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CHARACTERIZATION OF SMOOTH, COMPACT ALGEBRAIC CURVES IN $\mathbb{R}^2$

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0. Introduction. The classical Bernstein inequality for derivatives of trigonometric polynomials can be stated as follows: Let $p(x,y)$ be a polynomial of two real variables so that $q(\theta) \equiv p(\cos(\theta), \sin(\theta))$ is a trigonometric polynomial of degree equal to $\deg(p)$. Then

$$|q'(\theta)| \le (\deg q)\|q\|_{[0,2\pi]}, \qquad \theta \in [0, 2\pi],$$

which is equivalent to

$$|D_T p(x,y)| \le (\deg p)\|p\|_S, \qquad (x,y) \in S$$

where $S = \{(x,y) : x^2 + y^2 = 1\}$, $\|f\|_E$ is the supremum norm of a function f on a set E, and D_T denotes the unit tangential derivative along S. We note that by general Banach space theory, for *any* smooth compact curve K in the plane one gets an estimate of the form

$$\|D_T p(x,y)\|_K \le C\|p\|_K$$

where C depends in some unspecified way on $\deg(p)$ and K. The main purpose of this paper is to prove the following result giving a characterization of *algebraic*

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curves among all smooth (C^1) compact curves in terms of whether certain classical analytical results in approximation theory are valid.

Main Theorem. *Let K be a smooth compact connected curve in $\mathbb{R}^2$ and let $C(K)$ denote the continuous functions on K. The following are equivalent*:

1) *K is algebraic.*

2) *K satisfies a tangential Markov inequality with exponent one*, *i.e.*, *there exists $M = M(K) > 0$ such that*

$$(M_T) \qquad \|D_T p\|_K \le M(\deg p)\|p\|_K$$

for all polynomials p where D_T denotes the unit tangential derivative (*along K*).

3) *For some $0 < \alpha < 1$, K satisfies a Bernstein theorem*: *there exists $B = B(K) > 0$ such that for $f \in C(K)$*,

$$(B) \qquad \text{if } E_n(f) \le n^{-\alpha}, \text{ then } f \in \mathrm{Lip}(\alpha) \text{ and } \|f\|_\alpha \le B$$

where

$$E_n(f) \equiv \inf\{\|f - p_n\|_K : p_n \in P_n\}$$

and P_n = polynomials of degree at most n in two variables.

4) *For all $0 < \alpha < 1$, K satisfies a Bernstein theorem.*

Here $\|f\|_\alpha$ denotes the $\mathrm{Lip}(\alpha)$ norm of f (defined in Section 2). In the next three sections we will prove the main theorem. We fix a smooth compact curve K in $\mathbb{R}^2$ which we may take to be irreducible.

1. Proof that 1) implies 2), i.e., K algebraic implies (M_T) with exponent one. There is a beautiful characterization of *complex* algebraic subvarieties of $\mathbb{C}^N$ among the (complex-) analytic ones, due to Sadullaev [S]. We briefly describe his result. Let A be a complex analytic subvariety of $\mathbb{C}^N$ such that the regular points of A, A_{reg}, from a complex manifold of pure dimension $m < N$. Let K be a compact subset of A and form the extremal function

$$u_K(z) \equiv \sup\left\{\frac{1}{\deg(p)} \log \frac{|p(z)|}{\|p\|_K} : p \text{ polynomial}, \ \deg(p) > 0\right\}.$$

Then $u_K^*(z) \equiv \limsup_{\zeta \to z} u_K(\zeta) \equiv +\infty$; but clearly $u_K(z) \le 0$ for z in K and $u_K(z)$ may be finite at other points z as well. We say that K is *pluripolar in A* if K is pluripolar as a subset of the complex manifold A_{reg}.

