TOPICS IN COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 31
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

REGULARITY OF THE TANGENTIAL CAUCHY-RIEMANN COMPLEX AND APPLICATIONS

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In a series of papers Webster ([26], [27], [28]) has shown how $\mathcal{C}^k$-estimates for the tangential Cauchy-Riemann complex can be applied to several non-linear problems in Complex Analysis. For example, he gave a simplification of the proof of Kuranishi's embedding theorem and an application to the integrability problem for almost CR vector bundles.

Connected with Webster's approach are some regularity assertions, whose parameters follow from the $\mathcal{C}^k$-estimates. Therefore, an improvement of these estimates would lead to an improvement in the applications. Now the author of this article, together with Lan Ma, has given in [12] such an improvement. In two subsequent papers this has been applied to Webster's approach. The results are formulated in two theorems at the end of this introduction.

Here we want to briefly describe the estimates for the tangential CR equations. Let G be a strictly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^m$-smooth boundary and $0 \in bG$. By a local homotopy formula for the tangential Cauchy-Riemann operator $\overline{\partial}_b$ we mean the following. There exists a neighborhood base $\{M\}$ of relatively open sets M, with $0 \in M \subset bG$, and on each M there are given operators R_q ($q = 1, 2, \ldots, n-2$), which fullfill the equation

$$f = \overline{\partial}_b R_q f + R_{q+1} \overline{\partial}_b f,$$

for $f \in \mathcal{C}^0_{0,q}(\overline{M})$, $\overline{\partial}_b f \in \mathcal{C}^0_{0,q+1}(\overline{M})$, (with $\overline{\partial}_b f = 0$ for $q = n-2$). Our goal now is to construct homotopy formulae, which satisfy $\mathcal{C}^k$-estimates of the following kind ($k = 0, 1, \ldots, m-3$)

$$|R_q f|_{k,M} \leq c \left(|f|_{k,M} + |\overline{\partial}_b f|_{k,M} \right).$$

1991 *Mathematics Subject Classification*: 32F25, 32F10, 32H02.

The paper is in final form and no version of it will be published elsewhere.

Here $R_q f$ is not necessarily a linear operator, but with the aid of a so-called Seeley continuation operator one could always obtain one. For the applications we have in mind, a linear operator is not always appropriate. The standard local homotopy operators R_q go back to Henkin [6]. These operators are the starting point for our construction. After a rotation, bG can be given locally as the graph over its tangent space at 0 $T_0(bG) = \{(z', u) | z' \in \mathbb{C}^{n-1}, u \in \mathbb{R}\}$, $z_n = u + iv$. Choose a strictly convex domain W in $\mathbb{C}^{n-1} \times \mathbb{R}$ and suppose, that on W we are given a function $H(z', u)$, such that

$$M := \{(z', u + iH(z', u)) | (z', u) \in W\},$$

where $H(\cdot, u)$ is strictly convex for all fixed u (see the next section for more details). The above situation can always be achieved. Namely, after a local biholomorphic transformation bG can be assumed strictly convex. But because of the above mentioned applications, we take a slightly more general viewpoint.

For specially chosen W the Henkin operator satisfies $\mathcal{C}^0$-estimates. This shall also include that $R_q f$ is continuous up to the boundary of $\overline{M}$ (see Theorem 1). For $\mathcal{C}^k$-estimates we need a modification of R_q. Let $f \in \mathcal{C}^k_{0,q}(\overline{M})$, with $\overline{\partial}_b f \in \mathcal{C}^k_{0,q+1}(\overline{M})$. Then we choose $\mathcal{C}^k$-smooth continuations $e_1(f)$ of f, $e_2(\overline{\partial}_b f)$ of $\overline{\partial}_b f$, with $\operatorname{supp} e_i \subset\subset \widetilde{M}$ ($M \subset\subset \widetilde{M} \subset bG$). With e_1 and e_2 fixed, we can construct forms $R^*_q(e)(f) \in \mathcal{C}^k_{0,q-1}(\overline{M})$ and $R^*_{q+1}(e)(\overline{\partial}_b f) \in \mathcal{C}^k_{0,q}(\overline{M})$, such that

