CHARACTERIZATION OF SMOOTH, COMPACT
ALGEBRAIC CURVES IN $\mathbb{R}^2$

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0. Introduction. The classical Bernstein inequality for derivatives of trigonometric polynomials can be stated as follows: Let $p(x, y)$ be a polynomial of two real variables so that $q(\theta) \equiv p(\cos(\theta), \sin(\theta))$ is a trigonometric polynomial of degree equal to $\deg(p)$. Then

$$|q'(\theta)| \leq (\deg q) \|q\|_{[0, 2\pi]}, \quad \theta \in [0, 2\pi],$$

which is equivalent to

$$|D_T p(x, y)| \leq (\deg p) \|p\|_S, \quad (x, y) \in S$$

where $S = \{(x, y) : x^2 + y^2 = 1\}$, $\|f\|_E$ is the supremum norm of a function $f$ on a set $E$, and $D_T$ denotes the unit tangential derivative along $S$. We note that by general Banach space theory, for any smooth compact curve $K$ in the plane one gets an estimate of the form

$$\|D_T p(x, y)\|_K \leq C\|p\|_K$$

where $C$ depends in some unspecified way on $\deg(p)$ and $K$. The main purpose of this paper is to prove the following result giving a characterization of algebraic...
curves among all smooth \((C^1)\) compact curves in terms of whether certain classical analytical results in approximation theory are valid.

**Main Theorem.** Let \(K\) be a smooth compact connected curve in \(\mathbb{R}^2\) and let \(C(K)\) denote the continuous functions on \(K\). The following are equivalent:

1) \(K\) is algebraic.
2) \(K\) satisfies a tangential Markov inequality with exponent one, i.e., there exists \(M = M(K) > 0\) such that
\[
\|D_T p\|_K \leq M(\deg p)\|p\|_K
\]
for all polynomials \(p\) where \(D_T\) denotes the unit tangential derivative (along \(K\)).
3) For some \(0 < \alpha < 1\), \(K\) satisfies a Bernstein theorem: there exists \(B = B(K) > 0\) such that for \(f \in C(K)\),
\[
E_n(f) \leq n^{-\alpha}, \quad \text{then } f \in \text{Lip}(\alpha) \quad \text{and} \quad \|f\|_{\alpha} \leq B
\]
where
\[
E_n(f) = \inf\{\|f - p_n\|_K : p_n \in P_n\}
\]
and \(P_n\) = polynomials of degree at most \(n\) in two variables.
4) For all \(0 < \alpha < 1\), \(K\) satisfies a Bernstein theorem.

Here \(\|f\|_{\alpha}\) denotes the Lip(\(\alpha\)) norm of \(f\) (defined in Section 2). In the next three sections we will prove the main theorem. We fix a smooth compact curve \(K\) in \(\mathbb{R}^2\) which we may take to be irreducible.

1. **Proof that 1) implies 2), i.e., \(K\) algebraic implies \((M_T)\) with exponent one.** There is a beautiful characterization of complex algebraic subvarieties of \(\mathbb{C}^N\) among the (complex-) analytic ones, due to Sadullaev [S]. We briefly describe his result. Let \(A\) be a complex analytic subvariety of \(\mathbb{C}^N\) such that the regular points of \(A\), \(A_{\text{reg}}\), from a complex manifold of pure dimension \(m < N\). Let \(K\) be a compact subset of \(A\) and form the extremal function
\[
u_K(z) = \sup \left\{ \frac{1}{\deg(p)} \log \frac{|p(z)|}{\|p\|_K} : p \text{ polynomial}, \deg(p) > 0 \right\}.
\]
Then \(u_K(z) = \lim \sup_{\zeta \to z} u_K(\zeta) = +\infty\); but clearly \(u_K(z) \leq 0\) for \(z\) in \(K\) and \(u_K(z)\) may be finite at other points \(z\) as well. We say that \(K\) is pluripolar in \(A\) if \(K\) is pluripolar as a subset of the complex manifold \(A_{\text{reg}}\).

