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CARLEMAN'S FORMULAS AND CONDITIONS OF ANALYTIC EXTENDABILITY

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1. Criteria for analytic continuation into a domain of a function given on part of the boundary. The following classical assertion is well known. Let $\mathcal{D} \subset \mathbb{C}$ be a simply connected bounded domain with smooth boundary $\partial\mathcal{D}$ and $f \in C(\partial\mathcal{D})$. Then

$$\int_{\partial\mathcal{D}} f z_1^m dz_1 = 0, \qquad m = 0, 1, 2, \ldots \tag{1}$$

if and only if $f(z)$ extends into the domain $\mathcal{D}$ as a holomorphic function of class $A(\mathcal{D}) \cap C(\overline{\mathcal{D}})$. For the multidimensional case instead of the form $z_1^m dz_1$ we have the exterior differential form of class $Z^{\infty}_{n,n-1}(\overline{D})$.

THEOREM 1 (Weinstock–Aronov–Dautov). *Let $\mathcal{D}$ be a domain in $\mathbb{C}^n$ with smooth boundary and $f \in C(\partial\mathcal{D})$. Then there is a function $F \in A(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ such that $F|_{\partial\mathcal{D}} = f$ if and only if*

$$\int_{\partial\mathcal{D}} f\alpha = 0 \tag{2}$$

for every form $\alpha \in Z^{\infty}_{n,n-1}(\overline{\mathcal{D}})$.

If f is only defined on a part of the boundary of $\mathcal{D}$, then the existence of an analytic continuation into $\mathcal{D}$ cannot be decided by the vanishing of some family of continuous linear functionals (as in (1)–(2)). Solutions to this problem were given by G. Zin (1953), V. A. Fok–F. M. Kuni (1959), D. I. Patil (1972), M. G. Krein–P. Ya. Nudelman (1973), A. Steiner (1974), N. N. Tarkhanov (1989), O. V. Karepov–N. N. Tarkhanov (1990), A. A. Shlyapunov–N. N. Tarkhanov

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(1990), L. Zhamenskaya (1990), L. A. Aizenberg (1988, 1990, 1991, 1992), L. A. Aizenberg–A. M. Kytmanov (1990, 1991), L. A. Aizenberg–C. Rea (1991).

A very simple solution was given by L. A. Aizenberg (1990,1992):

Case 1: $n = 1$. Let $\mathcal{D}$ be the domain bounded by a part of the unit circle $\gamma_1 = \{z_1 : |z_1| = 1\}$ and a smooth open arc Γ connecting two points of γ_1 and lying inside γ_1. Let $0 \notin \overline{\mathcal{D}}$. We set

$$a_k = \int\limits_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta^{k+1}}, \qquad k = 0, 1, 2, \ldots$$

THEOREM 2. *If* $f \in C(\Gamma) \cap L^1(\Gamma)$, *then there is a function* $F \in A(\mathcal{D}) \cap C(\mathcal{D} \cup \Gamma)$ *such that* $F|_\Gamma = g$ *if and only if*

(3) $$\varlimsup_{k\to\infty} \sqrt[k]{|a_k|} \le 1.$$

If $f|_\Gamma$ *is not identically zero*, *then* (3) *is equivalent to*

(4) $$\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1.$$

Proof. *Necessity*. Put $\Gamma_\varepsilon = \{z_1 : |z_1| < 1 - \varepsilon\} \cap \Gamma$, where $0 < \varepsilon < 1$, and

$$a_k^\varepsilon = \int\limits_{\Gamma_\varepsilon} \frac{f(\zeta)d\zeta}{\zeta^{k+1}}.$$

Suppose there exists a function F as in the Theorem. Then the a_k^ε are equal to the corresponding integrals of F/ζ^{k+1} over the part of the circle $\gamma_{1-\varepsilon}$, therefore,

$$|a_k^\varepsilon| \le \frac{C(\varepsilon)}{(1-\varepsilon)^{k+1}}.$$

Also

$$a_k = a_k^\varepsilon + \int\limits_{\Gamma\setminus\Gamma_\varepsilon} \frac{f(\zeta)d\zeta}{\zeta^{k+1}},$$

hence,

$$|a_k| \le \frac{C(\varepsilon)}{(1-\varepsilon)^{k+1}} + \frac{C_1}{(1-\varepsilon)^{k+1}}.$$

Now we obtain

$$\varlimsup_{k\to\infty} \sqrt[k]{|a_k|} \le \frac{1}{1-\varepsilon},$$

whence we arrive at (3) as $\varepsilon \to +0$.