$$f = \overline{\partial}_b R^*_q(e)(f) + R^*_{q+1}(e)(\overline{\partial}_b f)$$

on $\overline{M}$ and which satisfy

$$|R^*_q(e)(f)|_{k,M} \le c(|e_1(f)|_{k,\widetilde{M}} + |e_2(\overline{\partial}_b f)|_{k,\widetilde{M}}),$$
$$|R^*_{q+1}(e)(\overline{\partial}_b f)|_{k,M} \le c|e_2(\overline{\partial}_b f)|_{k,\widetilde{M}},$$

where the constant only depends on the geometry and k (see Theorem 2).

Introducing a so-called Seeley continuation operator E and setting $e_1(f) = Ef$, $e_2(\overline{\partial}_b f) = E\overline{\partial}_b f$, then $R^*_q(e)$ turns to be a linear operator R^*_q, which also satisfies $\mathcal{C}^k$-estimates (see Theorem 3). In this case the above estimates are in terms of $|Ef|_{k,\widetilde{M}}$. From the definitions of E it follows that there are constants $c_k(E)$, with $|Ef|_{k,\widetilde{M}} \le c_k(E)|f|_{k,M}$. For solving the $\overline{\partial}_b$-equation, E is very useful. But for the applications we have in mind, the behavior of $c_k(E)$ is not appropriate, especially when $\widetilde{M}$ shrinks.

In his approach Webster has given in [25] interior estimates for $R_q(f)$. In $R_q(f)$ no continuation is involved. But a drawback is that the constants in the estimates contain negative powers of the boundary distance. This causes a great loss of regularity. In our approach the constant is harmless. The loss of regularity now occurs in passing over from the $\mathcal{C}^k$-norms of $e_i(f)$ to those of f. Local estimates, related to the $\overline{\partial}_b$-operator and using integral formula methods, were also given in [3], [4], [20], [22], [23], [24].

A non-mechanical application to Webster's method, by using the Nash-Moser iteration scheme, gives the following improvement (cf. [13], [14]) of Webster's results.

Kuranishi's Embedding Theorem. *Let $n \geq 4$. If M is a $(2n-1)$-dimensional strictly pseudoconvex CR-manifold of differentiability class $\mathcal{C}^m$, then M admits, locally near each point, a holomorphic embedding of class $\mathcal{C}^k$, provided*

$$m \geq k + 13, \qquad k \geq 18.$$

(Webster's result: $m \geq 6k + 5n - 2$, $k \geq 21$.)

Theorem on CR-bundles. *Let M be a $\mathcal{C}^m$-smooth strongly pseudoconvex real hypersurface in $\mathbb{C}^n$. Suppose that (E, D) is a $\mathcal{C}^m$-smooth almost CR vector bundle over M. Then every point of M has a neighborhood, over which a $\mathcal{C}^k$-smooth CR frame for (E, D) exists, if $m \geq k + 7, k \geq 6, n \geq 4$.*

(Webster's result: $m \geq 4n + 5k + 12, k \geq 8$.)

1. Local solutions. Our first goal is to define M as a graph over $\mathbb{R}^{2n-1}$. Let $m, n \in \mathbb{N}$, $m, n \geq 3$, $\mathbb{R}^{2n-1} = \mathbb{C}^{n-1} \times \mathbb{R} \ni (z', u)$, $z' = (z_1, z_2, \ldots, z_{n-1}), z_n = u + iv$.

We choose a convex domain $\widetilde{W} \subset\subset \mathbb{R}^{2n-1}$, with $0 \in \widetilde{W}$ and $\operatorname{diam}(\widetilde{W}) \leq 1$. On $\widetilde{W}$ we are given a $\mathcal{C}^m$-smooth function $H : \widetilde{W} \to \mathbb{R}$, with

$$H(z', u) = q_2(z', u) + h(z', u),$$

where q_2 is a quadratic form in (z', u) and $h(z', u) = O((|z'| + |u|)^3)$.