Theorem 1.1 [S]. *A is algebraic if and only if $u_K \in L^\infty_{\mathrm{loc}}(A)$ for some* (*and hence for each*) *non-pluripolar compact set K in A.*

For example, if $q(z, w)$ is a polynomial in two complex variables, then

$$A \equiv \{(z, w) : q(z, w) = 0\}$$

is an algebraic curve in $\mathbb{C}^2$. If we let

$$K = A \cap \mathbb{R}^2 = \{(z, w) \in A : \mathfrak{J}z = \mathfrak{J}w = 0\},$$

then locally the curve K looks like a piece of an interval in $\mathbb{R}^2$ and hence is not (pluri-) polar in A provided K is non-empty and non-singular. Thus u_K is locally bounded on A and Lip(1) near K. This will be the basis for the proof of our characterization of algebraicity.

We now proceed with the proof. Let $K = \{(x, y) \in \mathbb{R}^2 : k(x, y) = 0\}$ for some irreducible polynomial k with $\nabla k = (k_x, k_y) \neq (0, 0)$ on K. Fix (x_0, y_0) in K. Let A in $\mathbb{C}^2$ be the complexification of K, i.e.,

$$K = A \cap \mathbb{R}^2 = \{(z, w) \in A : \Im z = \Im w = 0\}.$$

Without loss of generality, we can use a linear change of coordinates to arrange that $(x_0, y_0) = (0, 0)$ and $\nabla k(0, 0) = (0, 1)$. Note then that the tangential derivative of a function at this point of K is just differentiation with respect to $x = \Re z$. Let $p = p(x, y) = \sum_{a+b\leq n} c_{ab} x^a y^b$ be a polynomial of degree n in the real variables x, y. We use the same notation $p = p(z, w) = \sum_{a+b\leq n} c_{ab} z^a w^b$ for the polynomial of degree n in the complex variables z, w.

Let $(u, v) = F(z, w) = (z, k(z, w))$. This is a non-singular algebraic change of coordinates valid between a ball B_{r_0} of radius r_0 about $(0, 0)$ in the (z, w) coordinates and a ball $B_{\tilde{r}_0}$ of radius $\widetilde{r}_0$ about $(0, 0)$ in the (u, v) coordinates. By the smoothness and compactness of K, there is a uniform r_0 (and $\widetilde{r}_0$) valid for all points (x_0, y_0) in K. A simple computation shows that

$$D_T p(0, 0) = \frac{\partial \widetilde{p}}{\partial u}(0, 0)$$

where $\widetilde{p}$ is p in the (u, v) coordinates.

By applying Cauchy's integral formula to $\partial\widetilde{p}/\partial u$ on the circle

$$C_{\tilde{r}} \equiv \{(u, 0) : |u| = \widetilde{r}\}, \qquad \widetilde{r} < \widetilde{r}_0,$$

we obtain

$$|D_T p(0, 0)| = \left| \frac{1}{2\pi i} \int_{C_{\tilde{r}}} \frac{\widetilde{p}(u, 0)}{u^2}, du \right| \leq \frac{\|\widetilde{p}\|_{C_{\tilde{r}}}}{\widetilde{r}} = \frac{\|p\|_{\gamma_r}}{\widetilde{r}}$$

where γ_r is the pre-image of $C_{\tilde{r}}$ under our coordinate change. Hence, by the definition of the extremal function u_K, we have

$$|D_T p(0, 0)| \leq \frac{1}{\widetilde{r}} \|p\|_K \exp[n \|u_K\|_{\gamma_r}].$$

It follows from Sadullaev's work that

$$\|u_K\|_{\gamma_r} \leq C \log(1 + \widetilde{r})$$

for some $C = C(F(K))$. Here we are using Corollary 3.3 and Proposition 3.4 of [S] which say that for a non-polar (real) algebraic curve E in a one (complex) dimensional variety V, the extremal function u_E is harmonic in $V - E$ and is the (one-variable) Green function for $V - E$. Furthermore, if V is smooth near E, then u_E is Lip(1) on a neighborhood of E in V.

