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CONVEXITY IN COMPLEX ANALYSIS

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Convexity is the key concept of functional analysis, but apart from some notable exceptions, it has played a relatively minor rôle in several complex variables theory. The work of Lempert has focused attention on convex domains, and in these two lectures I will present examples involving invariant metrics in complex analysis where convexity, whether realized as in the case of bounded symmetric domains or assumed as in the case of B_{l_p}, is essential. Functional analysis brings to problems a variety of developed concepts such as complex extreme points, complex uniform convexity and an approach which is often coordinate and dimension free. I hope to illustrate these points in my lectures.

Throughout this article X will denote a Banach space over the complex numbers $\mathbb{C}$, B_X will denote the open unit ball in X and $\overline{B_X}$ its closure. We denote by Δ the open unit disc in $\mathbb{C}$.

Our first example relates the maximum modulus theorem of complex analysis with the functional analytic concept of complex extreme point. A point x in X, ($\|x\| = 1$), is a complex extreme point (of the unit ball) if $\|x + \lambda y\| \le 1$ for all $\lambda \in \Delta$ implies $y = 0$. The strong maximum modulus theorem, due to Thorp and Whitely in 1965, states the following:

If $f : \Delta \to X$ is holomorphic and all unit vectors in X are complex extreme points, then $\|f\|$ is constant if and only if f is constant.

The converse is also true and the result has been extended in a variety of directions. The following is used in the study of complex geodesics:

If $f : \Delta \to \overline{B_X}$ is holomorphic and $f(\Delta) \cap \partial B_X$ is non-empty then $f(\Delta) \subset \partial B_X$ and if $f(\Delta)$ contains a complex extreme point, then f is a constant function (∂ denotes the boundary).

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Our second example concerns bounded symmetric domains. A bounded symmetric domain is a complex manifold modelled on a Banach space which is biholomorphically equivalent to a bounded domain and such that through each point x there exists a unique biholomorphic symmetry of the domain δ_x, $\delta_x^2 = 1$, with x as its unique fixed point. Convexity plays no rôle in this definition. Nevertheless, a deep result of Kaup shows that each bounded symmetric domain can be realized as the unit ball of a Banach space. Moreover, the Banach spaces X for which B_X is symmetric can be endowed with a triple product $\{x, y, z\}$ which satisfies axioms similar to those satisfied by $\mathcal{C}^*$ algebras.

EXAMPLE 1 (Simple but contains the 'generic' example.) Let H and K be complex Hilbert spaces and let $\mathcal{L}(H, K)$ denote the set of all continuous linear operators from H to K. The triple product which captures the symmetric structure of the open unit ball is given by

$$\{A, B, C\} = \frac{AB^*C + CB^*A}{2}.$$

If $H = K$ and $\dim(H) = n < \infty$ we have the space of all $n \times n$ matrices which is a $\mathcal{C}^*$ algebra while if $\dim(H) = n$ and $\dim(K) \neq n$ we obtain a Banach space which is not a $\mathcal{C}^*$ algebra.

We call X a JB^* triple if B_X is a bounded symmetric domain. If X is 'too large' then B_X does not admit any $\mathcal{C}^2$ (biholomorphically invariant) metrics. A functional analytic characterization of 'not too large' is reflexivity. We introduce tripotents to give an algebraic characterization. An element e in X is called a tripotent if $e^3 := \{e, e, e\} = e$. This is the triple product analogue of a partial isometry.

If we fix $y \in X$ and consider the operator

$$y \,\square\, y : x \to \{y, y, x\}$$

then y is a tripotent if and only if 1 is an eigenvalue of $y \,\square\, y$ and y belongs to its 1-eigenspace. The tripotent e is said to be minimal if the 1-eigenspace of $e \,\square\, e$ is 1-dimensional. If the JB^* triple is a dual space then it has a unique predual and there is a one to one correspondence between minimal tripotents in X and complex extreme points of the predual. Tripotents e and f are said to be orthogonal if

$$(e \,\square\, e)(f) = \{e, e, f\} = 0$$

i.e. if f is an eigenvector in the 0-eigenspace of $e \,\square\, e$.