Sufficiency. Consider the Cauchy type integral

(5) $$\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta - z} = F_\pm(z),$$

which defines a function F_+ holomorphic in $\mathcal{D}$ and a function F_- holomorphic in $\mathcal{D}_-$ (part of the unit disc after the removal of $\overline{\mathcal{D}}$) such that the difference between their limit values along normals (points z^+ and z^- equally distant from ζ) on Γ is equal to $f(\zeta)$: for $\zeta \in \Gamma$,

$$F_+(\zeta) - F_-(\zeta) \stackrel{\text{def}}{=} \lim_{z^\pm \to \zeta} [F_+(z^+) - F_-(z^-)] = f(\zeta). \tag{6}$$

Moreover, if one of the functions F_+ or F_- is continuous in the corresponding domain up to Γ, so is the other.

Expanding (5) into a power series in z in a neighborhood of zero, we find that the coefficients of this series are $a_k/(2\pi i)$. This and (3) imply that F_- is holomorphic on the whole unit disc. Then $F_+ - F_- \in A(\mathcal{D}) \cap C(\mathcal{D} \cup \Gamma)$, and by (6) we can take $F = F_+ - F_-$.

If $\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|} < 1$, then (5) is holomorphic in the disc with radius $R > 1$ and the singularity Γ is removable, hence the second part of Theorem 2 is true.

Corollary 1. *Let $f \in C(\Gamma)$. There is a function $F \in A(\mathcal{D}) \cap C(\mathcal{D} \cup \Gamma)$ such that $F|_\Gamma = f$ if and only if*

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k^\varepsilon|} \leq \frac{1}{1-\varepsilon} \qquad \textit{for } 0 < \varepsilon < \varepsilon_0 < 1.$$

Let us generalize this result to a simply connected domain Ω with Jordan boundary $\partial\Omega$ containing a smooth open arc Γ. We connect the ends of Γ by a curve $\mathcal{C}$, lying outside $\overline{\Omega}$, and let Ω_1 be the domain with boundary $\mathcal{C} \cup (\partial\Omega \setminus \Gamma)$, which we also assume to be Jordan. Let $w = \varphi(z_1)$ map Ω_1 conformally onto the unit disc so that the preimage of zero is in $\Omega_1 \setminus \overline{\Omega}$. Let

$$A_k^\varepsilon = \int_{\Gamma_\varepsilon} \frac{f(\zeta)d\varphi(\zeta)}{\varphi^{k+1}(\zeta)}.$$

Corollary 2. *If $f \in C(\Gamma)$, then there is a function $F \in A(\Omega) \cap C(\Omega \cup \Gamma)$ such that $F|_\Gamma = f$ if and only if*

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|A_k^\varepsilon|} \leq \frac{1}{1-\varepsilon} \qquad \textit{for } 0 < \varepsilon < \varepsilon_0 < 1. \tag{7}$$

We can apply the approach of this section to the case when $\Gamma = \partial\mathcal{D}$. Suppose 0 lies outside $\overline{\mathcal{D}}$, where $\mathcal{D}$ is a simply connected bounded domain with a smooth boundary. Then the classical condition for $f \in C(\partial\mathcal{D})$ to analytically continue to $\mathcal{D}$ is (1).

Corollary 3. *The (necessary and sufficient) condition for analytic continuation of $f \in C(\partial\mathcal{D})$ to $\mathcal{D}$ is*

$$\overline{\lim_{k\to\infty}} \sqrt[k]{\Big|\int_{\partial\mathcal{D}} f z_1^k dz_1\Big|} < \rho = \min_{z\in\partial\mathcal{D}} |z|. \tag{8}$$

If (8) is true, then (1) is true. If the integrals in (1) do not grow too rapidly, they vanish.