We assume that $H(\cdot, u)$ is a strictly convex function and that there exist positive constants κ_1, κ_2, with

$$\kappa_1 |z'|^2 \leq q_2(z', u) \leq \kappa_2(|z'|^2 + u^2).$$

Remark. In Webster's approach to the Kuranishi embedding theorem, q_2 only depends on z'. But if for example H is strictly convex, q_2 will also depend on u.

We define an embedding $Z : \widetilde{W} \to \widetilde{M} =: Z(\widetilde{W}) \subset \mathbb{C}^n$, by $Z(z', u) = (z', z_n)$, with $z_n = u + iv = u + iH(z', u)$. Therefore we have

$$\widetilde{M} = \{Z(z', u) | (z', u) \in \widetilde{W}\}.$$

Set $r(z) := -v + H(z', u)$. Then $\widetilde{M} = \{z \in \widetilde{W} \times \mathbb{R} | r(z) = 0\}$, $T_0\widetilde{M} = \mathbb{R}^{2n-1}$, $T_0^h\widetilde{M} = \mathbb{C}^{n-1}$.

Now we construct the desired neighborhood base of 0. Let $s = s(z_n)$ be a real valued $\mathcal{C}^m$-function of one complex variable, with $s(z_n) = v + O(|z_n|^2)$. For sufficiently small $0 < \rho < 1$, we assume with

$$W := W_\rho := \{(z', u) \in \widetilde{W} | s(u + iH(z', u)) < \rho\},$$
$$M := M_\rho := \{z \in \widetilde{M} | s(z_n) < \rho\} = Z(W_\rho),$$

that $W_\rho \subset\subset \widetilde{W}$. Moreover we assume that for a sequence of numbers ρ, converging to 0, W_ρ is strictly convex and $d\rho \wedge ds|_{bM_\rho} \neq 0$, $bW_\rho \in \mathcal{C}^m$.

If ρ is chosen sufficiently small, then M_ρ is a strictly pseudoconvex surface with $\mathcal{C}^m$-smooth boundary.

Remark. If H is strictly convex, $s = v$ would be a good choice. However, for the embedding theorem one has to take $s = Re(z_n^2 - iz_n)$.

Set for $\zeta, z \in \mathbb{C}^n$, and for $i = 1, 2, \dots n$, $\partial_i := \partial/\partial\zeta_i$, $r_i := \partial_i r$,

$$\Phi(\zeta, z) := \langle \partial r(\zeta), \zeta - z \rangle := \sum_{i=1}^n r_i(\zeta)(\zeta_i - z_i),$$

$$\Phi^*(\zeta, z) := \langle \partial r(z), \zeta - z \rangle := \sum_{i=1}^n r_i(z)(\zeta_i - z_i),$$

$$S := i \sum_{i=1}^n (\overline{r_i(\zeta)}\partial_i - r_i(\zeta)\overline{\partial}_i).$$

Φ, Φ^* are barrier functions, from which the kernels will be derived. S is a tangential vector field, which is transversal to the holomorphic tangent space. Now we assume that there exists a positive constant κ_3 with

$$|S\Phi(\zeta, z)| \geq \kappa_3, \qquad |S\Phi^*(\zeta, z)| \geq \kappa_3,$$
$$|\Phi(\zeta, z)| \geq \kappa_3 |\zeta - z|^2, \qquad |\Phi^*(\zeta, z)| \geq \kappa_3 |\zeta - z|^2,$$

for all $\zeta, z \in \widetilde{M}$. Set $\kappa := \frac{1}{2}\min(1, \kappa_1, \kappa_2^{-1}, \kappa_3)$.

For sufficiently small $\widetilde{W}$ this can be achieved if $\widetilde{M}$ is locally strictly convex or in the situation of [26], [28].

