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DEFORMATIONS OF A STRONGLY PSEUDO-CONVEX DOMAIN OF COMPLEX DIMENSION ≥ 4

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Introduction. Deformations of 1-convex spaces were considered in [B-K], where the authors proved the existence of a semi-universal family of deformations of germs along its exceptional set and gave the following conjecture on deformations of the whole 1-convex space.

CONJECTURE ([B-K], Bemerkungen (5.10)]). *Let X be a strongly pseudo-convex complex space such that $T := \mathrm{Supp}(\mathcal{T}^1(X, \mathcal{O}_X))$ is compact. Then there exists a convergent formally semi-universal deformation $\mathcal{X}$ of the whole space X with the following property: If $K \subset X$ is any strongly pseudo-convex compact subset* (1) *of X containing T, then the germ of $\mathcal{X}$ with respect to K is a semi-universal* (2) *deformation of (X, K).*

In this talk, I will approach this problem in the case where X is a strongly pseudo-convex manifold of $\dim_{\mathbb{C}} X \geq 4$ by a different method, so called the Kuranishi's method, relying on T. Akahori's harmonic analysis extablished in [Ak1,2]. Since he constructed there a versal family (in the sense of Kuranishi) of complex structures on a strongly pseudo-convex level subset, we will start with this family. Though the family of strongly pseudo-convex domains corresponding to this family of complex structures is formally versal, no formal principle working in our situation has been established. So the remaining problem is to show the versality. See [M2] for details.

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(1) A compact subset in a 1-convex space X is called a strongly pseudo-convex compact subset if it is the inverse image of a Stein compact subset in the Remmert quotient of X.

(2) Probably "versal" unless K contains the whole exceptional subset.

1. Definitions of deformation theory. Let X be a complex manifold and K a compact subset. We will consider deformations of X of the following two types: (I) deformations as a germ along K, (II) deformations of the whole X.

Definition 1.1. A *family of deformations* of X is a smooth map $\pi : \mathcal{X} \to T$ with an isomorphism $i : \pi^{-1}(o) \simeq X (o \in T)$, where (T, o) is a germ of a complex space.

Definition 1.2. Let (S, o) be a germ of a complex space and $o \in S' \subset S$ a subspace. A family $\pi : \mathcal{X} \to T$, with $\pi^{-1}(o) \simeq X (o \in T)$, has the *lifting property* for an extension $S' \to S$ if, for any other family $\varpi : \mathcal{Y} \to S$ with $j : \varpi^{-1}(o) \simeq X (o \in S)$, any holomorphic maps G' and τ' in the commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[d, "\pi"'] & \mathcal{Y}_{|S'} \arrow[l, "G'"'] \arrow[r] \arrow[d, "\varpi_{|S'}"'] & \mathcal{Y} \arrow[d, "\varpi"] \arrow[ll, dashed, bend right] \\
T & S' \arrow[l, "\tau'"'] \arrow[r] & S \arrow[ll, dashed, bend left]
\end{tikzcd}$$

with $\tau'(o) = o$ and $i \circ G'_{|\varpi^{-1}(o)} = j$ can be lifted to the dotted holomorphic maps.

Definition 1.3. A family is *versal* (resp. *formally versal*) if it has the lifting property for any extension $S' \to S$ (resp. $S' \to S$ with artinian $\mathcal{O}_S$).

For a family $\pi : \mathcal{X} \to T$, with $\pi^{-1}(o) \simeq X (o \in T)$, we have the Kodaira–Spencer map $\rho : T_oT \to H^1(X, \Theta_X)$ in the same way as in the case of deformations of compact complex manifolds.

Definition 1.4. A family of deformations of X is *effective* if ρ is injective.

Definition 1.5. A family of deformations of X is *semi-universal* (resp. *formally semi-universal*) if it is versal (resp. formally versal) and effective.

For deformations of X as a germ along K, or simply deformations of the germ (X, K), we make the same definitions as above except that we have to consider all $\mathcal{X}$, $\pi^{-1}(o)$, $\mathcal{Y}$, $\varpi^{-1}(o)$ as germs along K. In particular, the Kodaira–Spencer map is $\rho : T_oT \to \varinjlim_{U \supset K} H^1(U, \Theta_U)$.

