma(f_n) - B_\gamma(f)\|_{\mathbb{H}} \to 0 \quad \text{as } n \to \infty.$$

This together with (2.4) yields $g = B_\gamma(f)$ which ends the proof. ∎

The following main theorem involves the notion of a regular quasiconformal Teichmüller mapping. We recall that a quasiconformal self-mapping φ of the unit disc Δ is said to be a regular Teichmüller mapping if there exists an analytic function $\psi : \Delta \to \mathbb{C}$ and a constant k, $0 \le k < 1$, such that the complex dilatation of φ has the form

$$\frac{\overline{\partial}\varphi}{\partial\varphi} = k\frac{\overline{\psi}}{|\psi|}. \tag{2.5}$$

THEOREM 2.2. *If $f \in \mathbb{H}$, $\|f\|_{\mathbb{H}} > 0$, and φ is a regular quasiconformal Teichmüller extension of an automorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$ to Δ with the complex dilatation*

$$\frac{\overline{\partial}\varphi}{\partial\varphi} = -\frac{1}{\lambda}\frac{\overline{f'_\Delta}}{f'_\Delta} \tag{2.6}$$

where λ, $|\lambda| > 1$, is a real constant then $\lambda \in \Lambda^*_\gamma$ and $|\lambda|$ is the smallest positive eigenvalue of γ, i.e.

$$|\lambda| = \min\{|\mu| : \mu \in \Lambda^*_\gamma\} = \min\{|\mu| : \mu \in \Lambda_\gamma\}. \tag{2.7}$$

Moreover,

$$\|A_\gamma\| = K(\gamma) = \frac{|\lambda| + 1}{|\lambda| - 1} \tag{2.8}$$

and

$$(\lambda + 1)A(f) = (\lambda - 1)A_\gamma(f). \tag{2.9}$$

Proof. Let G be a complex function in the unit disc Δ such that

$$G \circ \varphi = \overline{f_\Delta} - \lambda f_\Delta. \tag{2.10}$$

Differentiating both sides of this equality with respect to z and $\overline{z}$ we get the simultaneous equations

$$(\partial G) \circ \varphi\partial\varphi + (\overline{\partial}G) \circ \varphi\partial\overline{\varphi} = -\lambda f'_\Delta,$$
$$(\partial G) \circ \varphi\overline{\partial}\varphi + (\overline{\partial}G) \circ \varphi\overline{\partial}\overline{\varphi} = \overline{f'_\Delta}.$$

Since $\partial\varphi\overline{\partial}\overline{\varphi} - \overline{\partial}\varphi\partial\overline{\varphi} = \partial\varphi\overline{\partial\varphi} - \overline{\partial}\varphi\overline{\overline{\partial}\varphi} = |\partial\varphi|^2 - |\overline{\partial}\varphi|^2 > 0$ a.e. in Δ, (2.6) implies that $\overline{\partial}G = 0$ a.e. in Δ. This way G is an analytic function in Δ, cf. [A]. Moreover, by the quasi–invariance of the Dirichlet integral we derive from the equality (2.10) that

$$\int_{\Delta} |G'|^2 dS = 4 \int_{\Delta} |\partial \operatorname{Re}((\overline{f_\Delta} - \lambda f_\Delta) \circ \varphi^{-1})|^2 dS$$
$$\le 4K(\varphi^{-1})|1-\lambda|^2 \int_{\Delta} |\partial \operatorname{Re} f_\Delta|^2 dS = K(\varphi)|1-\lambda|^2 \int_{\Delta} |f'_\Delta|^2 dS < \infty.$$

Thus there exists a function $g \in \mathbb{H}$ such that $G(z) = g_\Delta(z) + G(0)$, $z \in \Delta$, and by the equality (2.10) we get on Δ

$$\operatorname{Re} g_\Delta \circ \varphi + \operatorname{Re} G(0) = (1-\lambda) \operatorname{Re} f_\Delta,$$
$$\operatorname{Im} g_\Delta \circ \varphi + \operatorname{Im} G(0) = -(1+\lambda) \operatorname{Im} f_\Delta.$$

Hence, by the definition of the harmonic conjugation operator A and Lemma 2.1 we obtain

$$B_\gamma(g) = (1-\lambda) f \quad \text{and} \quad B_\gamma(A(g)) = -(1+\lambda)A(f).$$