Near $\widetilde{M}$ there exists the local frame of $(1,0)$-vector fields

$$Y_i := \partial/\partial z_i - r_i/r_n \partial/\partial z_n, \qquad i = 1, 2, \dots, n-1,$$
$$Y_n := i/r_n \partial/\partial z_n.$$

The dual coframe is

$$\omega_i := dz_i \quad (i < n), \quad \omega_n := \theta := -i\partial r.$$

Let $U \subset \mathbb{C}^n$ be an open neighborhood of $\overline{M}$ and $f \in \mathcal{C}^k_{0,q}(U)$. By using $\{\omega_1, \dots \dots, \omega_n\}$, f has the unique decomposition

$$f = f_t + f_n \wedge \overline{\theta},$$

where f_t, f_n do not contain $\overline{\theta}$.

Definition. We call $f \in \mathcal{C}^k_{0,q}(U)$ *tangential* when f decomposes into

$$f = \sum_{1 \leq i_\nu < n} f_{i_1, i_2, \dots, i_q} d\overline{z}_{i_1} \wedge d\overline{z}_{i_2} \wedge \dots \wedge d\overline{z}_{i_q}.$$

f is called *simply tangential* if additionally the coefficients do not depend on v.

On open sets in Euclidean space, we take the usual $\mathcal{C}^k$-norms. For a tangential form g, we define the *tangential* $\mathcal{C}^k$-norm

$$|g|_{k,M_\rho} := |Z^*g|_{k,W_\rho} := \max_I |g_I(Z(z',u))|_{k,W_\rho}$$

and for $f \in \mathcal{C}^k_{0,q}(U)$ we set

$$|f|_{k,M_\rho} := |f_t|_{k,M_\rho} + |f_n|_{k,M_\rho}.$$

Define

$$\widetilde{\mathcal{C}}^k_{0,q}(\overline{M_\rho}) := \{g | \exists U \supset \overline{M_\rho} : g \in \mathcal{C}^k_{0,q}(U), g \text{ simply tangential}\}.$$

To any $f \in \mathcal{C}^k_{0,q}(U)$ we can attach a simply tangential form $f_0 \in \widetilde{\mathcal{C}}^k_{0,q}(U)$, namely

$$f_0(z) := f_t(Z(z',u)).$$

$f_0 = f_t$ on M_ρ. We call two forms f and $\widetilde{f}$ equivalent if $f_0 = (\widetilde{f})_0$, the equivalence class is denoted by $[f]$. Clearly $f_0 \in [f]$, so every class contains a unique simply tangential representative. We define

$$[\mathcal{C}^k_{0,q}(\overline{M_\rho})] := \{[f] | f_0 \in \widetilde{\mathcal{C}}^k_{0,q}(\overline{M_\rho})\},$$

with norm

$$|[f]|_{k,M_\rho} := \inf_{g\in[f]} |g|_{k,M_\rho}.$$

This implies

$$|[f]|_{k,M_\rho} = |f_0|_{k,M_\rho} = |f_0|_{k,W_\rho}.$$

Then the mapping $[\mathcal{C}^k_{0,q}(\overline{M_\rho})] \to \widetilde{\mathcal{C}}^k_{0,q}(\overline{M_\rho}), [f] \to f_0$, is an isometry of Banach spaces. The classes $[f]$ are exactly those on which $\overline{\partial}_b$ operates. Now we define a representative for $\overline{\partial}_b$ on f_0. Set for a function

$$\partial_M f := \sum_{i<n} Y_i f dz_i, \quad \overline{\partial}_M f := \sum_{i<n} \overline{Y}_i f d\overline{z}_i.$$

So $\partial f = \partial_M f + Y_n f \theta$. For forms $f = \sum f_I d\overline{z}_I$ we set

$$\partial_M f = \sum_I \partial_M f_I \wedge d\overline{z}_I, \qquad \overline{\partial}_M f = \sum_I \overline{\partial}_M f_I \wedge d\overline{z}_I.$$

If f is simply tangential, then $\overline{\partial}_M f$ also. Now it is easy to see, that

$$\overline{\partial}_b[f] = \overline{\partial}_b[f_0] = [\overline{\partial} f_0] = [\overline{\partial}_M f_0]$$

and $(\overline{\partial} f)_0 = \overline{\partial}_M f_0$. So instead of working with $\overline{\partial}_b$ on classes, we can use $\overline{\partial}_M$ on simply tangential forms. The pull-back to W allows to give a transparent interpretation of the constants involved in the estimates.