We conclude that

$$|D_T p(0,0)| \le \frac{1}{\widetilde{r}}\|p\|_K \exp[nC\log(1+\widetilde{r})].$$

Taking $\widetilde{r} = \widetilde{r}_0/n$ in the above inequality we obtain

$$|D_T p(0,0)| \le \frac{n}{\widetilde{r}_0}\left(1+\frac{\widetilde{r}_0}{n}\right)^{nC}\|p\|_K \le \frac{n}{\widetilde{r}_0}e^{\tilde{r}_0 C}\|p\|_K.$$

2. Proof that 2) implies 4), i.e., (M_T) with exponent one implies (B) for each $0<\alpha<1$. Suppose we have a tangential Markov inequality

$$(M_T) \qquad \|D_T p\|_K \le M(\deg(p))\|p\|_K.$$

The proof of property (B) then follows very closely the proof of the classical Bernstein theorem using Bernstein's inequality on trigonometric polynomials (cf. [L], pp. 59–60).

For points $a,b\in K$, we denote by $\varrho(a,b)$ the geodesic distance along K between a and b. In the rest of this section, we assume for simplicity that our functions $f\in C(K)$ satisfy $\|f\|_K\le 1$.

LEMMA 2.1. *There exists a constant C depending only on K such that for any $f\in C(K)$ we have*

$$|f(a)-f(b)| \le C\varrho(a,b)\sum_{n\le 1/\varrho(a,b)} E_n(f), \qquad a,b\in K,$$

where $E_n(f) = \inf\{\|f-p_n\|_K : p_n\in P_n\}$.

Proof. Without loss of generality, we may assume $\varrho(a,b)<1$. First of all, from the mean-value theorem,

$$(1) \qquad |p(a)-p(b)| \le \varrho(a,b)\|D_T p\|_K$$

for any polynomial p (indeed, any C^1 function p). Now

$$|f(a)-f(b)| = |f(a)-p(a)+p(a)-p(b)+p(b)-f(b)|$$

so that, setting $p=p_n$ where $p_n\in P_n$ and $E_n(f)=\|f-p_n\|_K$, we get

$$(2) \qquad |f(a)-f(b)| \le |p_n(a)-p_n(b)| + 2E_n(f) \le \varrho(a,b)\|D_T p_n\|_K + 2E_n(f)$$

by (1).

For any $a\in K$ we have the identity

$$D_T p_{2^k}(a) = D_T p_1(a) - D_T p_0(a) + \sum_{i=1}^{k}[D_T p_{2^i}(a) - D_T p_{2^{i-1}}(a)].$$

By (M_T), the triangle inequality, and the fact that $E_{2^i}\le E_{2^{i-1}}$, we get

$$|D_T p_{2^i}(a) - D_T p_{2^{i-1}}(a)| \le M2^i\|p_{2^i}-p_{2^{i-1}}\|_K \le M2^i 2E_{2^{i-1}}(f).$$

Thus

$$\|D_T p_{2^k}\|_K \le 2ME_0(f) + M2^{1+1}\sum_{i=1}^{k} 2^{i-1}E_{2^{i-1}}(f).$$

Note that

$$\sum_{i=1}^{k} 2^{i-1}E_{2^{i-1}} \le 2\sum_{i=1}^{2^k-1} E_i \tag{3}$$

since E_k decreases with k so that

$$2E_2 \le 2E_1, \quad 4E_4 \le 2E_2 + 2E_3, \ \ldots, \ 2^{j-1}E_{2^{j-1}} \le 2E_{2^{j-2}} + \ldots + 2E_{2^{j-1}-1}.$$

We thus obtain

$$\|D_T p_{2^k}\|_K \le 8M \sum_{0\le n\le 2^k-1} E_n(f) \le 8M \sum_{0\le n\le 2^k} E_n(f).$$

