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A KONTINUITÄTSSATZ

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In this paper we study the envelopes of holomorphy of certain domains in $\mathbb{C}^N$ or in Stein manifolds.

Recall that, according to Rossi [6], if Ω is a domain in a Stein manifold $\mathcal{M}$, then Ω has an envelope of holomorphy: There is a Stein manifold $\widetilde{\Omega}$ with a locally biholomorphic projection $\pi : \widetilde{\Omega} \to \mathcal{M}$ for which there is a holomorphic injection $\iota : \Omega \to \widetilde{\Omega}$ such that $\pi \circ \iota$ is the identity on Ω and with the property that for each $f \in \mathcal{O}(\Omega)$ there is a unique $\widetilde{f} \in \mathcal{O}(\widetilde{\Omega})$ with $f = \widetilde{f} \circ \iota$. By Rossi's construction, $\widetilde{\Omega}$ can be identified with the space of all nonzero multiplicative linear funtionals $\psi : \mathcal{O}(\Omega) \to \mathbb{C}$ endowed with the weak$*$ topology.

The motivation for our work comes from the study of removable singularities for CR-functions. An account of the present state of this theory is given in [3]. We establish below a general theorem of *Kontinuitätssatz*-type and then apply it to deduce some information about removable sets in bD, D a strictly pseudoconvex domain.

Recall that a *holomorphic p-chain* on a complex manifold $\mathcal{M}$ is a current, c, of bidimension (p, p) that acts on forms by

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$$\langle c, \alpha \rangle = \sum n_j \int_{V_j} \alpha, \qquad \alpha \in \mathcal{D}^{p,p}(\mathcal{M}),$$

where $\{V_j\}$ is a locally finite family of purely p-dimensional complex analytic subsets of $\mathcal{M}$ and the n_j are integers. See, *e.g.*, [2, 5]. A holomorphic p-chain is *positive* if it can be represented as above with positive coefficients n_j. In this case, the *support* of c is the set $|c| = \bigcup V_j$.

THEOREM. *Let D and U be open sets in a Stein manifold $\mathcal{M}$ of dimension $N, U \subset D$. Denote by $(\widetilde{U}, \pi)$ the envelope of holomorphy of U. Assume that for the point $a \in D$ there is a family $\{c_t\}_{t\in(0,1]}$ of positive holomorphic p-chains, $p \geq 1$, in a neighborhood D' of $D \backslash U$ in D with the following properties*:

1) $\bigcup_{t\in(0,1]} |c_t| \backslash U$ *is a compact subset of D.*

2) $a \in |c_1|$.

3) $t \mapsto c_t$ *is continuous in the sense that for every $\alpha \in \mathcal{D}^{p,p}(D')$ the function $t \mapsto \langle c_t, \alpha \rangle$ is continuous.*

4) $\lim_{t\to 0^+} c_t = 0$ *in D' in the sense that for every $\alpha \in \mathcal{D}^{p,p}(D')$,*

$$\lim_{t\to 0^+} \langle c_t, \alpha \rangle = 0.$$

Then $a \in \pi(\widetilde{U})$.

If, in addition, we suppose that π is bijective from $\pi^{-1}(U)$ onto U, and if for every $a \in D$ a corresponding family $\{c_t\}_{t\in(0,1]}$ exists with $p \geq \frac{N}{2}$ (and with D' possibly depending on the family), then D is a holomorphic extension of U in the sense that each $f \in \mathcal{O}(U)$ extends holomorphically into D.

Remarks. 1. The condition 3) may be rephrased by saying that the map $t \to c_t$ from $(0,1]$ into $(\mathcal{D}^{p,p})' = \mathcal{D}'^{N-p,N-p}$ (on D') is continuous when the space $(\mathcal{D}^{p,p})'$ is endowed with the weak$*$ topology. A family of holomorphic p-chains that satisfies the condition 3) will be called *a continuous family* of *holomorphic p-chains*.