The algebraic characterization we are seeking is the following:

PROPOSITION 2. *If X is a JB^* triple then the following are equivalent*:

(a) *X is reflexive*,

(b) *there exists an integer r such that any set of mutually orthogonal tripotents contains at most r elements*,

(c) *there exists an integer s such that any polydisc P which sits isometrically in B_X (i.e. $\partial P \subset \partial B_X$) has dimension $\leq s$.*

(d) *B_X admits an invariant $\mathcal{C}^2$ metric.*

If these equivalent conditions are satisfied, then the maximum value of r that may occur in (b) is equal to the maximum of s that may occur in (c) and is called the rank of the bounded symmetric domain. By (c) one sees that this coincides with the dimension of the maximal torus contained in $\overline{B}_X$. The spin factors are examples of infinite dimensional bounded symmetric domains of finite rank. All finite dimensional bounded symmetric domains have finite rank.

In [1] and [5] the authors discuss the following problem:

Given a bounded symmetric domain B_X find an invariant $\mathcal{C}^2$ metric α on B_X which gives the best constant in the Schwarz Lemma.

The solution to this problem uses an invariant inner product defined using minimal tripotents. If the JB^* triple X has finite rank r, then for each $x \in X$ there exist nonnegative real numbers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq 0$ and minimal tripotents $e_1, \ldots, e_r$ such that

$$x = \sum_{i=1}^{r} \lambda_i e_i.$$

If ϕ_e is the complex extreme point in X' associated with e then

$$\langle x, y\rangle_a = \sum_{i=1}^{r} \lambda_i \phi_{e_i}(y)$$

defines an inner product on X which is invariant under (surjective) linear isometries of X i.e. if $L : X \to X$ is a linear mapping satisfying $\|Lx\| = \|x\|$ for all $x \in X$ then

$$\langle Lx, Ly\rangle_a = \langle x, y\rangle_a \quad \text{for all } x, y \in X.$$

We use this invariant inner product to define a metric on the tangent space at the origin. If m_x denotes a Möbius transformation which takes x to 0 then

$$\alpha^2(x, v) = \langle m_x'(0)(v), m_x'(0)(v)\rangle_a$$

defines an invariant $\mathcal{C}^2$ metric on B_X which gives the best constant in the Schwarz lemma. More precisely, we have the following result.

PROPOSITION 3. *If X is a JB^* triple of finite rank r then for any holomorphic $f : B_X \to B_X$, we have*

$$\alpha(f(x), f'(x)(v)) \leq \sqrt{r}\alpha(x, v).$$

Moreover, if β is an invariant $\mathcal{C}^2$ metric on B_X and

$$\beta(f(x), f'(x)(v)) \leq M\beta(x, v)$$

for all holomorphic $f : B_X \to B_X$ then $M \geq \sqrt{r}$ and $M = \sqrt{r}$ if and only if β is a positive multiple of α.

Before moving to the next topic we will mention just two further results. The first once more emphasizes the relationship between concepts in functional analysis and complex analysis.

Proposition 4. *If X is a JB^* triple system then the following sets coincide*;

(a) *the complex extreme points of B_X*,

(b) *the maximal tripotents in B_X* (*e is maximal if $e \square e$ is invertible*),

(c) *the distinguished boundary of B_X*,

(d) *the Shilov boundary of $H^\infty(B_X)$* (*the space of bounded $\mathbb{C}$-valued holomorphic functions on B_X*).

Our second remark is that B_X, X a JB^* triple, is irreducible—with respect to either its algebraic or holomorphic structure—if and only if $\langle\quad,\quad\rangle_a$ is the unique inner product, up to positive scalar multiples, which is invariant by isometries. This observation and the realization that the same property is satisfied by other Banach spaces is a key element in our next set of examples.

A well known Schwarz Lemma of H. Cartan says the following:

If $\mathcal{D}$ is a bounded domain in $\mathbb{C}^n$, $a \in \mathcal{D}$, and $f : \mathcal{D} \to \mathcal{D}$ is holomorphic with fixed point a, then $|\det(f'(a))| \le 1$ and $|\det(f'(a))| = 1$ if and only if f is biholomorphic.

One can consider the same problem with different domain and range.

Problem. *If $\mathcal{D}_1$ and $\mathcal{D}_2$ are bounded domains in finite dimensional Banach spaces*, $x_0 \in \mathcal{D}_1$, $y_0 \in \mathcal{D}_2$, *then*:

(i) *find* $\max\{|\det(f'(x_0))| : f : \mathcal{D}_1 \to \mathcal{D}_2$ *holomorphic and* $f(x_0) = y_0\}$;

(ii) *characterize all f which achieve the extremal value* (*we call f extremal in this case*).