Then, since $E_m(f) \le E_{m-1}(f)$,

$$\sum_{1\le n\le 2^k} E_n(f) \ge E_{2^k}(f) \sum_{1\le n\le 2^k} 1 = 2^k E_{2^k}(f)$$

so that using (2) with $n = 2^k$ we obtain

$$|f(a) - f(b)| \le \varrho(a,b)\|D_T p_{2^k}\|_K + 2E_{2^k}(f) \le C(\varrho(a,b) + 2^{-k}) \sum_{0\le n\le 2^k} E_n(f)$$

for some constant C. Now choose $k \in \{0,1,\ldots\}$ with $2^k \le \varrho(a,b)^{-1} < 2^{k+1}$. Then since $2\varrho(a,b) > 2^{-k}$ we get our result. Note that (M_T) with exponent *one* is *essential*; if the exponent of $\deg(p)$ were greater than 1, the above argument would fail.

LEMMA 2.2. *If* $\sum_{n=1}^{\infty} n^{-1}E_n(f) < \infty$, *then there exists* $C > 0$ *with*

$$E_n(f) \le C \sum_{j\ge[n/2]} j^{-1}E_j(f), \qquad n = 2,3,\ldots$$

Proof. We first note the following fact (cf. [L], p. 58):

$$\sum_{j=1}^{\infty} E_{2^j n} \le \sum_{j=n}^{\infty} \frac{1}{j}E_j. \tag{4}$$

To see this, simply note that in the sum on the right, the first n terms from E_n/n to $E_{2n-1}/(2n-1)$ are each at least $E_{2n-1}/n \ge E_{2n}/n$ and hence add to at least E_{2n}; the next $2n$ terms are each at least $E_{4n}/(2n)$ and hence add to at least E_{4n}, etc., yielding the result. Using (4), we thus obtain

$$E_n(f) \le \sum_{i=1}^{\infty} E_{2^{i-1}n}(f) \le C \sum_{j\ge[n/2]} j^{-1}E_j(f).$$

Note the following corollary.

COROLLARY 2.3. *If* $E_n(f) \le n^{-\alpha}$, $0 < \alpha < 1$, *then*

$$E_n(f) \le C \sum_{j \ge [n/2]} j^{-1-\alpha}.$$

Recall that for $I = [-1, 1]$, we say $f \in \mathrm{Lip}_I(\alpha)$ if

$$\|f\|_{0,\alpha} \equiv \|f\|_I + \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

For $f \in C(K)$, we write $f \in \mathrm{Lip}(\alpha)$ if for each x in K there exists a coordinate chart $\phi : I \to K$ with $x \in \phi(-1, 1)$ and $f \circ \phi \in \mathrm{Lip}_I(\alpha)$. Then

$$\|f\|_\alpha \equiv \sum_i \|f \circ \phi_i\|_{0,\alpha}$$

where the sum is over a finite collection of charts with $K = \bigcup_i \phi_i(I)$. We want to conclude, under the hypothesis of Corollary 2.3, that we actually have $f \in \mathrm{Lip}(\alpha)$ and $\|f\|_\alpha \le B$. To prove this, we use both Lemmas 2.1 and 2.2. First of all, by Lemma 2.1, for $a, b \in K$,

$$|f(a) - f(b)| \le C\varrho(a, b) \sum_{n \le 1/\varrho(a,b)} E_n(f).$$

Now from Lemma 2.2 (Corollary 2.3) we can estimate each term $E_n(f)$:

$$E_n(f) \le C \sum_{j \ge [n/2]} j^{-1-\alpha} \le C'\alpha(n/2)^{-\alpha}, \qquad n = 1, 2, \ldots$$

(by the integral test). Thus

$$|f(a) - f(b)| \le C\varrho(a, b) \sum_{n \le 1/\varrho(a,b)} C'\alpha(n/2)^{-\alpha} \le C''[\varrho(a, b)]^\alpha$$

where $C'' = C''(K, \alpha)$ is a constant depending only on K and α. We note that by compactness and smoothness of K, there exists a constant c depending only on K such that

$$\varrho(a, b) \le c\|a - b\|, \qquad a, b \in K.$$

Thus $f \in \mathrm{Lip}(\alpha)$ as desired. Moreover, we get a uniform bound on the $\mathrm{Lip}(\alpha)$ norms for f as in the corollary. Hence we have proved (B) for $0 < \alpha < 1$.