2. The arguments used to prove the theorem will be seen to depend on the order structure of the interval $(0,1]$. However, the theorem as stated admits an immediate extension to certain other parameter spaces. Suppose that Y is a locally compact space which is the image of $(0,1]$ under the *proper* map φ. (The map φ is to be proper in the usual sense that for each compact set $E \subset Y$, the set $\varphi^{-1}(E)$ is compact.) Suppose that $\{c_y\}_{y\in Y}$ is a family of positive holomorphic p-chains, $p \geq 1$, in D' with the properties that $\bigcup_{y\in Y} |c_y| \backslash U$ is compact, that for all $\alpha \in \mathcal{D}^{p,p}(D')$ the map $y \mapsto \langle c_y, \alpha \rangle$ is continuous and that for all $\alpha \in \mathcal{D}^{p,p}(D'), \langle c_y, \alpha \rangle \to 0$ as $y \to \infty$ in Y. If we define d_t by $d_t = c_{\varphi(t)}$, $t \in (0,1]$, then the family $\{d_t\}_{t\in(0,1]}$ has the properties 1), 3) and 4).

This is an obvious remark, but, *e.g.*, it lets us replace the one-dimensional parameter space $(0,1]$ by any Euclidean space of dimension at least two or by any separable finite dimensional manifold Σ that has a single end in the sense that there are arbitrarily large compact sets E in Σ such that $\Sigma\backslash E$ is connected. Among these manifolds are found the Stein manifolds of dimension at least two.

The proof of the theorem depends on a lemma that records some more or less well-known geometric consequences of the continuity hypotheses.

LEMMA. a) *If* $X \subset D'$ *is a compact set*, *then there is* $\tau = \tau(X) \in (0,1]$ *such that* $|c_t| \subset D'\backslash X$ *for* $t \in (0,\tau]$.

b) *For* $t \in (0,1], |c_t| \subset \overline{\bigcup_{s<t} |c_s|}$.

c) *If* $a \in D'$ *is the limit* $a = \lim_v a_v$ *for* $a_v \in |c_{t_v}|$ *with* $t_v \to t \in (0,1]$, *then* $a \in |c_t|$.

Proof. If a) is false, there is a sequence $\{t_v\}_{v=1}^{\infty}$ in $(0,1]$ with $\lim_v t_v = 0$ such that for a suitable sequence $\{q_v\}_{v=1}^{\infty}$ with $q_v \in |c_{t_v}| \cap X$, we have $\lim_v q_v = q_0 \in X$. Let $B \subset D'$ be a holomorphic chart containing q_0 with coordinates z, let $\omega = dd^c|z|^2$ be the fundamental form on B induced from the Euclidean metric determined by the coordinates z, and let $B_0 \Subset B$ be a ball with center q_0, radius r. Denote by χ a nonnegative smooth function on $\mathcal{M}$ with $\operatorname{supp}\chi \Subset B$, $\chi = 1$ on B_0. By Wirtinger's inequality, if

$$c_{t_\nu} = \sum_j n(\nu,j)[V_{\nu,j}]$$

with positive integers $n(\nu,j)$, then

$$\langle c_{t_\nu}, \chi\omega\rangle \geq \sum_j n(\nu,j) \int\limits_{V_{\nu,j}\cap B_0} \omega \geq cr^{2p}$$

for a constant $c > 0$ independent of ν. (See [1], [2], or [7].) This contradicts the assumption 4), so a) is proved.

For b), suppose that $p_0 \in |c_t|\backslash\overline{\bigcup_{s<t} |c_s|}$. There is then a small open set B as above that contains p_0 and that is disjoint from $\overline{\bigcup_{s<t} |c_s|}$. Let χ be a smooth nonnegative function on $\mathcal{M}$ with $\operatorname{supp}\chi \Subset B$, $\chi(p_0) > 0$. Then $\langle c_t, \chi\omega\rangle > 0$, but for all $s < t$, $\langle c_s, \chi\omega\rangle = 0$, a contradiction to 3).

Part c) of the lemma is proved in a similar way.

Proof of the Theorem. We have the envelope of holomorphy $(\widetilde{U}, \pi)$ of the open set U, and there is an inclusion $\iota : U \to \widetilde{U}$, which is a biholomorphic map from U onto the open set $\iota(U)$ in $\widetilde{U}$ and which satisfies $\pi \circ \iota =$ Identity on U.

Let a, D' and $\{c_t\}_{t\in(0,1]}$ be as in the statement of the theorem. We are going to lift the continuous family $\{c_t\}_{t\in(0,1]}$ of holomorphic p-chains to a family $\{\widetilde{c}_t\}_{t\in(0,1]}$ of holomorphic p-chains in $\pi^{-1}(D')$ in such a way that for each $t, \pi_*\widetilde{c}_t = c_t$.