COROLLARY 4. *Given a simply connected domain* Ω *with Jordan boundary and a smooth open arc* $\Gamma \subset \Omega, \Omega = \Omega_1 \cup \Omega_2 \cup \Gamma, \Omega_1$ *and* Ω_2 *simply connected domains. Let* $f \in C(\Gamma)$. *There exists a function* $F \in A(\Omega)$ *so that* $F|_\Gamma = f$ (*this is not a boundary problem, but an interior problem*) *if and only if two conditions of type* (7) *hold.*

EXAMPLE. Let Ω be the whole complex plane $\mathbb{C}$. In this case, a conformal mapping of $\mathcal{D}$ onto the unit disc does not exist, of course, but we do not need it. Let Γ be a simply smooth curve, dividing the plane $\mathbb{C}$ into two domains: Ω_1 and Ω_2. Let $i \in \Omega_1$ and $-i \in \Omega_2$, and let

$$a_k^\pm = \int_\Gamma \frac{f(\zeta)d\zeta}{(\zeta \pm i)^{k+1}}, \qquad k = 0, 1, 2, \ldots,$$

where $f \in C(\Gamma) \cap L^1(\Gamma)$. Then the function f can be extended to an entire function if and only if

$$\lim_{k\to\infty} \sqrt[k]{|a_k^+|} = \lim_{k\to\infty} \sqrt[k]{|a_k^-|} = 0.$$

Case 2: $n > 1$. Let $\Omega = \{\zeta : \psi(\zeta) < 0\}$ be a $(p_1, \ldots, p_n)$-circular domain in $\mathbb{C}^n$, where $p_1, \ldots, p_n$ are natural numbers, i.e., $z \in \Omega$ implies $(z_1, e^{itp_1}, \ldots, z_n e^{itp_n}) \in \Omega$ for $t \in \mathbb{R}$. In particular, for $p_1 = \ldots = p_n = 1$ this circular domain is a Cartan domain. Moreover, assume that Ω is convex and bounded and $\partial\Omega \in C^2$. Furthermore, let $\mathcal{D}$ be a domain bounded by a part of $\partial\Omega$ and by a hypersurface $\Gamma \in C^2$ dividing Ω into two parts, the complement of $\overline{\mathcal{D}}$ containing the origin. Let us consider the Cauchy–Fantappiè differential form

$$w(\zeta - z, w) = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{k=1}^n (-1)^{k-1} w_k dw[k] \wedge d\zeta}{\langle w, \zeta - z\rangle^n},$$

where $dw[k] = dw_1 \wedge \ldots \wedge dw_{k-1} \wedge dw_{k+1} \wedge \ldots \wedge dw_n, d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta_n, \langle a, b\rangle = a_1 b_1 + \ldots + a_n b_n$; then $\operatorname{grad}\psi = (\partial\psi/\partial\zeta_1, \ldots, \partial\psi/\partial\zeta_n)$. By the Sard Theorem, $\operatorname{grad}\psi \neq 0$ for almost all r on $\partial\Omega_r$, where $\Omega_r = r\Omega$ is a homothety of Ω with $0 < r < 1$. We will assume that $\operatorname{grad}\psi \neq 0$ on Γ. We set

$$C_q = \frac{(|q| + n - 1)!}{q!} \int_\Gamma f(\zeta) \left(\frac{\operatorname{grad}\psi}{\langle \operatorname{grad}\psi, \zeta\rangle}\right)^q \omega(\zeta, \operatorname{grad}\psi),$$

where $q = (q_1, \ldots, q_n)$, $q! = q_1! \ldots q_n!$, $|q| = q_1 + \ldots + q_n$, $w^q = w_1^{q_1} \ldots w_n^{q_n}$,

$$a_k = \sum b_{q,s} C_q \overline{C}_s,$$

where

$$b_{q,s} = \int_\Omega z^q \overline{z}^s dv,$$

dv is the volume element in Ω. We emphasize that the integral moments C_q depend on f and Γ, buth the moments $b_{q,s}$ depend only on Ω.

THEOREM 3. *For a function* $f \in C(\Gamma) \cap L^1(\Gamma)$ *to have a holomorphic continuation* $F \in A(\mathcal{D}) \cap C(\mathcal{D} \cup \Gamma)$ *with* $F|_\Gamma = f$, *it is necessary and sufficient that the following two conditions are fulfilled*:

(i) f *is a CR function on* Γ,
(ii) $\overline{\lim}_{k\to\infty} \sqrt[k]{a_k} \le 1$.

We managed to find a very simple solution, using a complete system of holomorphic functions (weighted homogeneous polynomial of degree k if it is homogeneous of degree k with respect to $z_1^{1/p_1}, \ldots, z_n^{1/p_n}$). If we used the basis, the answer would not be so easy.