**Theorem 1.1** [S]. \(A\) is algebraic if and only if \(u_K \in L^\infty_{\text{loc}}(A)\) for some (and hence for each) non-pluriharmonic compact set \(K\) in \(A\).

For example, if \(q(z, w)\) is a polynomial in two complex variables, then
\[
A = \{(z, w) : q(z, w) = 0\}
\]
is an algebraic curve in \(\mathbb{C}^2\). If we let
\[
K = A \cap \mathbb{R}^2 = \{(z, w) \in A : \Re z = \Re w = 0\},
\]
then locally the curve $K$ looks like a piece of an interval in $\mathbb{R}^2$ and hence is not (pluri-) polar in $A$ provided $K$ is non-empty and non-singular. Thus $u_K$ is locally bounded on $A$ and Lip(1) near $K$. This will be the basis for the proof of our characterization of algebraicity.

We now proceed with the proof. Let $K = \{ (x, y) \in \mathbb{R}^2 : k(x, y) = 0 \}$ for some irreducible polynomial $k$ with $\nabla k = (k_x, k_y) \neq (0, 0)$ on $K$. Fix $(x_0, y_0)$ in $K$. Let $A$ in $\mathbb{C}^2$ be the complexification of $K$, i.e.,

$$K = A \cap \mathbb{R}^2 = \{ (z, w) \in A : 3z = 3w = 0 \}.$$ 

Without loss of generality, we can use a linear change of coordinates to arrange that $(x_0, y_0) = (0, 0)$ and $\nabla k(0, 0) = (0, 1)$. Note then that the tangential derivative of a function at this point of $K$ is just differentiation with respect to $x = \Re z$.

Let $p = p(x, y) = \sum_{a+b\leq n} c_{ab} x^a y^b$ be a polynomial of degree $n$ in the real variables $x$, $y$. We use the same notation $p = p(z, w) = \sum_{a+b\leq n} c_{ab} z^a w^b$ for the polynomial of degree $n$ in the complex variables $z$, $w$.

Let $(u, v) = F(z, w) = (z, k(z, w))$. This is a non-singular algebraic change of coordinates valid between a ball $B_{r_0}$ of radius $r_0$ about $(0, 0)$ in the $(z, w)$ coordinates and a ball $B_{\tilde{r}_0}$ of radius $\tilde{r}_0$ about $(0, 0)$ in the $(u, v)$ coordinates. By the smoothness and compactness of $K$, there is a uniform $r_0$ (and $\tilde{r}_0$) valid for all points $(x_0, y_0)$ in $K$. A simple computation shows that

$$D_T p(0, 0) = \frac{\partial \tilde{p}}{\partial u}(0, 0)$$

where $\tilde{p}$ is $p$ in the $(u, v)$ coordinates.

By applying Cauchy’s integral formula to $\partial \tilde{p}/\partial u$ on the circle

$$C_\tilde{r} \equiv \{ (u, 0) : |u| = \tilde{r} \}, \quad \tilde{r} < \tilde{r}_0,$$

we obtain

$$|D_T p(0, 0)| = \left| \frac{1}{2\pi i} \int_{C_\tilde{r}} \frac{\tilde{p}(u, 0)}{u^2}, du \right| \leq \frac{\|\tilde{p}\|_{C_\tilde{r}}}{\tilde{r}} = \frac{\|p\|_{\gamma_\tilde{r}}}{\tilde{r}}$$

where $\gamma_\tilde{r}$ is the pre-image of $C_\tilde{r}$ under our coordinate change. Hence, by the definition of the extremal function $u_K$, we have

$$|D_T p(0, 0)| \leq \frac{1}{\tilde{r}} \|p\|_K \exp[n\|u_K\|_{\gamma_{\tilde{r}}}] .$$

It follows from Sadullaev’s work that

$$\|u_K\|_{\gamma_{\tilde{r}}} \leq C \log(1 + \tilde{r})$$

for some $C = C(F(K))$. Here we are using Corollary 3.3 and Proposition 3.4 of [S] which say that for a non-polar (real) algebraic curve $E$ in a one (complex) dimensional variety $V$, the extremal function $u_E$ is harmonic in $V - E$ and is the (one-variable) Green function for $V - E$. Furthermore, if $V$ is smooth near $E$, then $u_E$ is Lip(1) on a neighborhood of $E$ in $V$. 