For this purpose, define a number τ by

$\tau =$ sup$\{t'$: for each $t < t'$, there is a holomorphic p-chain $\widetilde{c}_t$ in $\pi^{-1}(D')$ with $\pi_*\widetilde{c}_t = c_t$ such that $|\widetilde{c}_t| \cap \iota(U) = \iota(|c_t| \cap U)$ and such that $|\widetilde{c}_t|\backslash\iota(U)$ is compact$\}$.

By the lemma and condition 1), $\tau > 0$.

Let $b \in |c_\tau|\backslash U$. The lemma provides points $b_\nu \in |c_{t_\nu}|\backslash U$ for $t_\nu < \tau$ with $b = \lim_\nu b_\nu$. By the definition of τ, for each ν there is a point $\widetilde{b}_\nu \in |\widetilde{c}_{t_\nu}|$ that satisfies $\pi(\widetilde{b}_\nu) = b_\nu$.

Let $V \subset D'$ be a relatively compact open set that contains the compact set $\bigcup_{t\in(0,1]} |c_t|\backslash U$.

The maximum principle implies that for $f \in \mathcal{O}(U)$,

$$|\widetilde{f}(\widetilde{b}_v)| \leq \max\{|\widetilde{f}(p)| : p \in |c_{t_v}| \cap \iota(bV)\}$$

for every v. The set $X = bV \cap \bigcup_{t\in(0,1]} |c_t|$ is a compact subset of U, so $\iota(X) \subset \iota U \subset \widetilde{U}$ is compact. All of the points $\widetilde{b}_v$ are contained in the $\mathcal{O}(\widetilde{U})$-hull of ιX, which is a compact set that we will denote by $\widehat{X}$. By passing to a subsequence if necessary, we can suppose that $\{\widetilde{b}_v\}_{v=1,2,\dots}$ converges to a point $\widetilde{b} \in \widehat{X}$. Then $b = \pi(\widetilde{b})$. We have shown that $|c_\tau| \subset \pi(\widetilde{U})$.

We also know that the analytic sets $|\widetilde{c}_{t_v}| \cap \pi^{-1}(V)$ are contained in $\widehat{X}$. The map π is locally biholomorphic and $\widehat{X}$ is compact, so there is also a $k \in (0, \infty)$ such that $\pi^{-1}(z) \cap \widehat{X}$ contains no more than k points when $z \in \pi(\widehat{X})$. Thus if we fix a Kähler metric on the Stein manifold $\mathcal{M}$ and lift it to $\widetilde{U}$ by way of the map π, it follows that the analytic sets $|\widetilde{c}_{t_v}| \cap \pi^{-1}(V)$ have uniformly bounded area, *i.e.*, $2p$-dimensional volume, with respect to the lifted metric.

By Bishop's theorem [1], [2], [7] there is a purely p-dimensional analytic set A_τ° in $\pi^{-1}(V) \cap \widehat{X}$ such that for a holomorphic p-chain c_τ° in V with $|c_\tau^\circ| = \pi(A_\tau^\circ)$, we have $\widetilde{c}_{t_v}|\pi^{-1}(V) \to c_\tau^\circ$ in the sense of currents. As $c_{t_v}|U \cap D' \to c_t|U \cap D'$, it follows that there is a holomorphic p-chain $\widetilde{c}_\tau$ on $\widetilde{U}$ that satisfies $\pi(\widetilde{c}_\tau) = c_\tau$ on $D'\backslash U$ and $\widetilde{c}_\tau = c_\tau^\circ$ on $\iota(V)$. By construction, $\pi(|\widetilde{c}_\tau|) = |c_\tau|$, $\widetilde{c}_\tau|\iota(U \cap D') = \iota_*(c_\tau|U \cap D')$, and $|\widetilde{c}_\tau|\backslash\iota(U)$ is compact.

Thus the supremum in the definition of τ is attained.

Since the map π is a local biholomorphism, we see that given t for which there is a $\widetilde{c}_t$ in $\pi^{-1}(D')$ with the properties used to define τ, there is also a $\widetilde{c}_{t}$, corresponding to c_t, provided $|t - t'|$ is sufficiently small.

The connectedness of $(0, 1]$ implies the existence of a $\widetilde{c}_1$ in $\pi^{-1}(D')$ that projects onto c_1. In particular, $a \in \pi(\widetilde{U})$ as we wished to show.

This completes the proof of the first part of the theorem.

We show next that for a given $p \geq 1$, the set of points in $\widetilde{U}$ that can be reached by a family $\{\widetilde{c}_t\}_{t\in(0,1]}$ of the kind we have been considering is open.

