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A NOTE ON COEFFICIENT MULTIPLIERS $(H^p, \mathcal{B})$ AND $(H^p, BMOA)$

MARIA NOWAK

Institute of Mathematics, M. Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland

1. Introduction and statement of results. For a function f analytic in $\mathbf{U} = \{z : |z| < 1\}$ let

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r,f) = \max_{0\le\theta\le 2\pi} |f(re^{i\theta})|,$$

where $0 \le r < 1$.

The Hardy class H^p, $0 < p \le \infty$, is the space of those f for which

$$\|f\|_p = \sup_{0\le r<1} M_p(r,f) < \infty.$$

A function $f \in H^1$ is said to be in the space $BMOA$ iff its boundary function $f(e^{i\theta})$ is of bounded mean oscillation.

The Bloch space $\mathcal{B}$ consists of all analytic function in $\mathbf{U}$ for which

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbf{U}} (1-|z|)|f'(z)| < \infty$$

The proper inclusions:

$$H^\infty \subset BMOA \subset \bigcap_{0<p<\infty} H^p, \qquad BMOA \subset \mathcal{B}$$

are well-known (e.g. [3]).

A complex sequence $\{\lambda_n\}$ is called a multiplier of a sequence space A into a sequence space B if $\{\lambda_n a_n\} \in B$ whenever $\{a_n\} \in A$. A space of analytic functions in $\mathbf{U}$ can be regarded as a sequence space by identifying each function with its

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sequence of Taylor coefficients. The set of all multipliers from A to B will be denoted by (A, B).

Recently Mateljevic and Pavlovic ([4], see also [5]) have characterized the multiplier spaces $(H^1, \mathcal{B})$ and $(H^p, BMOA)$, $1 \le p \le 2$. They have proved the following theorems:

THEOREM A. *Let* $1 \le p \le 2$ *and* $1/p + 1/q = 1$. *Then* $g \in (H^p, BMOA)$ *if and only if*

$$M_q(r, g') \le \frac{c}{1-r}, \qquad 0 < r < 1,$$

where c *denotes a constant.*

THEOREM B. $(H^1, \mathcal{B}) = (H^1, BMOA) = \mathcal{B}$

Here we extend the above theorems by describing the spaces $(H^p, \mathcal{B})$, $0 < p < \infty$ and $(H^p, BMOA)$, $0 < p < 1$.

Let c denote a general constant not necessarily the same in each case. We have

THEOREM 1. *If* $1 \le p < \infty$ *and* $1/p + 1/q = 1$ *then*

$$g \in (H^p, \mathcal{B})$$

if and only if there is a constant c *such that*

$$M_q(r, g') \le \frac{c}{1-r}, \qquad 0 < r < 1. \tag{1}$$

THEOREM 2. *If* $0 < p < 1$, n *is an integer such that* $1/p < n + 1$ *then*

$$\begin{aligned}(H^p, H^\infty) &= (H^p, BMOA) = (H^p, \mathcal{B}) \\ &= \left\{ g \ :\ M_\infty(r, g^{(n)}) < \frac{c}{(1-r)^{n+1-1/p}} \right\} = A_n.\end{aligned} \tag{2}$$

Note that for $0 < p \le 2$, $(H^p, \mathcal{B}) = (H^p, BMOA)$.

2. Proof of Theorem 1. For $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n$, $g(z) = \sum_{n=0}^\infty \widehat{g}(n) z^n$ analytic in $\mathbf{U}$ define

$$h(z) = f \star g(z) = \sum_{n=0}^{\infty} \widehat{f}(n) \widehat{g}(n) z^n. \tag{3}$$

Then

$$h(r^2 e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(re^{i(\varphi-\theta)}) d\theta, \qquad 0 < r < 1. \tag{4}$$

Assume that g satisfies (1) and $f \in H^p$, $1 \le p < \infty$. Differentiating (4) with respect to φ we obtain

$$rh'(r^2 e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g'(re^{i(\varphi-\theta)}) e^{-i\theta} d\theta.$$

Hence by Hölder's inequality

$$M_\infty(r^2, h') \le C M_p(r, f) M_q(r, g'), \tag{5}$$

which implies $h \in \mathcal{B}$.