We conclude that

\[ |D_T p(0,0)| \leq \frac{1}{\tilde{r}} \|p\|_K \exp[\nu C \log(1 + \tilde{r})]. \]

Taking \( \tilde{r} = \tilde{r}_0/n \) in the above inequality we obtain

\[ |D_T p(0,0)| \leq \frac{n}{\tilde{r}_0} \left( 1 + \frac{\tilde{r}_0}{n} \right)^{\nu C} \|p\|_K \leq \frac{n}{\tilde{r}_0} e^{\tilde{r}_0 C} \|p\|_K. \]

2. Proof that 2) implies 4), i.e., \((M_T)\) with exponent one implies \((B)\) for each \(0 < \alpha < 1\). Suppose we have a tangential Markov inequality

\[(M_T) \quad \|D_T p\|_K \leq M(\deg(p))\|p\|_K.\]

The proof of property \((B)\) then follows very closely the proof of the classical Bernstein theorem using Bernstein’s inequality on trigonometric polynomials (cf. [L], pp. 59–60).

For points \(a, b \in K\), we denote by \(\rho(a, b)\) the geodesic distance along \(K\) between \(a\) and \(b\). In the rest of this section, we assume for simplicity that our functions \(f \in C(K)\) satisfy \(\|f\|_K \leq 1\).

**Lemma 2.1.** There exists a constant \(C\) depending only on \(K\) such that for any \(f \in C(K)\) we have

\[ |f(a) - f(b)| \leq C\rho(a, b) \sum_{n \leq 1/\rho(a, b)} E_n(f), \quad a, b \in K, \]

where \(E_n(f) = \inf\{\|f - p_n\|_K : p_n \in P_n\}\).

**Proof.** Without loss of generality, we may assume \(\rho(a, b) < 1\). First of all, from the mean-value theorem,

\[ |p(a) - p(b)| \leq \rho(a, b) \|D_T p\|_K \]

for any polynomial \(p\) (indeed, any \(C^1\) function \(p\)). Now

\[ |f(a) - f(b)| = |f(a) - p(a) + p(a) - p(b) + p(b) - f(b)| \]

so that, setting \(p = p_n\) where \(p_n \in P_n\) and \(E_n(f) = \|f - p_n\|_K\), we get

\[ |f(a) - f(b)| \leq |p_n(a) - p_n(b)| + 2E_n(f) \leq \rho(a, b) \|D_T p_n\|_K + 2E_n(f) \]

by (1).

For any \(a \in K\) we have the identity

\[ D_T p_{2^k}(a) = D_T p_1(a) - D_T p_0(a) + \sum_{i=1}^{k} [D_T p_{2^i}(a) - D_T p_{2^{i-1}}(a)]. \]

By \((M_T)\), the triangle inequality, and the fact that \(E_{2^i} \leq E_{2^{i-1}}\), we get

\[ |D_T p_{2^i}(a) - D_T p_{2^{i-1}}(a)| \leq M2^{i}\|p_{2^i} - p_{2^{i-1}}\|_K \leq M2^{i}E_{2^{i-1}}(f). \]
Thus
\[ \| D_{TP_{2k}} \|_K \leq 2M E_0(f) + M 2^{1+1} \sum_{i=1}^{k} 2^{i-1} E_{2^{i-1}}(f). \]

Note that
\[ \sum_{i=1}^{k} 2^{i-1} E_{2^{i-1}} \leq 2 \sum_{i=1}^{2^{k-1}} E_i \]

since \( E_k \) decreases with \( k \) so that
\[ 2E_2 \leq 2E_1, \quad 4E_4 \leq 2E_2 + 2E_3, \ldots, \quad 2^{j-1}E_{2^{j-1}} \leq 2E_{2^j-2} + \ldots + 2E_{2^j-1}. \]

We thus obtain
\[ \| D_{TP_{2k}} \|_K \leq 8M \sum_{0 \leq n \leq 2^k} E_n(f) \leq 8M \sum_{0 \leq n \leq 2^k} E_n(f). \]