COROLLARY 1: statement of Theorem 2.

COROLLARY 2 (Aizenberg–Kytmanov). *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded convex n-circular domain. Set* $d_q(\Omega) = \max_{\overline{\Omega}} |z^q|$. *For a function* $f \in C(\Gamma) \cap L^1(\Gamma)$ *to have a holomorphic continuation in* $\mathcal{D}$ *as above it is necessary and sufficient that*

(i) f *is a CR function on* Γ,
(ii) $\overline{\lim}_{|q|\to\infty} \sqrt[|q|]{|C_q| d_q(\Omega)} \le 1$.

2. Carleman formulas. Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary $\partial\mathcal{D}$ and let M be a set of positive $(2n-1)$-dimensional Lebesgue measure in $\partial\mathcal{D}$. We consider the following problem: if f is a holomorphic function in $\mathcal{D}$ that is sufficiently well behaved at the boundary $\partial\mathcal{D}$, for example f is continuous in $\overline{\mathcal{D}}$, ($f \in A_C(\mathcal{D})$), or f is contained in the Hardy class $H^1(\mathcal{D})$, then how can it be reconstructed inside $\mathcal{D}$ by its values on M with the help of an integral formula? The problem makes sense because M is a uniqueness set for such functions (L. A. Aizenberg, 1959). Solutions to this problem were given by T. Carleman (1926), G. Goluzin–V. Krylov (1933), G. Zin (1953), V. Fock–F. Kuni (1959), D. Patil (1972), M. Krein–P. Nudelman (1973), L. Aizenberg–N. Tarkhanov (1988), A. Kytmanov–T. Nikitina (1989), L. Aizenberg (1984, 1985, 1990, 1991, 1992).

Three methods of solution are known, due to: 1) Carleman–Goluzin–Krylov, 2) Lavrent'ev, 3) Kytmanov. There is also a very general approach offered by N. Videnskiĭ–E. Gavurina–V. Khavin (1983) for $n = 1$.

A very simple solution:

Case 1: $n = 1$. If $M = \Gamma$ is an arc in the unit disc with ends on the unit circle, then we can give a simpler formula (see the beginning of example 3, sec. 1 in [1] and Goluzin–Krylov (1933))

$$f(z) = \lim_{m\to\infty} \frac{1}{2\pi i} \int_\Gamma f(\zeta) \left(\frac{z}{\zeta}\right)^m \frac{d\zeta}{\zeta - z}. \tag{1}$$

We show that this simple formula can be easily obtained, not only by introducing a quenching function into the Cauchy formula (the Carleman–Goluzin–Krylov method), but also by approximating the Cauchy kernel on $\partial\mathcal{D} \setminus \Gamma$ (the Lavrent'ev method). This also yields simple Carleman formulas with holomorphic kernels, generalizing (1), in the multidimensional case. So, let $\mathcal{D}$ be a domain in the disc $U(0,r)$ whose boundary consists of an arc $\gamma \subset \partial U$ and a smooth arc Γ lying in $U(0,r)$ and connecting the ends of γ, and let $0 \notin \overline{\mathcal{D}}$. By the Cauchy formula (γ and Γ oriented compatibly with $\partial\mathcal{D}$)

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(\zeta)d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\gamma} \frac{f(\zeta)d\zeta}{\zeta - z}. \tag{2}$$

The kernel of the second term on the right-hand side of (2) has a series expansion

$$\frac{1}{2\pi i}\frac{1}{\zeta - z} = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{z^k}{\zeta^{k+1}}$$

which, for fixed $z \in \mathcal{D}$, converges uniformly on γ with respect to ζ. Therefore,

$$\frac{1}{2\pi i}\frac{1}{\zeta - z} = \lim_{m\to\infty}\frac{1}{2\pi i}\sum_{k=0}^{m-1}\frac{z^k}{\zeta^{k+1}} = \lim_{m\to\infty}\frac{1}{2\pi i}\frac{1-(z/\zeta)^m}{\zeta - z}. \tag{3}$$

The last function under the limit is holomorphic with respect to ζ in $\overline{\mathcal{D}}$, hence

$$0 = \frac{1}{2\pi i}\int\limits_{\Gamma} f(\zeta)\frac{1-(z/\zeta)^m}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int\limits_{\gamma} f(\zeta)\frac{1-(z/\zeta)^m}{\zeta - z}d\zeta. \tag{4}$$

Then we subtract (4) from (2) and pass to the limit as $m \to \infty$. Then the integral over γ approaches zero by (3), and we have the formula (1).