In this case the Banach spaces must have the same finite dimension. The determinant is defined by taking a unit vector basis in each space. Carathéodory considered the problem of characterizing $f(\mathcal{D}_1)$ for extremal f, Alexander and Lempert looked at the problem for $\mathcal{D}_1 = B_{l_1}$, $\mathcal{D}_1 = B_{l_\infty}$ and $\mathcal{D}_2 = B_{l_2}$ and Kubota and Travaglini studied the case where $\mathcal{D}_1$ is a finite dimensional bounded symmetric domain and $\mathcal{D}_2 = B_{l_2}$. For detailed references we refer to Dineen and Timoney [2].

In [2] the problem is studied for $\mathcal{D}_1 = B_X$, $\mathcal{D}_2 = B_{l_2}$ and $x_0 = y_0 = 0$. Interchanging the domain and range leads to a dual problem. In this situation the classical Schwarz Lemma shows that $f : \mathcal{D}_1 \to \mathcal{D}_2$ implies $f'(0)(\mathcal{D}_1) \subset \mathcal{D}_2$ and so problem (i) reduces to a linear problem while problem (ii) is replaced by the more specialized problem of finding conditions under which the maximum is always achieved by a linear mapping. Let H denote a finite dimensional Hilbert space. The image of $\overline{B}_H$ by a linear mapping is called an ellipsoid (a different meaning is given to the term ellipsoid in complex analysis [6]). Let X denote a Banach space with $\dim(X) = \dim(H)$ and let $T : B_X \to B_H$ be linear. The problem (i) mentioned above is to find

$$\max\{|\det(T)|; T : B_X \to B_H\}.$$

Because of the classical formula relating volumes and determinants we see that the linear mapping T which maximizes the determinant is also the linear mapping which realizes, as $T^{-1}(B_H)$, the minimum volume ellipsoid containing B_X. A deep result of F. John (1948) says that this linear operator is unique up to composition by unitary operators of H and moreover the inner product on X given by $\langle x, y\rangle = \langle Tx, Ty\rangle_H$ is invariant under isometries of X.

Motivated by the bounded symmetric domain case we were led to consider Banach spaces with unique inner products, up to positive multiples, invariant under isometries. Examples of such spaces are l^p. Having considered such spaces it is natural to consider spaces of the form

$$X_1 \oplus X_2 \oplus \ldots \oplus X_n$$

where each X_i has the unique invariant inner product property (defined in an obvious way). For such spaces any inner product invariant under isometries can be shown to have the form

$$\langle\ ,\ \rangle = \sum_{i=1}^{n} c_i \langle\ ,\ \rangle_i$$

where $c_i > 0$ and $\langle\ ,\ \rangle_i$ is an inner product on X_i invariant under isometries of X_i.

Using Haar measure on the group of all isometries it is easily shown that each finite dimensional Banach space admits an inner product which is invariant under isometries.

At this stage we require a concept which is widely used in Banach space theory—the Banach-Mazur distance $d(\,\cdot\,,\,\cdot\,)$. For isomorphic Banach spaces X and Y this is defined in the following fashion

$$\begin{aligned} d(X,Y) &= \inf\{\|T\|\|T^{-1}\|; T : X \to Y \text{ is a linear isomorphism}\} \\ &= \inf\left\{\rho \geq 1 : \frac{1}{\rho} B_Y \subset T(B_X) \subset B_Y\right\}. \end{aligned}$$

The second formula for $d(X,Y)$ clearly indicates its relationship with the Schwarz Lemma.

Let $e(X) = d(X, l_2)$. Now, if X has the unique invariant inner product property then

$$e(X)^2 = \frac{\sup_{\|x\|=1}\langle x, x\rangle}{\inf_{\|x\|=1}\langle x, x\rangle}$$

where $\langle\ ,\ \rangle$ is any invariant inner product and if $X = X_1 \oplus X_2 \oplus \ldots \oplus X_n$, where each X_i has the unique invariant inner product property, then

$$e(X)^2 = \inf_{c_i>0}\left[\frac{\sup_{\|\sum x\|=1}\sum c_i\langle x_i, x_i\rangle_i}{\inf_{\|\sum x\|=1}\sum c_i\langle x_i, x_i\rangle_i}\right]$$

where each $\langle\ ,\ \rangle_i$ is an invariant inner product on X_i. Using Lagrange multipliers this gives a fairly useful computational tool. The fact that the minimum volume ellipsoid and the Banach-Mazur distance are both calculated using invariant inner products leads to the following proposition.