2. Theorems. Let X be a complex manifold with $n = \dim_{\mathbb{C}} X \geq 4$ and $r : X \to \mathbb{R}$ a continuous exhaustion function which is C^∞, $dr \neq 0$ and strictly plurisubharmonic outside a compact subset $K \subset X$. Assume $K = \bigcap_{\varepsilon > \alpha} \Omega_\varepsilon$ for some α with $\Omega_\varepsilon = \{x \in X \mid r(x) < \varepsilon\}$.

THEOREM 1 [(M2]). *There exists a semi-universal family of deformations of the germ* (X,K).

The following is a special case of Theorem 1.

COROLLARY 2. *There exists a semi-universal family of deformations of the germ* (X,E) *where* E *denotes the exceptional subset.*

For deformations of the whole X, [B-S-W] says that there does not always exist a finite-dimensional semi-universal family.

THEOREM 3 ([M2]). *There exists a formally semi-universal convergent family of deformations of the whole* X, *which is a versal family of deformations of* (X,K) *for any strongly pseudo-convex compact subset* K *of* X.

We will prove these theorems by means of the so-called Kuranishi method, which gives information on the smoothness of the (formally) semi-universal family.

COROLLARY 4. *If* $\varinjlim_{U\supset K} H^2(U,\Theta_U)=0$ (*resp.* $H^2(X,\Theta_X)=0$) *then the semi-universal family in Theorem* 1 (*resp. the formally semi-universal convergent family in Theorem* 3) *is non-singular.*

COROLLARY 5. *If the canonical bundle* K_U *is negative for some neighbourhood of* K (*resp.* K_X *is negative*) *then the semi-universal family in Theorem* 1 (*resp. the formally semi-universal convergent family in Theorem* 3) *is non-singular.*

THEOREM 6 ([M3]). *If the canonical bundle* K_U *is trivial for some neighbourhood of* K (*resp.* K_X *is trivial*) *then the semi-universal family in Theorem* 1 (*resp. the formally semi-universal convergent family in Theorem* 3) *is non-singular.*

3. Outline of the proof of Theorem 1. The proof consists of two parts: (I) construction of the canonical family and (II) proof of versality.

(I) *Construction of the canonical family*. Fix $\varepsilon>\alpha$. In [Ak1, 2], T. Akahori constructed a family $\phi_\varepsilon(t)$ of complex structures on $\overline{\Omega}_\varepsilon$ having the following properties:

(1) $\phi_\varepsilon(t)\in A^{0,1}(\overline{\Omega}_\varepsilon,T'X)[[t_1,\dots,t_q]]$ and is convergent in $\|\ \|'_{(0,k)}$-norm for any $k\ge n+2$ (refer to [Ak1] for $\|\ \|'_{(0,k)}$-norm),

(2) $\phi_\varepsilon(t)$ is real-analytic on $\overline{\Omega}_{\varepsilon_*}\times D$ for some $\alpha<\varepsilon_*<\varepsilon$ and a neighbourhood D of 0 in $\mathbb{C}^q$,

(3) $\phi_\varepsilon(0)=0$ and $\overline{\partial}\phi_\varepsilon(t)-\frac12[\phi_\varepsilon(t),\phi_\varepsilon(t)]\equiv 0 \bmod (h_1(t),\dots,h_l(t))$ where $\rho_{\mathbb{H}^2}[\phi_\varepsilon(t),\phi_\varepsilon(t)]=\sum_{\nu=1}^{l}h_\nu(t)\beta_\nu$ with $\rho_{\mathbb{H}^2}:A^{0,2}(\overline{\Omega}_\varepsilon,T'X)\to\mathbb{H}^2:=\operatorname{Ker}\Box$ denoting the orthogonal projection and $\beta_1,\dots,\beta_l$ a base of $\mathbb{H}^2$,

(4) $\phi_\varepsilon(t)=\sum_{\lambda=1}^{q}\alpha_\lambda t_\lambda+o(t^2)$, where $\overline{\partial}\alpha_1=\ldots=\overline{\partial}\alpha_q=0$ and $[\alpha_1],\dots,[\alpha_q]$ is a base of $H^1(\overline{\Omega}_\varepsilon,T'X)$.