For $g \in \widetilde{\mathcal{C}}^k_{0,q}(\overline{M}), \overline{\partial}_M g \in \widetilde{\mathcal{C}}^k_{0,q+1}(\overline{M})$, we choose fixed $\mathcal{C}^k$-continuations $e_1(g) \in \widetilde{\mathcal{C}}^k_{0,q}(\widetilde{M})$, $e_2(\overline{\partial}_M g) \in \widetilde{\mathcal{C}}^k_{0,q+1}(\widetilde{M})$, which are supported in $\widetilde{M}$.

If convenient, we can construct such extensions by a so called Seeley continuation operator (cf. also [10], [21]) $E : \mathcal{C}^0(\overline{W}_\rho) \to \mathcal{C}^0(\widetilde{W})$, with $\operatorname{supp} E \subset\subset \widetilde{W}$,

$E(\mathcal{C}^k(\overline{W}_\rho)) \subset \mathcal{C}^k(\widetilde{W})$, and which is bounded in $\mathcal{C}^k$-norms, such that

$$|Eg|_{k.\widetilde{W}} \leq c_k(E)|g|_{k,W_\rho}.$$

Next we describe the construction of the kernels for the homotopy operators. Set

$$\eta^0(\zeta,z) := \frac{d\zeta_n}{\zeta_n - z_n}, \quad \eta^+(\zeta,z) := \frac{\partial_\zeta r(\zeta)}{\Phi(\zeta,z)}, \quad \eta^-(\zeta,z) := \frac{\partial_\zeta r(z)}{\Phi^*(\zeta,z)},$$

$$\Delta_{0+-} := \{\lambda = (\lambda_0,\lambda_+,\lambda_-) \in \mathbb{R}^3 | \lambda_\nu \geq 0, \lambda_0 + \lambda_+ + \lambda_- = 1\},$$

$$\Delta_{0+} := \{\lambda \in \Delta_{0+-} | \lambda_- = 0\}, \qquad \Delta_0 := \{\lambda \in \Delta_{0+-} | \lambda_+ = \lambda_- = 0\}.$$

Δ_{0-}, Δ_{+-}, Δ_+, Δ_- are defined analogously. The simplices are oriented as follows

$$b\Delta_{0+-} = \Delta_{+-} - \Delta_{0-} + \Delta_{0+}, \qquad b\Delta_{ab} = \Delta_b - \Delta_a,$$

for $a,b \in \{0,+,-\}$. Δ_a are positively oriented points. Set

$$\eta(\zeta,\lambda,z) := \lambda_0\eta_0(\zeta,z) + \lambda_+\eta_+(\zeta,z) + \lambda_-\eta_-(\zeta,z)$$

and for $0 \leq q \leq n-1$

$$D_{n,q}(\eta) := c_{n,q}\eta \wedge ((\overline{\partial}_\zeta + d_\lambda)\eta)^{n-q-1} \wedge (\overline{\partial}_z\eta)^q,$$

$D_{n,n} = D_{n,-1} = 0$, with a constant $c_{n,q}$.

These kernels have the crucial property (cf. [19])

$$(*) \qquad (\overline{\partial}_\zeta + d_\lambda)D_{n,q}(\eta) = (d_\zeta + d_\lambda)D_{n,q}(\eta) = (-1)^q\overline{\partial}_z D_{n,q-1}(\eta).$$

Let $f \in \widetilde{\mathcal{C}}^0_{0,q}(\overline{M})$ be a simply tangential form. Set on $M = M_\rho$

$$R_qf(z) := \Big(-\int_{M\times\Delta_{+-}} f(\zeta) \wedge D_{n,q-1}(\eta)(\zeta,\lambda,z) + \int_{bM\times\Delta_{0+-}} f(\zeta) \wedge D_{n,q-1}(\eta)(\zeta,\lambda,z)\Big)_0.$$