Proposition 5. *If* $X = (X_1 \oplus X_2 \oplus \ldots \oplus X_n)_{l_p}$, $2 \le p \le \infty$, *each* X_i *has the unique invariant inner product property*, $1/r + 2/p = 1$, $d_j = \dim(X_j)$ *and* $d = \sum_{j=1}^n d_j$ *then*

$$\max\{|\det(f'(0))|; f : B_X \to B_H \textit{ holomorphic},\ f(0) = 0\}$$
$$= d^{-d/2r}\prod_{j=1}^{n}\left(\frac{d_j^{\frac{1}{r}}}{e(X_j)^2}\right)^{d_j/2}.$$

We now turn to problem (ii). The solution here depends on the set of points of contact, $\mathcal{C}(X)$, between the minimum volume ellipsoid containing B_X and ∂B_X, and on a further Schwarz Lemma due to Harris.

Proposition 6. *Every* $f : B_X \to \overline{B}_Y$, X *and* Y *Banach spaces*, *with* $f(0) = 0$ *and* $f'(0) = T$ *is linear if and only if* T *is a complex extreme point of* $H^\infty(B_X, Y)$.

We have let $H^\infty(B_X, Y)$ denote the Banach space of all bounded Y-valued holomorphic functions on B_X.

Proposition 7. *If* $\mathcal{C}(X)$ *is not contained in the zero set of a non-zero homogeneous polynomial then every extremal* f *in* $H(B_X, B_{l_2})$ *satisfying* $f(0) = 0$ *is linear and*

$$|\det(f'(0))| = \max\{|\det(g'(0))|; g : B_X \to B_{l_2} \textit{ holomorphic and } g(0) = 0\}.$$

The conditions in the above proposition are satisfied by l_p, $2 \le p \le \infty$. If $1 \le p < 2$ and $0 < \epsilon < 2^{4/p}(1 - 2^{1-2/p})/3$, then

$$f(z_1, z_2) = (z_1 + \epsilon z_1 z_2^2, z_2^2)$$

is a non-linear holomorphic function which achieves the maximum.

For $1 \le p < \infty$, $p \ne 2$, all biholomorphic automorphisms of B_{l_p} have the origin as a fixed point and consequently are the restrictions to the unit ball of linear isometries of l_p. Hence B_{l_p} is not a bounded symmetric domain for $p \ne 2$ and we do not have the use of Mobius transformations. We have compensated for this lack of global symmetry by confining ourselves to mappings which take the origin to the origin and to spaces which admit sufficiently many symmetries through the origin.

Finally, we consider complex geodesics. We let ρ denote the Poincaré metric or distance on Δ—it will be clear, from the context, to which we are referring.

We let $K_{\mathcal{D}}(\,\cdot\,,\,\cdot\,)$ and $C_{\mathcal{D}}(\,\cdot\,,\,\cdot\,)$ denote the Kobayashi and Carathéodory distances respectively on the domain $\mathcal{D}$ in the Banach space X and let $k_{\mathcal{D}}$ and $c_{\mathcal{D}}$ denote the corresponding infinitesimal metrics. For detailed references we refer to [3].

PROPOSITION 8. *If $\phi : \Delta \to \mathcal{D}$ is a holomorphic mapping then the following are equivalent*:

(i) *ϕ is a complex geodesic*, *i.e. for all p, $q \in \Delta$ we have*

$$\rho(p,q) = C_{\mathcal{D}}(\phi(p),\phi(q)),$$

(ii) *there exists $p \neq q$ in Δ such that*

$$\rho(p,q) = C_{\mathcal{D}}(\phi(p),\phi(q)),$$

(iii) *there exists $p \in \Delta$ such that*

$$\rho(p,1) = c_{\mathcal{D}}(\phi(p),\phi'(p)),$$

(iv) *ϕ is biholomorphic onto its image and $\phi(\Delta)$ is a holomorphic retract of $\mathcal{D}$, i.e. there exists $r \in H(\mathcal{D},\mathcal{D})$ such that $r^2 = r \circ r = r$ and $\phi(\Delta) = r(\mathcal{D})$*,

(v) $K_{\mathcal{D}|\phi(\Delta)} = C_{\mathcal{D}\big|\phi(\Delta)}$ *and ϕ is biholomorphic onto its image*,