By (1)–(3), we have a family of deformations of Ω_{ε_*}, $\pi_{\varepsilon_*}:\mathcal{X}_{\varepsilon_*}\to T_\varepsilon$ as a realization of the family of complex structures $\phi_\varepsilon(t)$ $(t\in T_\varepsilon)$, where T_ε is a complex space defined by $h_1(t) = \ldots = h_l(t) = 0$. Effectivity of this family follows from (4).

(II) *Proof of versality.* The key step is to prove the following proposition.

PROPOSITION 3.1. *Let $\varpi : \mathcal{Y} \to S$ be a family with $\overline{\Omega}_\varepsilon \subset \varpi^{-1}(o) \subset X (o \in S)$. Then for any subspace $o \in S' \subset S$, any holomorphic map $\tau' : (S', o) \to (T, o)$ and any family of embeddings of $\mathcal{A}_k$-class* (cf. [Ak1]) *$g' : \overline{\Omega}_\varepsilon \times S' \to \mathcal{Y}_{|S'}$ depending complex analytically on S' and satisfying*

(1)′ $\varpi \circ g' = pr_{S'}$,
(2)′ $(\overline{\partial} - \phi_\varepsilon(\tau'(s)))g' \equiv 0 \bmod \mathcal{I}_{S'}$,

there exist liftings $\tau : (S, o) \to (T, o)$ and $g : \overline{\Omega}_\varepsilon \times S \to \mathcal{Y}$ satisfying

(1) $\varpi \circ g = pr_S$,
(2) $(\overline{\partial} - \phi_\varepsilon(\tau(s)))g \equiv 0 \bmod \mathcal{I}_S$,

where

$$(\overline{\partial} - \phi_\varepsilon(t))g = \sum_{\alpha,\beta} \left(\frac{\partial g^\alpha(s)}{\partial \overline{z}^\beta} - \sum_\gamma (\phi_\varepsilon)^\gamma_{\overline{\beta}}(t) \frac{\partial g^\alpha(s)}{\partial z^\gamma} \right) \frac{\partial}{\partial z^\alpha} \otimes d\overline{z}^\beta$$

for the local expressions $\phi_\varepsilon(t) = \sum_{\alpha,\beta} (\phi_\varepsilon)^\gamma_{\overline{\beta}}(t) \frac{\partial}{\partial z^\gamma} \otimes d\overline{z}^\beta$ and $g : \zeta^\alpha = g^\alpha(z, s)$ $(\alpha = 1, \ldots, n)$ in local charts $(z^1, \ldots, z^n)$ of X and $(\zeta^1, \ldots, \zeta^n, t^1, \ldots, t^q)$ of $\mathcal{X}$ respectively.

Proof. $\tau(s) - \tau'(s)$ and $g(s) - g'(s)$ are constructed inductively on $\bmod(\mathcal{I}_S + \mathfrak{m}_S^{\mu+1} \mathcal{I}_{S'})$ using (3) and (4). We can perform this process using the division theorem by a submodule (cf. [Ga]).

By the same argument of [M1, Proposition 2.5], we can show that Proposition 3.1 implies the following

PROPOSITION 3.2. *Any $\pi_{\varepsilon_*} : \mathcal{X}_{\varepsilon_*} \to T_\varepsilon$ $(\alpha < \varepsilon)$ provides a formally semi-universal family of deformations of (X, K).*

Using K. Spallek's theorem on complex-analyticity of C^k-functions on a reduced complex space (cf. [Sp1, 2]), we infer from Proposition 3.1 the following

PROPOSITION 3.3. *All $\pi_{\varepsilon_*} : \mathcal{X}_{\varepsilon_*} \to T_\varepsilon$ $(\alpha < \varepsilon)$ have the lifting property for any extension $S' \to S$ with S being a germ of a reduced complex space.*

Then the versality of $\pi_{\varepsilon_*} : \mathcal{X}_{\varepsilon_*} \to T_\varepsilon$ follows from the following criterion of versality, which is an easy consequence of an argument of [E, pp. 415–416] based on [Fl, Satz (3.2)].