Denote by $\mathcal{C}$ the collection of all continuous families $\{c_t\}_{t\in(0,1]}$ in D' that satisfy 1), 3) and 4) for all choices of D'.

We may suppose that the ambient Stein manifold $\mathcal{M}$ is a closed submanifold of $\mathbb{C}^L$ for some L by the embedding theorem for Stein manifolds. Let ρ be a holomorphic retraction of a neighborhood W of $\mathcal{M}$ into $\mathbb{C}^L$ as provided by the theorem of Docquier and Grauert. See, *e.g.*, [4].

Fix a point $a \in D$ and a point $a' \in \widetilde{U}$ that projects onto a under π and that is in $\bigcup_{t\in(0,1]} |\widetilde{c}_t|$ for some $\{c\}_{t\in(0,1]} \in \mathcal{C}$ that is associated to a neighborhood D' of $D \backslash U$. Shrinking D' suitably lets us suppose $\bigcup_{t\in(0,1]} |c_t|$ to be relatively compact in $\mathcal{M}$.

By continuity there is a $\delta_0 > 0$ so small that if $w \in \mathbb{C}^L$ satsifies $|w| < \delta_0$, then for each $t \in (0,1]$ the translate $c_t + w$ of c_t by w is contained in W, and thus ρ is defined on $c_t + w$.

Let $D'' \subset D$ be a neighborhood of $D\backslash U$ such that $\overline{D}'' \backslash bD \subset D'$. Having fixed D'' we define a continuous family $\{d_t\}_{t\in(0,1]}$ of holomorphic p-chains as follows.

For $w \in \mathbb{C}^L$ *small*, say $|w| < \delta_0$, we have that for every $t \in (0,1]$, the boundary $b|c_t|$ of $|c_t|$ satisfies $b|c_t| + w \subset D \backslash D'$. Set

$$\alpha_w = \inf\{s : \text{ if } s' \in (s,1] \text{ then } \rho(c_s + w) \cap D''| \neq \emptyset\}.$$

For a monotonic continuous map $\psi : (\alpha_w, 1] \to (0,1]$ with $\psi(1) = 1$ we put

$$d_t = c_{\psi(t)} | D''.$$

This is a famliy of positive holomorphic p-chains in D'' that enjoys properties 1), 3) and 4) with respect to D''.

The projection π is a local homeomorphism, so if $|w|$ is small enough, then the lift $\{\widetilde{d}_t\}_{t\in(0,1]}$ of $\{d_t\}_{t\in(0,1]}$ contains a point that projects onto $\rho(a + w)$.

We have proved that the set of points in $\widetilde{U}$ that can be reached by one of our continuous families of holomorphic p-chains is open.

To continue with the proof of the second part of the theorem we show that since $p \geq \frac{N}{2}$, π is one-to-one on each $|\widetilde{c}_t|$. For small t, $\widetilde{c}_t = \iota c_t$, so as $\pi \circ \iota =$ Identity on U, it follows that π is one-to-one on $|\widetilde{c}_t|$. Let

$$t_0 = \sup\{t : \pi \text{ is one-to-one on } |\widetilde{c}_s| \text{ for all } s \leq t\}.$$

Then π is one-to-one on $|\widetilde{c}_{t_0}|$. If not, then there are distinct points $q', q'' \in |\widetilde{c}_{t_0}|$ with $\pi(q') = \pi(q'')$. As $\dim |\widetilde{c}_t| = p \geq \frac{N}{2}$ for all t and as $|\widetilde{c}_t|$ is for all t with $t < t_0$ and $|t - t_0|$ small, a small perturbation of $|\widetilde{c}_{t_0}|$, it follows from a Rouché-like result — see [2] – that there are distinct points $q'_t, q''_t \in |\widetilde{c}_t|$ near q' and q'' respectively with $\pi(q'_t) = \pi(q''_t)$. This contradicts the choice of t_0. Thus, π is one-to-one on $|\widetilde{c}_{t_0}|$. The set $|\widetilde{c}_{t_0}|$ is closed in $\pi^{-1}(D')$, and $\pi : \widetilde{U} \to \mathcal{M}$ is a local homeomorphism, so it follows that π is one-to-one on a neighborhood of $\pi^{-1}(|\widetilde{c}_{t_0}|)$ in D'. As π is one-to-one on $\iota(U)$, we see that π is one-to-one on $|c_t|$ for $t > t_0$ if $t - t_0$ is small enough. Connectedness implies that π is one-to-one on each $|\widetilde{c}_t|, t \in (0,1]$.