In [25] it was shown for $1 \leq q \leq n-2$ and $\overline{\partial}_M f \in \widetilde{\mathcal{C}}^0_{0,q+1}(\overline{M})$ ($\overline{\partial}_M f = 0$ for $q = n-2$), $z \in M$ that

$$f(z) = \overline{\partial}_M R_q f(z) + R_{q+1}(\overline{\partial}_M f)(z).$$

Now we modify R_q. Set $S := \widetilde{M}\backslash M$. Then $bS = b\widetilde{M} - bM$. We define for $z \in M$

$$R^*_q(e)f(z) := R_qf(z) + H_q(e)f(z),$$
$$R^*_{q+1}(e)\overline{\partial}_M f(z) := R_{q+1}\overline{\partial}_M f(z) + H_{q+1}(e)\overline{\partial}_M f(z),$$

with

$$H_q(e)f(z) := \Big(\int_{S\times\Delta_{0+-}} e_2(\overline{\partial}_M f) \wedge D_{n,q-1}(\eta)\Big)_0 - \overline{\partial}_M\Big(\int_{S\times\Delta_{0+-}} e_1(f) \wedge D_{n,q-2}(\eta)\Big)_0 + \Big(\int_{S\times\Delta_{0+}} e_1(f) \wedge D_{n,q-1}(\eta)\Big)_0,$$

$$H_{q+1}(e)(\overline{\partial}_M f)(z) := -\overline{\partial}_M \left(\int_{S\times\Delta_{0+-}} e_2(\overline{\partial}_M f)(\zeta) \wedge D_{n,q-1}(\eta)(\zeta,\lambda,z) \right)_0 .$$

The rightmost integral in the first equation is holomorphic in z. Evidently we have $\overline{\partial}_M H_q(e) f = -H_{q+1}(e)(\overline{\partial}_M f)$. This implies

$$f(z) = \overline{\partial}_M R^*_q(e) f(z) + R^*_{q+1}(e)(\overline{\partial}_M f)(z).$$

For $f \in \widetilde{\mathcal{C}}^1_{0,q}(\overline{M})$, $R^*_q(e)$ can be given in a more suitable way. Namely, for $z \in M$, we get by using $(*)$, Stokes formula and $\triangle := \overline{\partial}_M e_1(f) - e_2(\overline{\partial}_M f)$

$$R^*_q(e) f(z) = \left(- \int_{S\times\Delta_{0+-}} \triangle(\zeta) \wedge D_{n,q-1}(\eta)(\zeta,\lambda,z) - \int_{\widetilde{M}\times\Delta_{+-}} e_1 f(\zeta) \wedge D_{n,q-1}(\eta)(\zeta,\lambda,z) \right)_0 .$$

An analogous formula holds for $R^*_{q+1}(e)(\overline{\partial}_M f)(z)$. For the estimates it is crucial that $\triangle$ vanishes on M and that the supports of e_1 and e_2 are contained in $\widetilde{M}$.

Then the following theorems are true:

Theorem 1. *Let M be a strictly pseudoconvex $\mathcal{C}^m$-smooth surface as defined above, $m \geq 3$, $1 \leq q \leq n-2$. $f \in \widetilde{\mathcal{C}}^0_{0,q}(\overline{M})$, $\overline{\partial}_M f \in \widetilde{\mathcal{C}}^0_{0,q+1}(\overline{M})$. Then $R_q f \in \widetilde{\mathcal{C}}^0_{0,q-1}(\overline{M})$ and there exist positive constants $c(n)$ and $m(n) \in \mathbb{N}$, independent of f, such that*

$$|R_q f|_{0,M} \leq c(n) \left(\frac{1}{\kappa} + |h|_{3,W} \right)^{m(n)} (|f|_{0,M} + |\overline{\partial}_M f|_{0,M}).$$