3. Proof that 3) implies 1), i.e., (B) for *some* α implies K algebraic.

In order to prove that (B) implies K algebraic, we need some preliminaries. The first result we need is a generalization of Jackson's theorem on the decay of the approximation numbers $E_n(f)$ for $f \in \mathrm{Lip}(\alpha)$.

THEOREM 3.1. (Corollary 2.2 of [R]). *Let* $0 < \alpha \le 1$. *There exists* $C(\alpha) > 0$ *such that* $f \in \mathrm{Lip}(\alpha)$ *implies* $E_n(f) \le C(\alpha)\|f\|_\alpha n^{-\alpha}$.

Given a set A in a Banach space X, if X_n is an n-dimensional subspace of X, we call the number

$$E_{X_n}(A) \equiv \sup_{f\in A}\{\inf_{p\in X_n}\|f-p\|_X\} \equiv \sup_{f\in A} E_{X_n}(f)$$

the *degree of approximation* to A by X_n; this is the "worst" best approximation for elements in A by elements of X_n. Then the *n-width* of A in X is given by

$$d_n(A) \equiv \inf_{X_n} E_{X_n}(A)$$

where the infimum is taken over all n-dimensional subspaces of X. This is, in an obvious sense, the closest distance from A to all n-dimensional subspaces of X. To get upper bounds on the n-widths of sets A in X is easy; merely estimate $E_{X_n}(A)$ for an appropriate space X_n (e.g., polynomials of degree at most $n-1$ in one-variable settings). Thus, from the Jackson theorem, if we let

$$U = \{f \in C(K) : \|f\|_\alpha \le 1\}$$

be the unit ball in $\mathrm{Lip}(\alpha)$, then

$$d_{\delta(n)}(U) \le C(\alpha)n^{-\alpha} \tag{5}$$

where $\delta(n)$ is the dimension of the space $P_n|_K$ of polynomials in P_n restricted to K.

We call X_n *extremal* for A if $d_n(A)=E_{X_n}(A)$. For *full approximation sets* A, it is easy to find extremal subspaces. Such sets are constructed as follows. Take a sequence $p_1, p_2, \ldots$ of linearly independent elements in X and a decreasing sequence of positive numbers $a_1 \ge a_2 \ge \ldots$ with $a_m \to 0$. Let $X_m = \mathrm{span}\{p_1, \ldots, p_m\}$. Finally, let

$$A \equiv \{x \in X : E_{X_n}(x) \le a_n,\ n = 1, 2, \ldots\}.$$

The set A is called a *full approximation set*. We state without proof the following.

PROPOSITION 3.2 (Theorem 3, p.139 of [L]). *$d_n(A) = a_n$, $n = 1, 2, \ldots,$ and X_n is extremal for A.*

Sketch of proof. Clearly from the definitions of d_n and A, we have $d_n(A) \le E_{X_n}(A) \le a_n$; to prove the reverse inequality, one considers

$$A_n \equiv \{x \in X_{n+1} : \|x\|_X \le a_n\}$$

and shows that $d_n(A_n) = a_n$ (Theorem 2, p. 137 of [L]). Since $A_n \subset A$, we have $d_n(A_n) \le d_n(A)$, which yields the result.

We can now state the key result from [R].

THEOREM 3.3 [R]. *Suppose for some $0 < \alpha \le 1$ there exists B such that*

$$E_n(f) \le \frac{1}{n^\alpha} \quad \textit{implies} \quad \|f\|_\alpha \le B. \tag{6}$$

Then $1/n^\alpha = O(d_{\delta(n)}(U))$.