(vi) *ϕ is biholomorphic onto its image and all $g \in H(\phi(\Delta),\Delta)$ can be extended to $\tilde{g} \in H(\mathcal{D},\Delta)$*,

(vii) *there exists $\psi \in H(\mathcal{D},\Delta)$ such that $\psi \circ \phi = 1_{\Delta}$.*

Condition (ii) is useful as it tells us that it is only necessary to check that the distance between two distinct points is preserved. Condition (iii) is an infinitesimal version of (ii) and restricts attention to a single point. By (vi) we see that the ranges of complex geodesics are precisely the 1-dimensional submanifolds for which we have a holomorphic Hahn-Banach extension theorem and condition (iv) tells us that the ranges of complex geodesics are also the ranges of holomorphic projections. The final condition tells us that complex geodesics give precisely the ways in which discs can be passed through the domain.

If the domain $\mathcal{D}$ is a convex bounded domain then any pair of points in $\mathcal{D}$ lie in the range of a complex geodesic and in this situation condition (v) implies that $K_{\mathcal{D}} = C_{\mathcal{D}}$.

The main problems connected with complex geodesics are existence, uniqueness, continuity to the boundary, estimates near the boundary and specific formulae for prescribed domains.

I shall confine myself here to the problem of continuity to the boundary, and to the case $\mathcal{D} = B_X$. Afterwards I consider complex geodesics on B_{l_p}. A complex geodesic $\phi : \Delta \to B_X$ is said to be continuous (or continuous to the boundary) if there exists a continuous function $\widetilde{\phi} : \overline{\Delta} \to \overline{B}_X$ such that $\widetilde{\phi}|_{\Delta} = \phi$.

Gentili has shown that if all complex geodesics are continuous, then all unit vectors in X are complex extreme points. A partial converse is known ([3], [7]) which involves a qualitative concept of extreme point. If x is a unit vector in X

and $y \in X$, then for $0 < r < 1$ we may consider

$$\delta_r(x,y) = \sup\{\alpha; \alpha > 0, rx + \alpha\Delta y \subset \overline{B}_X\}.$$

Clearly $\lim_{r\to 1^-} \delta_r(x,y) = 0$ for all y if and only if x is a complex extreme point. If the rate at which this tends to zero is uniform with respect to unit vectors x and y then we say that X is a complex uniformly convex space. In a more precise fashion let us put

$$\delta_X(\epsilon) = \sup\{r > 0; \exists z, v \in X, \|z\| \geq 1-\epsilon, \|v\| = 1, \text{ such that } z + rv\Delta \subset B_X\}.$$

If $\delta_X(\epsilon) \to 0$ as $\epsilon \to 0$ then we say that X is complex uniformly convex. If X is a finite dimensional space then using compactness one sees that X is complex uniformly convex if and only if every unit vector is a complex extreme point.

Example 9. If $2 \leq p < \infty$ then $\delta_{l_p}(\epsilon) \leq A\epsilon^{1/p}$ for some positive constant A and if $1 \leq p < 2$ then there exists $B > 0$ such that

$$\delta_{l_p}(\epsilon) \leq B\epsilon^{1/2}$$

If X is the predual of a $\mathcal{C}^*$ algebra, for instance if $X = L^1$, then there exists $C > 0$ such that

$$\delta_X(\epsilon) \leq C\epsilon^{1/2}.$$

Proposition 10. *If there exists $A > 0$ and $s > 0$ such that $\delta_X(\epsilon) \leq A\epsilon^s$ then every complex geodesic on B_X is continuous.*

We now consider complex geodesics on B_{l_p}, $1 \leq p < \infty$. Initial contributions to this analysis were made by Poletskii and Gentili and the final solution was found using different methods and independently in [3] and [6]. An important rôle in the solution in [3] is played by a sufficient criterion on a holomorphic function in order that it be a complex geodesic. The main ideas for this are due to Lempert. We first need the concept of supporting hyperplane. If X is a Banach space and x is a unit vector in X then the Hahn-Banach theorem assures us of the existence of at least one $N_x \in X'$ satisfying $N_x(x) = \|N_x\| = 1$. We use the notation N_x for any choice of such a functional. The set $\{y : N_x(y) = 1\}$ is called a supporting hyperplane at the point x.