Theorem 2. *Let M be a strictly pseudoconvex $\mathcal{C}^m$-smooth surface as defined above, $m \geq 3$, $1 \leq q \leq n-2$, $k = 0,1,2,\ldots,m-3$, $f \in \widetilde{\mathcal{C}}^k_{0,q}(\overline{M}), \overline{\partial}_M f \in \widetilde{\mathcal{C}}^k_{0,q+1}(\overline{M})$. Then $R^*_q(e) f \in \widetilde{\mathcal{C}}^k_{0,q-1}(\overline{M})$, $R^*_{q+1}(e)(\overline{\partial}_M f) \in \widetilde{\mathcal{C}}^k_{0,q}(\overline{M})$ and there exist positive constants $c(n,k)$ and $m(n,k) \in \mathbb{N}$, independent of f, such that*

$$|R^*_q(e) f|_{k,M} \leq c(n,k) \left(\frac{1}{\kappa} + |h|_{k+3,\widetilde{W}} \right)^{m(n,k)} (|e_1(f)|_{k,\widetilde{M}} + |e_2(\overline{\partial}_M f)|_{k,\widetilde{M}}),$$

$$|R^*_{q+1}(e)(\overline{\partial}_M f)|_{k,M} \leq c(n,k) \left(\frac{1}{\kappa} + |h|_{k+3,\widetilde{W}} \right)^{m(n,k)} |e_2(\overline{\partial}_M f)|_{k,\widetilde{M}}.$$

The properties of the Seeley operator E yield

Theorem 3. *Let M be a strictly pseudoconvex $\mathcal{C}^m$-smooth surface as defined above, $m \geq 3$, $1 \leq q \leq n-2$, $k = 0,1,2,\ldots,m-3$, $f \in \widetilde{\mathcal{C}}^k_{0,q}(\overline{M}), \overline{\partial}_M f \in \widetilde{\mathcal{C}}^k_{0,q+1}(\overline{M})$. Then $R^*_q f := R^*_q(E) f \in \widetilde{\mathcal{C}}^k_{0,q-1}(\overline{M})$ and there exist positive constants*

$c(n,k)$ and $m(n,k)\in\mathbb{N}$, independent of f, such that

$$|R_q^* f|_{k,M} \le c(n,k)c_k(E)\Big(\frac{1}{\kappa}+|h|_{k+3,\widetilde W}\Big)^{m(n,k)}(|f|_{k,M}+|\overline\partial_M f|_{k,M}).$$

For $R_{q+1}^*(\overline\partial_M f)$ an analogous estimate holds.

2. Sketch of the proof. $R_q(f)$ decomposes into

$$R_q(f)=\int_{bM} f\wedge K_q^I+\int_M f\wedge K_q^{II}.$$

$\mathcal{C}^0$-estimates for the second integral are easy to show. For the first integral we proceed as follows:

$$\int_{bM} f\wedge K_q^I = c\int_{bM} f\wedge \frac{d\zeta_n}{\zeta_n-z_n}\wedge\frac{\partial_\zeta r(\zeta)}{\varPhi}\wedge\frac{\partial_\zeta r(z)}{\varPhi^*}\wedge\ldots$$

Obviously, the singularity of the kernel is of mixed type, which causes a considerable difficulty. The main difficulties occur at the so-called characteristic points, where

$$\frac{\partial r}{\partial\zeta_i}=0,$$

for all $i<n$. At characteristic points, the holomorphic tangent space to M coincides with the tangent space to bM. In our situation, these points form a nowhere dense closed subset. So it suffices to show uniform continuity and the estimates only at the non-characteristic points. For such a point z, which is near bM, we can apply the Stokes theorem and obtain

$$\int_{bM} f\wedge K_q^I=\lim_{\varepsilon\to0}\int_{bM,|\zeta_n-z_n|\ge\varepsilon} f\wedge K_q^I = T_1+T_2+T_3,$$

with

$$T_1:=\int_M \overline\partial_M f\wedge K_q^I,\quad T_2:=(-1)^q\int_M f\wedge\overline\partial_M K_q^I,$$
$$T_3:=\pm\lim_{\varepsilon\to0}\int_{M,|\zeta_n-z_n|=\varepsilon} f\wedge K_q^I.$$