A CRITERION FOR VERSALITY ([M2]). *Let $p : F \to \mathcal{C}$ be a fibred groupoid satisfying the Schlessinger's conditions $(S1')$ and $(S2)$, and $w \in F$ a formally versal element. Then w is versal if and only if w has the lifting property for any extension $S \to S' = S'_{\mathrm{red}}$.*

Concerning the Schlessinger's conditions: $(S1')$ holds since it holds for deformations of complex manifolds, and $(S2)$ follows from the coherence of $\varinjlim_{\varepsilon>\alpha} R^1(\pi_{\varepsilon *})_* \Theta_{\mathcal{X}_{\varepsilon *}/T_\varepsilon}$, which is due to [Si].

4. Outline of the proof of Theorem 3. Here we consider the simplest case where the exceptional subset E is connected and $E \subset K$. The general case can be treated by a slight modification. Let $r : X \to [-\infty, +\infty)$ be a strictly plurisubharmonic C^∞ exhaustion function such that $dr \neq 0$ on $X \setminus E$ and $E = \bigcap_{-\infty<\varepsilon} \Omega_\varepsilon$. Choose an increasing sequence $\alpha < \varepsilon_1 < \varepsilon_2 < \ldots \to +\infty$. Since each $\pi_{\varepsilon_{i*}} : \mathcal{X}_{\varepsilon_{i*}} \to T_{\varepsilon_i}$ is a semi-universal family of deformations as a germ along $\overline{\Omega}_\varepsilon$ for all $\varepsilon < \varepsilon_{i*}$, we have a family $\pi : \mathcal{X} \to T$ of deformations of X by patching them together.

We see that $\pi : \mathcal{X} \to T$ provides a semi-universal family of deformations of (X, K) for any strongly pseudo-convex compact subset K of X. Suppose that $K = \bigcap_{i=1}^\infty U^{(i)}$ with $U^{(i)}$ the inverse images of Stein spaces in the Remmert quotient. By Theorem 1, we have a sequence of strongly pseudo-convex domains $K \subset \ldots \subset \Omega_{\varepsilon^{(i+1)}} \subset \Omega_{\varepsilon^{(i)}} \subset \ldots$ with $K = \bigcap_{i=1}^\infty \Omega_{\varepsilon^{(i)}}$ and families $\pi^{(i)} : \mathcal{X}^{(i)} \to T^{(i)}$ which a semi-universal family of deformations as a germ along $\overline{\Omega}_{\varepsilon^{(i)}}$.

Since $\pi : \mathcal{X} \to T$ and $\pi^{(i)} : \mathcal{X}^{(i)} \to T^{(i)}$ $(i = 1, 2, \ldots)$ are semi-universal families of deformations of (X, E) and since $\pi^{(i)} : \mathcal{X}^{(i)} \to T^{(i)}$ is a semi-universal family of deformations of $(X, \overline{\Omega}_{\varepsilon^{(i)}})$, $\pi^{(i)} : \mathcal{X}^{(i)} \to T^{(i)}$ is an open part of $\pi : \mathcal{X} \to T$. Hence the semi-universality of $\pi : \mathcal{X} \to T$ as a family of deformations of (X, K) follows from the semi-universality of each $\pi^{(i)} : \mathcal{X}^{(i)} \to T^{(i)}$ as a family of deformations of $(X, \overline{\Omega}_{\varepsilon^{(i)}})$.

Formal versality of $\pi : \mathcal{X} \to T$ follows by comparing it with the formally versal formal family obtained by M. Schlessinger's formal existence theorem; indeed, the functor of deformations of the whole X satisfies (H_1)–(H_3) of [Sch].