Then, since \( E_m(f) \leq E_{m-1}(f) \),
\[ \sum_{1 \leq n \leq 2^k} E_n(f) \geq E_{2^{k}}(f) \sum_{1 \leq n \leq 2^k} 1 = 2^k E_{2^k}(f) \]

so that using (2) with \( n = 2^k \) we obtain
\[ |f(a) - f(b)| \leq \| D_{TP_{2k}} \|_K + 2E_{2^k}(f) \leq C(g(a,b) + 2^{-k}) \sum_{0 \leq n \leq 2^k} E_n(f) \]

for some constant \( C \). Now choose \( k \in \{0, 1, \ldots\} \) with \( 2^k \leq g(a,b) < 2^{k+1} \). Then since \( 2g(a,b) > 2^{-k} \) we get our result. Note that \( M_T \) with exponent one is essential; if the exponent of \( \deg(p) \) were greater than 1, the above argument would fail.

**Lemma 2.2.** If \( \sum_{n=1}^{\infty} n^{-1} E_n(f) < \infty \), then there exists \( C > 0 \) with
\[ E_n(f) \leq C \sum_{j \geq \lfloor n/2 \rfloor} j^{-1} E_j(f), \quad n = 2, 3, \ldots \]

**Proof.** We first note the following fact (cf. [L], p. 58):
\[ \sum_{j=1}^{\infty} E_{2jn} \leq \sum_{j=n}^{\infty} 1 \cdot E_j. \]

To see this, simply note that in the sum on the right, the first \( n \) terms from \( E_n/n \) to \( E_{2n-1}/(2n-1) \) are each at least \( E_{2n-1}/n \geq E_{2n}/n \) and hence add to at least \( E_{2n} \); the next \( 2n \) terms are each at least \( E_{4n}/(2n) \) and hence add to at least \( E_{4n} \), etc., yielding the result. Using (4), we thus obtain
\[ E_n(f) \leq \sum_{i=1}^{\infty} E_{2^{-i}n}(f) \leq C \sum_{j \geq \lfloor n/2 \rfloor} j^{-1} E_j(f). \]

Note the following corollary.
Corollary 2.3. If \( E_n(f) \leq n^{-\alpha}, 0 < \alpha < 1 \), then
\[
E_n(f) \leq C \sum_{j \geq \lceil n/2 \rceil} j^{-1-\alpha}.
\]

Recall that for \( I = [-1,1] \), we say \( f \in \text{Lip}_I(\alpha) \) if
\[
\|f\|_{0,\alpha} \equiv \|f\|_I + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty.
\]

For \( f \in C(K) \), we write \( f \in \text{Lip}(\alpha) \) if for each \( x \) in \( K \) there exists a coordinate chart \( \phi: I \to K \) with \( x \in \phi(-1,1) \) and \( f \circ \phi \in \text{Lip}_I(\alpha) \). Then
\[
\|f\|_{\alpha} \equiv \sum_i \|f \circ \phi_i\|_{0,\alpha}
\]
where the sum is over a finite collection of charts with \( K = \bigcup \phi_i(I) \). We want to conclude, under the hypothesis of Corollary 2.3, that we actually have \( f \in \text{Lip}(\alpha) \) and \( \|f\|_{\alpha} \leq B \). To prove this, we use both Lemmas 2.1 and 2.2. First of all, by Lemma 2.1, for \( a, b \in K \),
\[
|f(a) - f(b)| \leq C\varrho(a,b) \sum_{n \leq 1/\varrho(a,b)} E_n(f).
\]