Proposition 11. *Let X be a complex Banach space and let $\phi : \overline{\Delta} \to \overline{B}_X$ denote a continuous map satisfying*

(i) *$\phi|_\Delta$ is holomorphic and $\phi(\Delta) \subset B_X$,*
(ii) *$\phi(\partial\Delta) \subset \partial B_X$,*
(iii) *there exists a choice of $N_{\phi(e^{i\theta})}$ for almost all $e^{i\theta} \in \partial\Delta$, a measurable function $p : \partial\Delta \to R^+$ and $h \in H^\infty(\Delta, X')$ such that*

$$[e^{i\theta} p(e^{i\theta}) N_{\phi(e^{i\theta})}](x) = \lim_{r\to 1^-} h(re^{i\theta})(x)$$

for all $x \in X$ and almost all $e^{i\theta} \in \partial\Delta$.

Then ϕ is a complex geodesic.

Proposition 12. *Let B_p denote the open unit ball of l_p, $1 \le p \le \infty$. Any two distinct points of B_p can be joined by a continuous complex geodesic which is unique up to reparametrization of Δ. Moreover, a non-zero mapping $\phi = (\phi_j)_{j=1}^{\infty}$, is a complex geodesic in B_p if and only if*

$$\phi_j(\xi) = c_j\left(\frac{\xi - \alpha_j}{1 - \overline{\alpha}_j \xi}\right)^{\beta_j}\left(\frac{1 - \overline{\alpha}_j \xi}{1 - \overline{\gamma}\xi}\right)^{2/p}, \qquad \xi \in \Delta,$$

where $\gamma \in \Delta$, $\alpha_j \in \overline{\Delta}$, $c_j \in \mathbb{C}$, $\beta_j = 0$ or 1

$$\sum_{j=1}^{\infty} |c_j|^p(1 + |\alpha_j|^2) = 1 + |\gamma|^2 \tag{1}$$

$$\sum_{j=1}^{\infty} |c_j|^p \alpha_j = \gamma. \tag{2}$$

We now give some new results obtained by applying proposition 12.

For $z \in B_p$ and $v \in l_p$ let

$$\phi(0) = z \tag{3}$$

and

$$\phi'(0) = \frac{v}{k_{B_p}(z,v)}. \tag{4}$$

A formula for $k_{B_p}(z,v)$ could be found by solving equations (1), (2), (3) and (4) in terms of z and v. These equations, however, do not appear to yield simple solutions except in the case $\beta_j = 0$ for all j i.e. only in the case where the component functions of the complex geodesic have no zeros. We shall write $k(z,v)$ in place of $k_{B_p}(z,v)$.

For $p \ge 2$ let $N_z = (|z_i|^{p-2} z_i)_{i=1}^{\infty}$. We have $N_z \in (l_p)'$ and if $\|z\| = 1$ then N_z is the unique supporting hyperplane at z. We define a 'quasi inner product' on $l_p \times l_p$ by the formula

$$(N_z, v) = \sum_{i=1}^{\infty} |z_i|^{p-2} z_i \overline{v_i}$$

where $z = (z_i)_{i=1}^{\infty}$ and $v = (v_i)_{i=1}^{\infty}$ belong to l_p.

If $p = 2$ then $N_z = z$ and this reduces to the usual inner product on the Hilbert space l_2. For all p and all z we have $(N_z, z) = \|z\|^p$.

We now assume $\beta_j = 0$ for all j, $p \ge 2$, $\phi_j(0) = z_j \ne 0$, and $\phi_j'(0) = v_j / k(z,v)$ where $(z_j)_j \in B_p$ and $(v_j)_j \in l_p$. By (3) we have $\phi_j(0) = c_j = z_j$ for all j. By (4)

$$\phi_j'(0) = \frac{v_j}{k(z,v)} = \frac{2}{p} z_j(\overline{\gamma} - \overline{\alpha}_j). \tag{5}$$

Hence

$$\frac{\sum_{j=1}^{\infty} |z_j|^{p-2} z_j \overline{v_j}}{k(z,v)} = \frac{2}{p}\gamma \sum_{j=1}^{\infty} |z_j|^{p-2} z_j \overline{z_j} - \frac{2}{p}\sum_{j=1}^{\infty} |z_j|^{p-2} z_j \overline{z_j}\alpha_j$$

and by (2)

$$\frac{(N_z, v)}{k(z,v)} = \frac{2}{p}\gamma\|z\|^p - \frac{2}{p}\gamma = -\frac{2}{p}\gamma(1-\|z\|^p).$$