To prove the converse suppose that g is an analytic function such that $f \star g \in \mathcal{B}$ whenever $f \in H^p$, $1 < p < \infty$. Without loss of generality we may assume that $f(0) = 0$. Then $f_1(z) = f(z)/z$ also belongs to H^p and $\|f_1\|_p = \|f\|_p$. It follows from the closed graph theorem that $T_g(f) = f \star g$ is a bounded linear operator from H^p to $\mathcal{B}$. So there is a constant c such that for any $f \in H^p$

$$\|T_g(f)\|_{\mathcal{B}} = \sup_{\substack{0\le r<1\\ 0\le\varphi\le2\pi}} (1-r^2)\left|\int_0^{2\pi} \frac{e^{-i\theta}}{r} f(re^{i\theta}) g'(re^{i(\varphi-\theta)})\,d\theta\right| \le c\|f\|_p. \tag{6}$$

This implies

$$\left|\int_0^{2\pi} \frac{f(re^{i\theta})}{re^{i\theta}} g'(re^{-i\theta})\,d\theta\right| \le \frac{c\|f\|_p}{1-r^2}, \qquad 0 \le r < 1. \tag{7}$$

Let $W(e^{i\theta}) = \sum_{k=-n}^{n} a_k e^{ik\theta}$ be a trigonometric polynomial with $\|W\|_{L^p[0,2\pi]} \le 1$. It follows from the M. Riesz theorem that its analytic projection $w(e^{i\theta}) = \sum_{k=0}^{n} a_k e^{ik\theta}$ satisfies

$$\|w\|_{L^p[0,2\pi]} \le A_p \|W\|_{L^p[0,2\pi]} \le A_p.$$

Also note that

$$\left|\int_0^{2\pi} W(e^{i\theta}) g'(r^2 e^{-i\theta})\,d\theta\right| = \left|\int_0^{2\pi} w(re^{i\theta}) g'(re^{-i\theta})\,d\theta\right| \le \frac{cA_p}{1-r^2}. \tag{8}$$

If we denote $g'_{r^2}(z) = g'(r^2 z)$, $0 < r < 1$, then taking the supremum over all W with $\|W\|_{L^p[0,2\pi]} \le 1$ we get

$$\|g'_{r^2}\|_q = M_q(r^2, g') \le \frac{c}{1-r^2},$$

and this proves Theorem 1.

3. Proof of Theorem 2. The following property of integral means is well known (cf. [1], p. 80): if $0 < p \le \infty$, $\beta > 0$ and f is analytic in $\mathbf{U}$ then

$$M_p(r, f) = O\left(\frac{1}{(1-r)^\beta}\right) \quad \text{if and only if} \quad M_p(r, f') = O\left(\frac{1}{(1-r)^{\beta+1}}\right)$$

Hence the set A_n in formula (2) does not depend on n if only $n + 1 > 1/p$.

Now assume that $0 < p < 1$. It was proved by Duren and Shields [2] that $(H^p, H^\infty) = A_n$. So to prove our theorem it is enough to show that $(H^p, \mathcal{B}) \subset A_n$. Suppose that g is an analytic function such that $f \star g \in \mathcal{B}$ whenever $f \in H^p$. Then the closed graph theorem implies

$$\|f \star g\|_{\mathcal{B}} \le c\|f\|_p.$$

For $g(z)=\sum_{k=0}^{\infty}\hat{g}(k)z^k$ we define

$$D^n g(z)=\sum_{k=0}^{\infty}(k+1)^n\widehat{g}(k)z^k.$$

Let $f(z)=\sum_{k=0}^{\infty}(k+1)^n z^k$. Then we have

$$\|D^n g_r\|_{\mathcal{B}}=\|g\star f_r\|_{\mathcal{B}}\le c\|f_r\|_p,$$

where $f_r(z)=f(rz)$, $0<r<1$.

Because $f(z)=P_n(z)/(1-z)^{n+1}$, where P_n is a polynomial of degree n,

$$\|D^n g_r\|_{\mathcal{B}}\le c\left\|\frac{1}{(1-rz)^{n+1}}\right\|_p=O\left(\frac{1}{(1-r)^{n+1-1/p}}\right). \tag{9}$$

Hence

$$\sup_{0<\rho<1}(1-\rho)M_\infty(\rho,(D^n g_r)')\le\frac{c}{(1-r)^{n+1-1/p}}. \tag{10}$$

It was shown in [4] that the integral means of $D^n g$ and $g^{(n)}$ have "the same behaviour". So by Lemma 1 of [4] (10) implies

$$M_\infty(\rho,D^{n+1}g_r)\le\frac{c}{(1-r)^{n+1-1/p}(1-\rho)},\qquad 0<r,\rho<1, \tag{11}$$

which is equivalent to

$$M_\infty(\rho r,g^{(n+1)})\le\frac{c}{(1-r)^{n+1-1/p}(1-\rho)},\qquad 0<r,\rho<1.$$

Hence

$$M_\infty(r,g^{(n+1)})\le\frac{c}{(1-r)^{n+2-1/p}}$$

and this means that $g\in A_n$.

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