Now from Lemma 2.2 (Corollary 2.3) we can estimate each term \( E_n(f) \):
\[
E_n(f) \leq C \sum_{j \geq \lceil n/2 \rceil} j^{-1-\alpha} \leq C'\alpha(n/2)^{-\alpha}, \quad n = 1, 2, \ldots
\]
(by the integral test). Thus
\[
|f(a) - f(b)| \leq C\varrho(a,b) \sum_{n \leq 1/\varrho(a,b)} C'\alpha(n/2)^{-\alpha} \leq C''[\varrho(a,b)]^{\alpha}
\]
where \( C'' = C''(K,\alpha) \) is a constant depending only on \( K \) and \( \alpha \). We note that by compactness and smoothness of \( K \), there exists a constant \( c \) depending only on \( K \) such that
\[
\varrho(a,b) \leq c\|a - b\|, \quad a, b \in K.
\]
Thus \( f \in \text{Lip}(\alpha) \) as desired. Moreover, we get a uniform bound on the \( \text{Lip}(\alpha) \) norms for \( f \) as in the corollary. Hence we have proved \((B)\) for \( 0 < \alpha < 1 \).

3. Proof that 3) implies 1), i.e., \((B)\) for some \( \alpha \) implies \( K \) algebraic.
In order to prove that \((B)\) implies \( K \) algebraic, we need some preliminaries. The first result we need is a generalization of Jackson’s theorem on the decay of the approximation numbers \( E_n(f) \) for \( f \in \text{Lip}(\alpha) \).

Theorem 3.1. (Corollary 2.2 of \( R \)). Let \( 0 < \alpha \leq 1 \). There exists \( C(\alpha) > 0 \) such that \( f \in \text{Lip}(\alpha) \) implies \( E_n(f) \leq C(\alpha)\|f\|_{\alpha}n^{-\alpha} \).
Given a set $A$ in a Banach space $X$, if $X_n$ is an $n$-dimensional subspace of $X$, we call the number
\[ E_{X_n}(A) \equiv \sup_{f \in A} \left\{ \inf_{p \in X_n} \| f - p \|_X \right\} \equiv \sup_{f \in A} E_{X_n}(f) \]
the degree of approximation to $A$ by $X_n$; this is the “worst” best approximation for elements in $A$ by elements of $X_n$. Then the $n$-width of $A$ in $X$ is given by
\[ d_n(A) \equiv \inf_{X_n} E_{X_n}(A) \]
where the infimum is taken over all $n$-dimensional subspaces of $X$. This is, in an obvious sense, the closest distance from $A$ to all $n$-dimensional subspaces of $X$.

To get upper bounds on the $n$-widths of sets $A$ in $X$ is easy; merely estimate $E_{X_n}(A)$ for an appropriate space $X_n$ (e.g., polynomials of degree at most $n - 1$ in one-variable settings). Thus, from the Jackson theorem, if we let
\[ U = \{ f \in C(K) : \| f \|_\alpha \leq 1 \} \]
be the unit ball in $\text{Lip}(\alpha)$, then
\[ d_\delta(U) \leq C(\alpha) n^{-\alpha} \]
where $\delta(n)$ is the dimension of the space $P_n|K$ of polynomials in $P_n$ restricted to $K$.

We call $X_n$ extremal for $A$ if $d_n(A) = E_{X_n}(A)$. For full approximation sets $A$, it is easy to find extremal subspaces. Such sets are constructed as follows. Take a sequence $p_1, p_2, \ldots$ of linearly independent elements in $X$ and a decreasing sequence of positive numbers $a_1 \geq a_2 \geq \ldots$ with $a_m \to 0$. Let $X_m = \text{span}\{p_1, \ldots, p_m\}$. Finally, let
\[ A \equiv \{ x \in X : E_{X_n}(x) \leq a_n, \ n = 1, 2, \ldots \}. \]
The set $A$ is called a full approximation set. We state without proof the following.

**Proposition 3.2 (Theorem 3, p.139 of [L]).** $d_n(A) = a_n$, $n = 1, 2, \ldots$, and $X_n$ is extremal for $A$.

**Sketch of proof.** Clearly from the definitions of $d_n$ and $A$, we have $d_n(A) \leq E_{X_n}(A) \leq a_n$; to prove the reverse inequality, one considers
\[ A_n \equiv \{ x \in X_{n+1} : \| x \|_X \leq a_n \} \]
and shows that $d_n(A_n) = a_n$ (Theorem 2, p. 137 of [L]). Since $A_n \subset A$, we have $d_n(A_n) \leq d_n(A)$, which yields the result.