We now show that given $\{c_t\}_{t\in(0,1]}$ satisfying conditions 1), 3) and 4) in D', the projection π is one-to-one on $\bigcup_{t\in(0,1]} |\widetilde{c}_t|$. To this end, suppose that π is one-to-one on $\bigcup_{t\in(0,t_0)} |\widetilde{c}_t|$ but not on $\bigcup_{t\in(0,t_0]} |\widetilde{c}_t|$. Then as π is one-to-one on $|\widetilde{c}_{t_0}|$, there are points $q_0 \in |\widetilde{c}_{t_0}|$ and $q_1 \in |\widetilde{c}_{t_1}|$ for a $t_1 < t_0$ with $\pi(q_0) = \pi(q_1)$. But then as $|\widetilde{c}_t|$ is, for $|t - t_0|$ small, a small deformation of $|\widetilde{c}_{t_0}|$, it follows that there is a point $q_t \in |\widetilde{c}_t|$ with $\pi(q_t) \in \pi(|\widetilde{c}_{t_1}|)$ if $t < t_0$ is sufficiently near t_0. This contradicts the assumption that π is one-to-one on $\bigcup_{t<t_0} |\widetilde{c}_t|$. Thus, π is one-to-one on $\bigcup_{t\leq t_0} |\widetilde{c}_t|$.

As π is a local homeomorphism and $\bigcup_{t\in(0,1]} |\widetilde{c}_t|\backslash\pi^{-1}(U)$ is compact, it follows that π is one-to-one on $\bigcup_{t<t_0+\delta} |\widetilde{c}_t|$ for sufficiently small $\delta > 0$. Connectedness of $(0,1]$ implies that π is one-to-one on $\bigcup_{t\in(0,1]} |\widetilde{c}_t|$ as claimed.

Finally, we show that π is one-to-one on the set $\bigcup_{c\in\mathcal{C}} \bigcup_{t\in(0,1]} |\widetilde{c}_t|$. Call this set Ω.

Suppose that π is not one-to-one on Ω. Then there are distinct c's in $\mathcal{C}$, say c' and c'' such that there exist points $p' \in |\widetilde{c}'_{t'}|$ and $p'' \in |\widetilde{c}''_{t''}|$ with $\pi(p') = \pi(p'') = p_0$. By hypothesis, π is injective over U, so $p_0 \in D\backslash U$. With no loss of generality we can suppose that $t' = t'' = 1$. (This assumption has the sole function of simplifying notation.)

Let c' be defined in the neighborhood D' of $D\backslash U$ and c'' in the neighborhood D''. Put $V = D' \cap D''$, which is again a neighborhood of $D\backslash U$. Let

$$\beta' = \inf\{s : \text{ if } \tau \in (s,1] \text{ then } |c'_\tau| \cap V \neq \emptyset\},$$

and define β'' similarly in terms of c''. Let $\psi' : (\beta',1] \to (0,1]$ be a homeomorphism with $\psi'(1) = 1$, and let $\psi'' : (\beta'',1] \to (0,1]$ be a homeomorphism with $\psi''(1) = 1$. Define d', $d'' \in \mathcal{C}$ by $d'_t = c'_{\psi'(t)}|V$ and $d''_t = c''_{\psi''(t)}|V$. Finally define d by $d_k = d'_t + d''_t$. Then $d \in \mathcal{C}$, but π is not one-to-one on $|\widetilde{d}_1|$, for this set contains both p' and p''. This contradicts what we have done above, and we have proved that π is injective on Ω. (It is for this last argument that we have chosen to work with chains rather than merely with continuous families of varieties.)

We have shown that π is injective from the open set Ω onto D. As Ω is an open set in the envelope of the holomorphy of U, it follows that each $f \in \mathcal{O}(U)$ extends in a natural way to all of D, *i.e.*, D is a holomorphic extension of U as claimed.

As a corollary we have the following result.