We thus have

$$\gamma = -\frac{p}{2}\frac{(N_z,v)}{k(z,v)} \cdot \frac{1}{1-\|z\|^p} \tag{6}$$

and by (5) we have

$$\alpha_j = \gamma - \frac{p}{2}\frac{\overline{v}_j}{\overline{z_j}k(z,v)}. \tag{7}$$

Substituting (6) and (7) into (1) yields

$$\sum_{j=1}^{\infty}|z_j|^p + \sum_{j=1}^{\infty}|z_j|^p\frac{(-\frac{p}{2})^2}{k^2(z,v)}\left|\frac{(N_z,v)}{1-\|z\|^p} + \frac{\overline{v_j}}{\overline{z_j}}\right|^2 = 1 + \left(\frac{p}{2}\right)^2\frac{|(N_z,v)|^2}{k^2(z,v)}\cdot\frac{1}{(1-\|z\|^p)^2}.$$

Hence

$$\begin{aligned}\|z\|^p + \frac{\left(\frac{p}{2}\right)^2}{k^2(z,v)}&\left[\frac{\|z\|^p|(N_z,v)|^2}{(1-\|z\|^p)^2} + \sum_{j=1}^{\infty}|z_j|^{p-2}|v_j|^2 + 2\operatorname{Re}\sum_{j=1}^{\infty}\frac{|z_j|^p(N_z,v)}{1-\|z\|^p}\frac{v_j}{z_j}\right]\\
&= \|z\|^p + \frac{\left(\frac{p}{2}\right)^2}{k^2(z,v)}\left[\frac{\|z\|^p|(N_z,v)|^2}{(1-\|z\|^p)^2} + \sum_{j=1}^{\infty}|z_j|^{p-2}|v_j|^2 + \frac{|(N_z,v)|^2}{1-\|z\|^p}\right]\\
&= 1 + \left(-\frac{p}{2}\right)^2\frac{|(N_z,v)|^2}{k^2(z,v)}\cdot\frac{1}{(1-\|z\|^p)^2}.\end{aligned}$$

Collecting terms we get

$$\frac{\left(\frac{p}{2}\right)^2}{k^2(z,v)}\left[|(N_z,v)|^2\left(\frac{\|z\|^p}{(1-\|z\|^p)^2} + \frac{2}{1-\|z\|^p} - \frac{1}{(1-\|z\|^p)^2}\right) + \sum_{j=1}^{\infty}|z_j|^{p-2}|v_j|^2\right] = 1-\|z\|^p.$$

Hence

$$k^2(z,v) = \left(\frac{p}{2}\right)^2\left(\frac{\sum_{j=1}^{\infty}|z_j|^{p-2}|v_j|^2}{1-\|z\|^p} + \frac{|(N_z,v)|^2}{(1-\|z\|^p)^2}\right).$$

We now consider another special case. Let $A = \{j \in N; z_j = 0\}$ and suppose $\beta_j = 0$ for all $j \notin A$. If A is the empty set this reduces to the previous case. Let

$$k_1^p(z,v) = \left(\frac{p}{2}\right)^2\left(\frac{\sum_{j=1}^{\infty}|z_j|^{p-2}|v_j|^2}{1-\|z\|^p} + \frac{|(N_z,v)|^2}{(1-\|z\|^p)^2}\right)$$

If A is the empty set then $k_1^p(z,v) = k^2(z,v)$.

If $(\phi_j)_{j=1}^{\infty}$ is a complex geodesic with

$$\phi(0) = z, \quad \phi'(0) = \frac{v}{k(z,v)} \quad \text{and} \quad \phi_j(\xi) = c_j\left(\frac{\xi-\alpha_j}{1-\overline{\alpha}_j\xi}\right)^{\beta_j}\left(\frac{1-\overline{\alpha}_j\xi}{1-\overline{\gamma}\xi}\right)^{2/p}$$

then we may suppose $\beta_j = 1$ and $\alpha_j = 0$ for all $j \in A$. Otherwise, $c_j = 0$ and this is covered by the previous case. Hence, for $j \in A$, we have

$$\phi_j(\xi) = \frac{c_j \xi}{(1-\overline{\gamma}\xi)^{2/p}} \quad \text{and} \quad \phi_j'(0) = c_j = \frac{v_j}{k(z,v)}.$$

If $j \notin A$ then $\phi_j(0) = c_j = z_j$ and

$$\phi_j'(0) = \frac{v_j}{k(z,v)} = \frac{2}{p} z_j(\overline{\gamma} - \overline{\alpha}_j).$$