We can now state the key result from [R].

**Theorem 3.3 [R].** Suppose for some $0 < \alpha \leq 1$ there exists $B$ such that
\[ E_n(f) \leq \frac{1}{n^\alpha} \quad \text{implies} \quad \| f \|_\alpha \leq B. \]
Then $1/n^\alpha = O(d_\delta(U))$. 

This says that if we have a Bernstein theorem for $K$, then $P_n|_K$ is (essentially) extremal, i.e., we automatically get an estimate from BELOW on the $\delta(n)$-widths of $U$, at least asymptotically. For the reader’s convenience, we reproduce Ragozin’s proof.

Proof. Let

$$A \equiv \{ f \in C(K) : E_n(f) \leq 1/n^\alpha, \, n = 1, 2, \ldots \}.$$ 

By Proposition 3.2, $d_{\delta(n)}(A) = 1/n^\alpha$. By (6), $A \subset BU \equiv \{ f \in C(K) : \| f \| \leq B \}$.

Hence

$$1/n^\alpha = d_{\delta(n)}(A) \leq d_{\delta(n)}(BU) = Bd_{\delta(n)}(U)$$

from obvious properties of $n$-widths. This completes the proof.

Recall by (5) we have

$$d_{\delta(n)}(U) \leq C(\alpha)n^{-\alpha}$$

so that

$$d_{\delta(n)}(U) \asymp 1/n^\alpha.$$ 

Next we relate $n$-widths of $U$ to $n$-widths of things we can compute. By comparing pieces of $K$ to intervals $I$ and patching together — it is known that $d_{\alpha}(U) \asymp 1/n^\alpha$ for $U = \{ f \in C(I) : \| f \|_{0,\alpha} \leq 1 \}$ — we get the following result.

Theorem 3.4 [R]. $d_{\alpha}(U) \asymp 1/n^\alpha$.

Combining Theorem 3.4 with (7), we see that (B) implies $d_{\alpha}(U) \asymp d_{\delta(n)}(U)$ so that $\delta(n) = O(n)$. This implies $K$ is algebraic since, for large $n$, we have shown that the dimension of $P_n|_K$ is of order $n$, not $n^2$. Indeed, $\delta(n) = O(n)$ if and only if $K$ is contained in an algebraic variety of dimension 1.

4. Remarks and examples. We mention that the main theorem remains true for $K$ a smooth, compact $m$-dimensional submanifold of $\mathbb{R}^N$, $m = 1, \ldots, N-1$ (cf. [BLMT]). In the non-smooth case, one must replace $(M_T)$ by a condition which “makes sense.” For example, as in Section 1, suppose that $A$ is a complex analytic subvariety of $\mathbb{C}^N$ of pure dimension $m < N$ in a neighborhood of $K \equiv A \cap \mathbb{R}^N$. Suppose for simplicity that $K$ is compact but not necessarily smooth. Then for each regular point $(x_0, y_0) \in K$, there is a tangential Markov inequality $(M_T)$ of the form

$$(M_T) \quad |D_T p(f(t))|_{t=0} \leq M_T(\deg p)\| p \|_K$$

with exponent 1 for all analytic disks $f : \{ t \in \mathbb{C} : |t| < 1 \} \to A$ with $f(0) = (x_0, y_0)$. This result and related problems will not be discussed here.

For a curve $K$ with singularities, we can require that $(M_T)$ holds for all tangential derivatives in 2). With this interpretation, we have the following result.
PROPOSITION 4.1. Let \( K \subset \mathbb{R}^2 \) be a curve consisting of finitely many line segments and arcs of circles. Then \( K \) satisfies a tangential Markov inequality with exponent \( r \leq 2 \).