This says that if we have a Bernstein theorem for K, then $P_n|_K$ is (essentially) extremal, i.e., we automatically get an estimate from BELOW on the $\delta(n)$-widths of U, at least asymptotically. For the reader's convenience, we reproduce Ragozin's proof.

Proof. Let

$$A \equiv \{f \in C(K) : E_n(f) \le 1/n^\alpha,\ n = 1, 2, \ldots\}.$$

By Proposition 3.2, $d_{\delta(n)}(A) = 1/n^\alpha$. By (6), $A \subset BU \equiv \{f \in C(K) : \|f\|_\alpha \le B\}$. Hence

$$1/n^\alpha = d_{\delta(n)}(A) \le d_{\delta(n)}(BU) = B d_{\delta(n)}(U)$$

from obvious properties of n-widths. This completes the proof.

Recall by (5) we have

$$d_{\delta(n)}(U) \le C(\alpha) n^{-\alpha}$$

so that

$$d_{\delta(n)}(U) \asymp \frac{1}{n^\alpha}. \tag{7}$$

Next we relate n-widths of U to n-widths of things we can *compute*. By comparing pieces of K to intervals I and patching together — it is known that $d_n(U) \asymp 1/n^\alpha$ for $U = \{f \in C(I) : \|f\|_{0,\alpha} \le 1\}$ — we get the following result.

THEOREM 3.4 [R]. $d_n(U) \asymp 1/n^\alpha$.

Combining Theorem 3.4 with (7), we see that (B) implies $d_n(U) \asymp d_{\delta(n)}(U)$ so that $\delta(n) = O(n)$. This implies K is algebraic since, for large n, we have shown that the dimension of $P_n\,|_K$ is of order n, *not* n^2. Indeed, $\delta(n) = O(n)$ if and only if K is contained in an algebraic variety of dimension 1.

4. Remarks and examples. We mention that the main theorem remains true for K a smooth, compact m-dimensional submanifold of $\mathbb{R}^N$, $m = 1, \ldots, N-1$ (cf. [BLMT]). In the non-smooth case, one must replace (M_T) by a condition which "makes sense." For example, as in Section 1, suppose that A is a complex analytic subvariety of $\mathbb{C}^N$ of pure dimension $m < N$ in a neighborhood of $K \equiv A \cap \mathbb{R}^N$. Suppose for simplicity that K is compact but not necessarily smooth. Then for each *regular* point $(x_0, y_0) \in K$, there is a tangential Markov inequality (M_T) of the form

$$|D_T p(f(t))|_{t=0} \le M_f(\deg p)\|p\|_K \tag{M_T'}$$

with exponent 1 for all analytic disks $f : \{t \in \mathbb{C} : |t| < 1\} \to A$ with $f(0) = (x_0, y_0)$. This result and related problems will not be discussed here.

For a curve K with singularities, we can require that (M_T) holds for *all* tangential derivatives in 2). With this interpretation, we have the following result.

PROPOSITION 4.1. *Let $K \subset \mathbb{R}^2$ be a curve consisting of finitely many line segments and arcs of circles. Then K satisfies a tangential Markov inequality with exponent $r \le 2$.*

Proof. Clearly if L is a line segment forming a part of K, then by the univariate case, at any point (x,y) in L,

$$|D_T p(x,y)| \le M(\deg p)^2 \|p\|_L \le M(\deg p)^2 \|p\|_K$$

for any polynomial $p = p(x,y)$. Thus it suffices to show that if E is an arc of a circle forming a part of K, then for any point (x,y) in E and any polynomial $p = p(x,y)$,

$$|D_T p(x,y)| \le M(\deg p)^2 \|p\|_E.$$

Without loss of generality we let E be an arc on the unit circle. Let $p = p(x,y)$ be a polynomial of degree n. Then p restricts to a trigonometric polynomial on E. By setting $z = e^{i\theta}$, we may write $p(z) = z^{-n} P_{2n}(z)$ for some holomorphic polynomial P_{2n} of degree $2n$. A simple calculation reveals that at a point z in E,