5. Smoothness of semi-universal families. First, I remark that the smoothness of semi-universal families is a problem of formal deformation theory. The proof of Corollary 4 is standard.

Proof of Corollary 5. Since there is an isomorphism $T'X \simeq \bigwedge^{n-1} T'X^* \otimes K_X^{-1}$, we have $H^2(X, \Theta_X) \simeq H^2(X, \Omega^{n-1}(K_X^{-1}))$. Hence, if $K_X < 0$ then $H^2(X, \Theta_X) = 0$ by S. Nakano's vanishing theorem on weakly 1-complete manifolds (cf. [N]).

Proof of Theorem 6. The key proposition is the following, which is proved using T. Ohsawa's reduction theorem for cohomology of strongly pseudo-convex manifolds (cf. [O]).

PROPOSITION 5.1 ($\partial\overline{\partial}$-Lemma, [M3]). *If $p + q \geq n + 1$, then any d-closed and ∂-exact (p, q)-form is $\partial\overline{\partial}$-exact.*

Since there exists a cohomology base of $H^1_{\overline{\partial}}(X, \Omega_X^{n-1})$ consisting of d-closed

$(n-1,1)$-forms, we can carry out the Tian–Todorov argument (cf. [Ti], [To]) using Proposition 5.1.

References

[Ak1] T. Akahori, *The new Neumann operator associated with deformations of strongly pseudo-convex domains and its applications to deformation theory*, Invent. Math. 68 (1982), 317–352.

[Ak2] —, *A criterion for the Neumann type problem over a differential complex on a strongly pseudo-convex domain*, Math. Ann. 264 (1983), 525–535.

[B-K] J. Bingener and S. Kosarew, *Lokale Modulräume in der analytischen Geometrie*, Aspects of Math., D2, D3, Vieweg-Verlag, Braunschweig, 1987.

[B-S-W] D. Burns, S. Schnider and R. O. Wells, *Deformations of strictly pseudo-convex domains*, Invent. Math. 46 (1978), 237–253.

[E] H. von Essen, *Nonflat deformations of modules and isolated singularities*, Math. Ann. 287 (1990), 413–427.

[Fl] H. Flenner, *Ein Kriterium für die Offenheit der Versalität*, Math. Z. 178 (1981), 449–473.

[G] A. Galligo, *Théorème de division et stabilité en géométrie analytique locale*, Ann. Inst. Fourier (Grenoble) 29 (1979), 107–184.

[M1] K. Miyajima, *Deformations of a complex manifold neear a strongly pseudo-convex real hypersurface and a realization of Kuranishi family of strongly pseudo-convex CR structures*, Math. Z. 205 (1990), 593–602.

[M2] —, *Kuranishi family of strongly pseudo-convex domains*, Osaka Math. J., to appear.

[M3] —, in preparation.

[N] S. Nakano, *Vanishing theeorems for weakly* 1*-complete manifolds, II*, Publ. R.I.M.S. Kyoto Univ. 10 (1974), 101–110.

[O] T. Ohsawa, *A reduction theorem for cohomology groups of very strongly q-convex Kähler manifolds*, Invent. Math. 63 (1981), 335–354.

[Sch] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. 130 (1968), 208–222.

[Si] Y.-T. Siu, *The* 1*-convex generalization of Grauert's direct image theorem*, Math. Ann. 190 (1971), 203–214.

[Sp1] K. Spallek, *Differenzierbare und holomorphe Funktionen auf analytischen Mengen*, Math. Ann. 161 (1965), 143–162.

[Sp2] —, *Zum Satz von Osgood und Hartogs für analytische Moduln. II*, Math. Ann. 182 (1969), 77–94.

[Ti] C. Tian, *Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Petersson–Weil metric*, in: Mathematical Aspects of String Theory, S. T. Yau (ed.), World Scientific, 1987, 629–646.

[To] A. N. Todorov, *The Weil–Petersson geometry of the moduli space of $SU(\geq 3)$ (Calabi–Yau) manifolds I*, Comm. Math. Phys. 126 (1989), 325–346.