In T_1 and T_2 the kernels are integrable. Passing to the limit yields

$$T_3=\int_{M,\zeta_n=z_n} f\wedge\frac{\mathcal{E}_1(\zeta',z')\partial' r(\zeta')\wedge\ldots}{\varPhi(\zeta',z_n;z)^{n-q-1}\varPhi^*(\zeta',z_n;z)^q},$$

with $|\mathcal{E}_1(\zeta',z')|\le c|\zeta'-z'|$. Therefore, the domain of integration depends on z. By assumption on H, the surface $\{M|\zeta_n=z_n\}$ is a closed strictly convex hypersurface in $\mathbb{C}^{n-1}$ and the restrictions of $\varPhi$, $\varPhi^*$ are the respective barrier functions. A local problem now has turned to a global one for a strictly convex domain. A careful analysis now yields

$$|T_3| \le c|f|_{0,M}$$

and the uniform convergence of T_3 to a continuous limit, when z tends to a characteristic point.

For T_1 and T_2 we need lemmata like this:

LEMMA. *For* $\varepsilon > 0$, $i < n$

$$\int\limits_{M,|\zeta_n - z_n|<\varepsilon} \frac{|d\zeta_1 \wedge \ldots \wedge d\zeta_{i-1} \wedge d\zeta_{i+1} \wedge \ldots \wedge d\zeta_n \wedge d\overline{\zeta_1} \wedge \ldots \wedge d\overline{\zeta_n}|}{|\zeta_n - z_n||\Phi^*|^{n-1}} \le c\varepsilon^{1/8}.$$

This lemma can be shown by applying a transformation of Bruna and Burgués (cf. [5]). Set $q(z) := (|r_1(z)|^2 + \cdots + |r_{n-1}(z)|^2)^{1/2}$ and assume $q(z) > 0$. Define an affine isometric isomorphism

$$\widetilde{\varphi}_z : \begin{cases} \mathbb{C}^{n-1} & \to \mathbb{C}^{n-1} \\ \zeta' & \mapsto w' \end{cases}$$

with

$$w_1 = (\widetilde{\varphi}_z)_1(\zeta') = \frac{1}{q(z)} \sum_{i=1}^{n-1} \partial_i r(z)(\zeta_i - z_i), \quad |w'| = |\zeta' - z'|.$$

Then we obtain the following affine isometric isomorphism of $\mathbb{C}^n$:

$$\varphi_z(\zeta) := \left(\widetilde{\varphi}_z(\zeta'), \frac{r_n(z)}{|r_n(z)|}(\zeta_n - z_n)\right) =: (w_1, \ldots, w_n).$$

In the φ_z-coordinates we have $\Phi^*(\zeta, z) = q(z)w_1 + |r_n(z)|w_n$. A rather explicit calculation now gives the desired estimate.

For $\mathcal{C}^k$-estimates we proceed as follows. Similarly as for R_q

$$R_q^*(e_1)(f) = \int\limits_S \Delta \wedge K_q^{III} + \int\limits_{\widetilde{M}} e_1(f) \wedge K_q^{IV}.$$

The estimates for the second integral are simple. By carefully applying several times Stokes formula to the first integral, one eventually arrives at terms which are similar to those of the $\mathcal{C}^0$-estimates for $R_q(f)$. The proof needs a series of lemmas, which cannot be given here. The main ideas are taken from [10], [16], [17]. We confine ourselves to the following remark. If

$$\frac{\partial}{\partial \zeta_i} + \frac{\partial}{\partial z_i}$$

is applied to $\zeta_n - z_n$, Φ and Φ^*, the result has some vanishing order for $\zeta = z$. So one can try to transform z-derivatives to ζ-derivatives. Then Stokes theorem should be applied. But the above derivation is a non-tangential vector field, so an appropriate switch from vector fields in $\mathbb{C}^n$ to tangential vector fields of M is necessary.

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