COROLLARY 1. *Let $\mathcal{M}$ be a Stein manifold and $U \subset \mathcal{M}$ an open subset. Assume the closed set $\mathcal{M}\backslash U$ to be convex with respect to complex hypersurfaces. If for every $a \in \mathcal{M}$ there is a family $\{c_t\}_{t\in(0,1]}$ of positive holomorphic p-chains with $p \geq \frac{N}{2}$ that satisfies the conditions* 1)–4), *then $\mathcal{M}$ is the envelope of holomorphy of U.*

In the case that $p = N-1$, the convexity condition we impose is that if $z \in U$, there exists a complex hypersurface Σ_z in $\mathcal{M}$ with $z \in \Sigma_z$, $\Sigma_z \cap (\mathcal{M}\backslash U) = \emptyset$. In the absence of the condition that $H^2(\mathcal{M},\mathbb{Z}) = 0$, not every complex hypersurface in $\mathcal{M}$ is the zero locus of a function holomorphic on $\mathcal{M}$. Thus, our convexity condition, which is geometric in character, is not obviously equivalent to the analytic condition that the closed set $\mathcal{M}\backslash U$ be "rationally convex" with respect to the algebra $\mathcal{O}(\mathcal{M})$ in the usual sense that given $z \in U$, there exists $f \in \mathcal{O}(\mathcal{M})$ with $f(z) = 0$ and without zeros on $\mathcal{M}\backslash U$.

The corollary is an immediate consequence of the theorem once we recognize that if $(\widetilde{U},\pi)$ denotes the envelope of holomorphy of U, then π is injective on $\pi^{-1}(U)$.

To see this, notice first that if W is a domain in a Stein manifold, $\mathcal{N}$, W with envelope of holomorphy $(\widetilde{W},\eta)$, then every complex hypersurface A in $\widetilde{W}$ meets W. (By abuse of notation we are here identifying W with its image $\iota W\subset\widetilde{W}$.) Suppose not so that the complex hypersurface $A\subset\widetilde{W}$ is disjoint from W. Then $(\widetilde{W}\backslash A,\eta)$ is a Riemann domain spread over the Stein manifold $\mathcal{N}$. As A is a complex hypersurface, $\widetilde{W}\backslash A$ is a Stein manifold with the property that for each $f\in\mathcal{O}(W)$ there is $F\in\mathcal{O}(\widetilde{W}\backslash A)$ such that $F|W=f$. This means that $\widetilde{W}\backslash A$ is the envelope of holomorphy of W. As $\widetilde{W}\backslash A$ is a proper subset of $\widetilde{W}$, this is impossible.

To continue, suppose that the projection $\pi:\widetilde{U}\to\mathcal{M}$ is not injective over U so that for some $p\in U$, there is a point $p'\in\pi^{-1}(p)$ other than ιp. By hypothesis there is an irreducible complex hypersurface $\Sigma_p\subset\mathcal{M}$ disjoint from $\mathcal{M}\backslash U$ that passes through the point p. Denote by Σ' a global branch of the complex hypersurface $\pi^{-1}(\Sigma_p)$ that passes through the point p'. By the last paragraph, Σ' meets ιU. As the map π is a local biholomorphism, this implies that $\iota\Sigma_p$ and Σ' share an open set and so coincide. As $\iota\Sigma_p$ is an analytic subvariety of $\mathcal{M}$ that is disjoint from $\mathcal{M}\backslash U$, it follows that $p'\in U$, which contradicts the choice of p'. Thus, as desired, π is injective over U.

A particular case of this, the case with which this work began, is the following.

Corollary 2. *Let D be a relatively compact strictly pseudoconvex domain with boundary of class $\mathcal{C}^2$ in an N-dimensional Stein manifold $\mathcal{M}$. Let $K\subset bD$ be a compact set. Assume that for every point $a\in D$ there is a continuous family $\{c_t\}_{t\in(0,1]}$ of positive holomorphic p-chains, $p\geq\frac{N}{2}$, in D with $a\in|c_1|$, with $\overline{\bigcup_{t\in(0,1]}|c_t|}\cap K=\emptyset$ and $\lim_{t\to0^+}c_t=0$. Then if f is a continuous CR-function on $bD\backslash K$ there is $F\in\mathcal{O}(D)$ that assumes continuously the boundary values f along $bD\backslash K$.*

In the usual terminology (see [3]) the set K is *removable*.

Proof. Denote by $\widehat{K}$ the hull of K with respect to the algebra $\mathcal{O}(\overline{D})$. It is known [3], [8] that each continuous CR-function on Γ extends holomorphically into $D\backslash\widehat{K}$. As the set $\widehat{K}\cap D$ is convex with respect to complex hypersurfaces in the Stein manifold D, Corollary 1 implies our result.

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