$$\begin{aligned} |D_T p(z)| &= \left|\frac{d}{dz} z^{-n} P_{2n}(z)\right| = \left| z^{-n} \frac{d}{dz} P_{2n}(z) - n z^{-n-1} P_{2n}(z)\right| \\ &\le \left|\frac{d}{dz} P_{2n}(z)\right| + |n P_{2n}(z)| \le \frac{e}{2} \frac{1}{\operatorname{cap}(E)} (2n)^2 \|p_{2n}\|_E + n\|P_{2n}\|_E. \end{aligned}$$

Here $\operatorname{cap}(E)$ denotes the logarithmic capacity of E and we have used Theorem 1 of Pommerenke [P].

The example of the boundary of a square shows that the exponent $r = 2$ is, in general, best possible. We conclude this note by sketching an alternate proof of 2) implies 1) which illustrates the significance of the exponent 2.

PROPOSITION 4.2. *Let K be a smooth compact connected curve in $\mathbb{R}^2$ satisfying (M_T) with exponent strictly less than 2, i.e., there exists $M = M(K) > 0$ and $1 \le r < 2$ such that*

$$(M_T) \qquad \|D_T p\|_K \le M(\deg p)^r \|p\|_K$$

for all polynomials p. Then K is algebraic.

Proof. Let $\gamma : [0, L] \to \mathbb{R}^2$ be the arclength parameterization of K. Note by the mean-value theorem and the fact that γ is smooth, for any function f which is differentiable on a neighborhood of K in $\mathbb{R}^2$ and for each pair of points $\gamma(t_1)$, $\gamma(t_2)$ on K,

$$(8) \qquad |f(\gamma(t_2)) - f(\gamma(t_1))| \le c[\|\gamma(t_2) - \gamma(t_1)\|] \|D_T f\|_K$$

for some constant $c = c(K)$. Suppose K is not algebraic. Fix a positive integer n and let $N = N(n) = \binom{n+2}{2}$ = dimension of P_n. Choose $N/2$ points $\{a_j\} \in K$ with $\|a_j - a_{j-1}\| < 4L/N$ for successive points a_{j-1}, a_j. Here L = arclength of K. We can find a non-zero polynomial $q_n \in P_n$ which vanishes at each point a_i.

By (M_T) applied to q_n,

$$\|D_T q_n\|_K \le M n^r \|q_n\|_K.$$

Now choose $a \in K$ with $|q_n(a)| = \|q_n\|_K$. Let a_i be a nearest point to a among the $\{a_j\}$. Using (8) and (M_T) we obtain

$$\|q_n\|_K = |q_n(a) - q_n(a_i)| \le c\frac{4L}{N}\|D_T q_n\|_K \le c\frac{4L}{N} M n^r \|q_n\|_K.$$

But $N > n^2/2$ so we have

$$\|q_n\|_K \le (8LcM) n^{r-2} \|q_n\|_K. \tag{9}$$

Since K is not algebraic, for each n we can chose $q_n \in P_n$ satisfying (9). Since $r < 2$, letting $n \to +\infty$ we obtain a contradiction.

References

[BLMT] L. Bos, N. Levenberg, P. Milman and B. A. Taylor, *Tangential Markov inequalities characterize algebraic submanifolds of* $\mathbb{R}^n$, Indiana Univ. Math. J., to appear.

[L] G. Lorentz, *Approximation of Functions*, Holt, Rinehart, and Winston, 1966.

[P] C. Pommerenke, *On the derivative of a polynomial*, Michigan Math. J. 6 (1959), 373–375.

[R] D. Ragozin, *Polynomial approximation on compact manifolds and homogeneous spaces*, Trans. Amer. Math. Soc. 150 (1970), 41–53.

[S] A. Sadullaev, *An estimate for polynomials on analytic sets*, Math. USSR-Izv. 20 (1983), 493–502.