This gives by (1.1)

$$(1+\lambda)A(f) = (\lambda-1)B_\gamma A B_\gamma^{-1}(f) = (\lambda-1)A_\gamma(f)$$

which proves the equality (2.9). This means that $\lambda \in \Lambda_\gamma^*$. It follows from the assumption of Lemma 2.1 that φ is a K-quasiconformal regular Teichmüller extension of the automorphism γ on Δ with $K = (|\lambda|+1)/(|\lambda|-1)$. Thus

$$\|A_\gamma\| \le K(\gamma) \le K. \tag{2.11}$$

If $\lambda < -1$ then by the property (vi) $|\lambda| = -\lambda \in \Lambda_\gamma^*$ as well. Therefore there exists $h \in \mathbb{H}$, $\|h\|_{\mathbb{H}} = 1$, satisfying

$$(|\lambda|+1)A(h) = (|\lambda|-1)A_\gamma(h).$$

Hence

$$\|A_\gamma(h)\|_{\mathbb{H}} = \|AA_\gamma(h)\|_{\mathbb{H}} = \frac{|\lambda|+1}{|\lambda|-1} = K$$

because $A^2 = -I$ and A is an isometry of the space $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ onto itself, cf. [P2, Theorem 1.3]. Thus $\|A_\gamma\| \ge K$. This together with (2.11) gives the equality (2.8) which yields the equality (2.7) because of the properties (ii) and (iii). This completes the proof. ■

Remark. This result seems to be closely related to that in [Kü, p. 302]. The equality (2.8) states additionally that the mapping φ in Theorem 2.2 is the extremal quasiconformal extension of γ to the unit disc Δ, i.e. an extension with the smallest maximal dilatation. This way we have proved, by the way, Strebel's theorem, cf. [St1], [St2], [L], in the case when the function ψ in (2.5) is a square of some analytic function, square integrable on Δ.

The smallest positive eigenvalue of a quasisymmetric automorphism γ of the unit circle plays a particularly important role among other eigenvalues of γ. Namely, every positive eigenvalue $\lambda \in \Lambda_\gamma^*$ can be expressed as the smallest positive eigenvalue of some other quasisymmetric automorphism $\sigma \in \mathbb{Q}_{\mathbf{T}}$. This interesting fact is the subject of the following corollary to Theorem 2.2.

COROLLARY 2.3. *If* $\lambda \in \Lambda_\gamma^*$, $\lambda > 0$ *is any eigenvalue of an automorphism* $\gamma \in \mathbb{Q}_{\mathbf{T}}$ *then there exists an automorphism* $\sigma \in \mathbb{Q}_{\mathbf{T}}$ *such that*

$$|\lambda| = \min\{|\mu| : \mu \in \Lambda_\sigma^*\} = \min\{|\mu| : \mu \in \Lambda_\sigma\} \tag{2.12}$$

and

$$K(\sigma) = \|A_\sigma\| = \frac{\lambda+1}{\lambda-1}. \tag{2.13}$$

Moreover, *the automorphism* $\sigma \in \mathbb{Q}_{\mathbf{T}}(K)$ *admits a* $K(\sigma)$*-quasiconformal extension* φ *to the unit disc* Δ *with a complex dilatation*

$$\frac{\overline{\partial}\varphi}{\partial\varphi} = -\frac{1}{\lambda}\frac{\overline{f'_\Delta}}{f'_\Delta} \tag{2.14}$$

where

$$(\lambda+1)A(f) = (\lambda-1)A_\sigma(f). \tag{2.15}$$

Proof. Assume $\lambda \in \Lambda_\gamma^*$, $\lambda > 0$ is an eigenvalue of an automorphism $\gamma \in \mathbb{Q}_{\mathbf{T}}$. Then there exists a function $f \in \mathbb{H}$, $\|f\|_{\mathbb{H}} = 1$ satisfying (1.3). It follows from the mapping theorem, cf. [LV; p. 194], also [B1], [B2], [ŁK] that there exists a solution φ of the Beltrami equation (2.14) being a K-quasiconformal self-mapping of Δ with $K = (\lambda+1)/(\lambda-1)$. It is well known that φ has a continuous extension to $\mathbf{T}$ as a quasisymmetric automorphism $\sigma \in \mathbb{Q}_{\mathbf{T}}$, cf. [LV]. Then the assumptions of Theorem 2.2 are satisfied for the automorphism $\sigma \in \mathbb{Q}_{\mathbf{T}}$ which satisfies the equalities (2.12), (2.13) and (2.15). This ends the proof. ∎

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