Case 2: $n>1$. Let Ω be a circular convex bounded domain (Cartan domain) with boundary of class C^2 and let Γ be a piecewise smooth hypersurface intersecting Ω and cutting from it the domain $\mathcal{D}, 0 \notin \overline{\mathcal{D}}$, i.e., the boundary $\partial\mathcal{D}$ is the union of Γ and of a part of the boundary of Ω, which we denote by γ ($\gamma \in C^2$ is sufficient). Then there exists a Cauchy–Fantappiè formula for the domain $\mathcal{D}$ with kernel holomorphic in z. It can be constructed by using a "barrier" for γ and a "barrier" for Γ, and the "glueing" of the barrier in the case where they do not match sufficiently smoothly at the joining of Γ and γ (cf. the proof of Theorems 12.1 and 12.3 in [1], method of Norguet) being passed over when integrating over some cycle in $\mathbb{C}^{2n}$ lying on $\Gamma \cap \gamma$. I.e., in fact, integration will be over γ, Γ and the faces $\gamma \cap \Gamma$. We write the last two cases conditionally as the integral over $\tilde{\Gamma}$ of some form $R(z,\zeta,d\zeta,d\overline{\zeta})$. Then, for $f \in A_C(\mathcal{D})$ and the points $z \in \mathcal{D}$,

$$f(z) = \int\limits_{\tilde{\Gamma}} f(\zeta)R(z,\zeta,d\zeta,d\overline{\zeta}) + \int\limits_{\gamma} f(\zeta)\omega(\zeta - z, \operatorname{grad}\rho), \tag{5}$$

where $\Omega = \{\zeta : \rho(\zeta) < 0\}$, $\rho \in C^2(\overline{\Omega})$.

A circular convex domain Ω is also linearly convex, i.e., the analytic tangent plane $\{z : \langle\rho'(\zeta), \zeta - a\rangle = 0\}$, where $\zeta \in \partial\Omega$, does not overlap Ω. In other words, for $\zeta \in \partial\Omega$ and $z \in \Omega$ the inequality $\langle \rho'(\zeta), \zeta - z\rangle \neq 0$ holds, or

$$\frac{\langle \rho'(\zeta), z\rangle}{\langle \rho'(\zeta), \zeta\rangle} \neq 1. \tag{6}$$

Moreover, if $z \in \Omega$, then $ze^{it} \in \Omega$, where $0 \leq t \leq 2\pi$. Therefore, (6) implies that

$$\left|\left\langle \frac{\rho'(\zeta)}{\langle \rho'(\zeta), \zeta\rangle}, z \right\rangle\right| < 1, \qquad z \in \Omega,\ \zeta \in \partial\Omega,$$

where $\rho'(\zeta) = \operatorname{grad} \rho(\zeta)$.

Consequently, the kernel of the second integral in (5) has a series expansion for $z \in \mathcal{D}, \zeta \in \gamma$,

$$\omega(\zeta - z, \rho'(\zeta)) = \frac{(n-1)!}{(2\pi i\langle \rho', \zeta\rangle)^n} \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!(n-1)!} \left\langle \frac{\rho'(\zeta)}{\langle \rho'(\zeta), \zeta\rangle}, z \right\rangle^k \sigma,$$

$$\sigma = \sum_{j=1}^{n} (-1)^{j-1} \rho' d\rho'[j] \wedge d\zeta,$$

which uniformly converges with respect to ζ on γ for fixed $z \in \mathcal{D}$.