Hence equations (6) and (7), for $j \notin A$, are still valid. Precisely the same method of analysis leads to the equation

$$k^p(z,v) - k^{p-2}(z,v)k_1^p(z,v) - \frac{\sum_{j\in A}|v_j|^p}{1-\|z\|^p} = 0 \tag{8}$$

Proposition 13, below, is a summary of the results we have just proved together with some other results which can be derived from equation (8). The formulae we have obtained for $k(\,\cdot\,,\,\cdot\,)$ satisfy equations (1), (2), (3) and (4) but only apply when $|\alpha_j| \leq 1$ and the associated mapping ϕ is not constant. In our case, various necessary conditions in terms of z and v are easily derived from (6) and (7). This limits the range of applicability of the formulae but it is still possible to obtain non-trivial sets of pairs $\{(z,v)\}$ where we have obtained the precise form of the infinitesimal Kobayashi metric. In using proposition 13 the formula in (a),…,(d) is first used to obtain a value for $k(z,v)$ and substitution into (9) confirms if the value obtained is valid.

In the following proposition we let

$$k_2^p(z,v) = \sum_{j, z_j = 0} \frac{|v_j|^p}{1-\|z\|^p}.$$

PROPOSITION 13. *Let $A = \{j; z_j = 0\}$. Suppose that*

$$\sup_{j\notin A}\left|\frac{(N_z,v)}{1-\|z\|^p} + \frac{z_j\overline{v}_j}{|z_j|^2}\right|^2 \leq \frac{k^2(z,v)}{\left(\frac{p}{2}\right)^2}. \tag{9}$$

(a) *If $p \geq 2$ and A is the empty set, then*

$$k(z,v) = \frac{p}{2}\left[\frac{\sum_{j=1}^{\infty}|z_j|^{p-2}|v_j|^2}{1-\|z\|^p} + \frac{|(N_z,v)|^2}{1-\|z\|^p}\right]^{1/2}.$$

(b) *If $p = 3$ and A is arbitrary, then*

$$k(z,v) = \sum_{\epsilon=\pm 1}\left(\frac{k_2^3(z,v)}{2} + \epsilon\frac{(81(k_2^3(z,v))^2 - 12(k_1^3(z,v))^{1/2}}{18}\right)^{1/3}.$$

(c) *If* $p=4$ *and* A *is arbitrary*, *then*

$$\begin{aligned}k(z,v) &= \left[\frac{k_1^4(z,v)+((k_1^4(z,v))^2+4k_2^4(z,v))^{1/2}}{2}\right]^{1/2}\\ &= \left\{2\left\{\frac{\sum_{j=1}^{\infty}|z_j|^2|v_j|^2}{1-\|z\|^4}+\frac{|(N_z,v)|^2}{(1-\|z\|^4)^2}\right\}\right.\\ &\quad\left.+\left\{4\left[\frac{\sum_{j=1}^{\infty}|z_j|^2|v_j|^2}{1-\|z\|^4}+\frac{|(N_z,v)|^2}{(1-\|z\|^4)^2}\right]^2+\frac{\sum_{j\in A}|v_j|^4}{1-\|z\|^4}\right\}^{1/2}\right\}^{1/2}\end{aligned}$$

(d) *Explicit formula* (*rather complicated*) *can also be found for arbitrary* A *under the same conditions when* $p=6$ *and* $p=8$.

(e) *If* $A=N$ *then* $z=0$ *and*, *for all* p, *the above reduce to*

$$(k(0,v))^p=\sum_{j=1}^{\infty}|v_j|^p.$$

The next step is to consider the situation in which $\beta_j=1$ and $\alpha_j\neq 0$ for some j. This leads to quite complicated, but interesting, equations of the form

$$Ar^p+Br^{p-1}+Cr^{p-2}+D=0$$

where A, B, C and D are parameters involving the α_j for which $\beta_j=1$. So far no elegant solution, in fact no solution of any kind, is emerging and it appears doubtful if there exists one single explicit formula which prescribes $k(z,v)$ for all $z\in B_p$ and $v\in l_p$.

We urge the reader to consult the references in the articles and book [4] we have given in our very short bibliography and take the opportunity to point out (apologetically) that the bar signs—denoting mainly complex conjugates and closures—disappeared in the final electronic transmission of [3] to the editors and did not appear in print.

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