Proof. Clearly if \( L \) is a line segment forming a part of \( K \), then by the univariate case, at any point \((x, y)\) in \( L \),

\[
|D_T p(x, y)| \leq M(\deg p)^2 \|p\|_L \leq M(\deg p)^2 \|p\|_K
\]

for any polynomial \( p = p(x, y) \). Thus it suffices to show that if \( E \) is an arc of a circle forming a part of \( K \), then for any point \((x, y)\) in \( E \) and any polynomial \( p = p(x, y) \),

\[
|D_T p(x, y)| \leq M(\deg p)^2 \|p\|_E.
\]

Without loss of generality we let \( E \) be an arc on the unit circle. Let \( p = p(x, y) \) be a polynomial of degree \( n \). Then \( p \) restricts to a trigonometric polynomial on \( E \). By setting \( z = e^{it} \), we may write \( p(z) = z^{-n}P_{2n}(z) \) for some holomorphic polynomial \( P_{2n} \) of degree \( 2n \). A simple calculation reveals that at a point \( z \) in \( E \),

\[
|D_T p(z)| = \left| \frac{d}{dz} z^{-n} P_{2n}(z) \right| = \left| z^{-n} \frac{d}{dz} P_{2n}(z) - n z^{-n-1} P_{2n}(z) \right|
\]

\[
\leq \left| \frac{d}{dz} P_{2n}(z) \right| + |nP_{2n}(z)| \leq e \frac{1}{2 \text{cap}(E)} (2n)^2 \|p_{2n}\|_E + n \|P_{2n}\|_E.
\]

Here \( \text{cap}(E) \) denotes the logarithmic capacity of \( E \) and we have used Theorem 1 of Pommerenke [P].

The example of the boundary of a square shows that the exponent \( r = 2 \) is, in general, best possible. We conclude this note by sketching an alternate proof of 1) which illustrates the significance of the exponent 2.

PROPOSITION 4.2. Let \( K \) be a smooth compact connected curve in \( \mathbb{R}^2 \) satisfying \((M_T)\) with exponent strictly less than 2, i.e., there exists \( M = M(K) > 0 \) and \( 1 \leq r < 2 \) such that

\[
(M_T) \quad \|D_T p\|_K \leq M(\deg p)^r \|p\|_K
\]

for all polynomials \( p \). Then \( K \) is algebraic.

Proof. Let \( \gamma : [0, L] \to \mathbb{R}^2 \) be the arclength parameterization of \( K \). Note by the mean-value theorem and the fact that \( \gamma \) is smooth, for any function \( f \) which is differentiable on a neighborhood of \( K \) in \( \mathbb{R}^2 \) and for each pair of points \( \gamma(t_1), \gamma(t_2) \) on \( K \),

\[
|f(\gamma(t_2)) - f(\gamma(t_1))| \leq \epsilon ||\gamma'(t_2)| - |\gamma'(t_1)||(||D_T f||_K)
\]

for some constant \( \epsilon = \epsilon(K) \). Suppose \( K \) is not algebraic. Fix a positive integer \( n \) and let \( N = N(n) = \left( \frac{n+2}{2} \right) \) = dimension of \( P_n \). Choose \( N/2 \) points \( a_j \in K \) with \( ||a_j - a_{j-1}|| < 4L/N \) for successive points \( a_{j-1}, a_j \). Here \( L = \text{arclength of } K \). We can find a non-zero polynomial \( q_n \in P_n \) which vanishes at each point \( a_j \).
By \((M_T)\) applied to \(q_n\),
\[
\|D_T q_n\|_K \leq M n^r \|q_n\|_K.
\]
Now choose \(a \in K\) with \(|q_n(a)| = \|q_n\|_K\). Let \(a_i\) be a nearest point to \(a\) among the \(\{a_j\}\). Using (8) and \((M_T)\) we obtain
\[
\|q_n\|_K = |q_n(a) - q_n(a_i)| \leq c_4 \frac{4L}{N} \|D_T q_n\|_K \leq c_4 \frac{4L}{N} M n^r \|q_n\|_K.
\]
But \(N > n^2/2\) so we have
\[
\|q_n\|_K \leq (8LcM) n^{r-2} \|q_n\|_K. \tag{9}
\]
Since \(K\) is not algebraic, for each \(n\) we can chose \(q_n \in P_n\) satisfying (9). Since \(r < 2\), letting \(n \to +\infty\) we obtain a contradiction.

References