Let the function ρ defining the domain Ω be such that every domain $\Omega(r) = \{z : \rho(\zeta) < r\}$ is also convex, $0 < r < 1$, while $\min \rho$ is attained at the point 0. Then, everywhere in $\overline{\Omega} \setminus \{0\}$ the inequality $\langle \rho'(\zeta), \zeta\rangle \neq 0$ holds, and the form

$$\varphi_k = \frac{1}{\langle \rho', \zeta\rangle^n} \left\langle \frac{\rho'(\zeta)}{\langle \rho', (\zeta), \zeta\rangle}, z \right\rangle^k \sigma \tag{7}$$

is of class $Z^1_{n,n-1}(\mathcal{D})$for every k. Hence, this form is orthogonal to the holomorphic functions when integrating over $\partial\Omega$:

$$0 = \int_{\Gamma} f(\zeta) \sum_{k=0}^{m} \frac{(k+n-1)!}{k!} \varphi_k + \int_{\gamma} f(\zeta) \sum_{k=0}^{m} \frac{(k+n-1)!}{k!} \varphi_k. \tag{8}$$

We subtract the equality (8) from (5) and pass to the limit as $m \to \infty$. Then the second integral in the obtained equality approaches zero, and we obtain the following assertion:

THEOREM 4. *If γ is a part of the boundary of a circular convex bounded domain Ω, Γ is a piecewise smooth hypersurface intersecting Ω, and $\mathcal{D}$ is the domain with boundary $\partial\mathcal{D} = \gamma \cup \Gamma$, and $0 \notin \overline{\mathcal{D}}$, then for every function $f \in A_C(\mathcal{D})$ and $z \in \mathcal{D}$, the following Carleman formula with holomorphic kernel is valid*:

$$f(z) = \int_{\tilde{\Gamma}} f(\zeta) R(z, \zeta, d\zeta, d\bar{\zeta}) - \lim_{m\to\infty} \int_{\Gamma} f(\zeta) \frac{1}{(2\pi i)^n} \sum_{k=0}^{m} \frac{(k+n-1)!}{k!} \varphi_k, \tag{9}$$

where the φ_k are given by the equality (7).

We note that if $n = 1$, $\rho(\zeta) = |\zeta|^2 - r$, then $\rho'(\zeta) = \overline{\zeta}$ and

$$\left\langle \frac{\rho'(\zeta)}{\langle \rho', (\zeta), \zeta\rangle}, z \right\rangle = \frac{z}{\zeta},$$

and from (9) we obtain (1). So, (9) is a direct generalization of (1) to the multidimensional case. Each differential form φ_k is a homogeneous polynomial of degree k in z.

Corollary 1. *If there exists a vector-valued function* (*a "barrier"*) $w = w(z,\zeta), z \in \mathcal{D}, \zeta \in \Gamma$, *such that* $\langle w, \zeta - z\rangle \neq 0$, $w \in C^1_\zeta(\Gamma)$, *and* w *smoothly extends to* ρ' *on* $\gamma \cap \Gamma$, *then*

$$f(z) = \lim_{m\to\infty} \int_\Gamma f(\zeta)\left[\omega(\zeta - z, w) - \frac{1}{(2\pi i)^n}\sum_{k=0}^m \frac{(k+n-1)!}{k!}\varphi_k\right].$$

Now let Ω be an n-circular domain (a Reinhardt domain). Then the series for $\omega(\zeta - z, \rho')$ with respect to a homogeneous polynomial in z can be replaced by a power series in z:

Corollary 2. *If under the conditions of Corollary* 1, γ *is part of the boundary of an* n*-circular convex bounded domain*, *then*

$$f(z) = \lim_{m\to\infty} \int_\Gamma f(\zeta)\left[\omega - \frac{1}{(2\pi i\langle \rho', \zeta\rangle)^n}\sum_{|\alpha|=0}^m \frac{(|\alpha|+n-1)!}{\alpha!}\left(\frac{\rho'(\zeta)}{\langle \rho', \zeta\rangle}\right)^\alpha z^a \sigma\right].$$

Corollary 3. *If* $\Omega = \{z : |z| < r\}$ *is a ball*, *then*

$$f(z) = \lim_{m\to\infty} \int_\Gamma f(\zeta)\left[\omega - \frac{(n-1)!}{(2\pi i)^n}\frac{(|\zeta|^{2m} - \langle \overline{\zeta}, z\rangle^m)^n}{|\zeta|^{2mn}\langle \overline{\zeta}, \zeta - z\rangle^n}\sum_{j=1}^n (-1)^{j-1}\overline{\zeta}_j d\overline{\zeta}[j] \wedge d\zeta\right]. \tag{10}$$

Corollary 4. *If* $n = 1$, $\Omega = U(0, r)$ *is a disc*, *from formula* (10) *we obtain formula* (1) *again.*

References

[1] L. A. Aizenberg, *Carleman Formulas in Complex Analysis*, Kluwer Academic Publishers, 1993.