TOPICS IN COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 31
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1995

COMPLEX GEODESICS AND FINSLER METRICS

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0. Introduction. In the study of intrinsic metrics and distances on complex manifolds, a crucial role is played by the notion of complex geodesic introduced by Vesentini [V]. Roughly speaking a complex geodesic is a holomorphic embedding of the unit disk with the hyperbolic metric which is an isometry with respect to the intrinsic metric or distance (or both) which is defined on the manifold under consideration.

As it is well known, the problem of existence of complex geodesics is satisfactory solved only for convex domains by the work of Lempert [L] who also proves uniqueness for strictly convex domains (see also [A, chapter 2.6]).

In order to find a different approach to this problem, which may eventually lead to an understanding of it on a larger class of complex manifolds, in [AP1] and [AP2] it was studied the same problem from a differential geometric point view looking for minimal conditions on an abstract complex Finsler metric which imply the existence and uniqueness of complex geodesics. A complete solution to the problem was achieved in terms of the holomorphic curvature of the metric, which must be a negative constant, and the vanishing of suitable torsion tensors. We give a brief account of these results at the beginning of section 2.

In this general framework it is very natural to ask whether it is possible to solve the same kind of problems for isometric holomorphic embeddings of $\mathbb{C}$ with the euclidean metric and of $\mathbb{P}_1$ with the Fubini-Study metric. In this paper we show that the methods of [AP1] and [AP2] work also in this case and that it is

1991 *Mathematics Subject Classification*: 32H15, 53C60.
The paper is in final form and no version of it will be published elsewhere.

possible to characterize the metrics which allow this type of complex geodesics. Not surprisingly, the conditions on the metrics are very similar: the same torsion must vanish and the holomorphic curvature must be constant zero for $\mathbb{C}$ and constant positive for $\mathbb{P}_1$.

As an application, in section 3 we use the existence of isometric embeddings of $\mathbb{C}$ to characterize $\mathbb{C}^n$. It is of interest that it appears a relation with complex Monge-Ampère equations very much as it happens for hyperbolic complex geodesics. We hope to come back to this aspect at a later date.

1. Finsler metrics and their curvature. In this section we shall recall some notions about complex Finsler metrics which will be used later in the paper. For further details we refer the reader to [AP1] and [AP2].

Given a real manifold M a function $F \in C^0\big(TM\big) \cap C^\infty\big(TM \setminus \{\text{Zero section}\}\big)$ is called a *real* (*smooth*) *Finsler metric* if:

$$F(p;v) > 0 \quad \text{for all } p \in M \text{ and } v \in T_pM \text{ with } v \neq 0, \tag{1.1}$$

$$F(p;\lambda v) = |\lambda| F(p;v) \quad \text{for all } p \in M,\ v \in T_pM \text{ and } \lambda \in \mathbb{R}. \tag{1.2}$$

For such metrics it is possible to define in the usual way the length of curves and therefore to recover the Euler-Lagrange equation for the stationary curves of the length functional. Such stationary curves are the *geodesics* of F. If $x^1, \ldots, x^n$ are coordinates on M, $v^1, \ldots, v^n$ the corresponding tangential coordinates and $G = F^2$, a curve γ is a geodesic iff it satisfies the following system of equations:

$$\frac{\partial^2 G}{\partial v^\alpha \partial v^\beta}(\gamma;\dot\gamma)\ddot\gamma^\beta = \frac{\partial G}{\partial x^\alpha}(\gamma;\dot\gamma) - \frac{\partial^2 G}{\partial v^\alpha \partial x^\beta}(\gamma;\dot\gamma)\dot\gamma^\beta, \tag{1.3}$$

for $\alpha = 1, \ldots, n$, where we adhere to the summation convention.

Then, as it may be expected, if G is strictly convex with respect to the tangential variables, one shows the existence and uniqueness of geodesics through any point and tangential direction (cfr. [AP1] and [Ru] for details). The metric F is said to be *complete* if M with the distance function associated to F is a complete metric space, which is equivalent to require that geodesics are defined on the whole real line.

This is all we need about real metrics. If M is a complex manifold, a *complex* (*smooth*) *Finsler metric* on M is a function $F \in C^0\big(T^{1,0}M\big) \cap C^\infty\big(T^{1,0}M \setminus \{\text{Zero section}\}\big)$ such that:

$$F(p;v) > 0 \quad \text{for all } p \in M \text{ and } v \in T^{1,0}_pM \text{ with } v \neq 0; \tag{1.4}$$

$$F(p;\lambda v) = |\lambda| F(p;v) \quad \text{for all } p \in M,\ v \in T^{1,0}_pM \text{ and } \lambda \in \mathbb{C}. \tag{1.5}$$

Naturally a complex Finsler metric defines a real Finsler metric by means of the canonical real bundle isomorphism between $T^{1,0}M$ and TM. It is therefore meanigful to talk about geodesics also for a complex Finsler metric. Of course it is not a natural assumption to require that the function $G = F^2$ is strictly convex on the tangential directions. We shall instead say that F is *strictly pseudoconvex* iff for all $p \in M$

$$\left(\frac{\partial G(p;v)}{\partial v^\alpha \partial \overline{v^\beta}} \right) > 0 \quad \text{on } T_p^{1,0}M \setminus \{0\}, \tag{1.6}$$

where again the v^α's are tangential coordinates. This is equivalent to requiring that the *indicatrix* of F

$$I(p) = \{ v \in T_p^{1,0}M \mid F(p;v) = 1 \} \tag{1.7}$$

is strictly pseudoconvex. This is a very natural assumption on F and we shall impose it throughout the paper often without saying it. It should be noted that *a priori* this assumption does not guarantee the existence of geodesics for the metric F.

We must introduce some more characters in order to run a decent show and unfortunately this forces us to use some complicated notations. If $z^1, \dots, z^n$ are coordinates on M, $v^1, \dots, v^n$ the corresponding tangential coordinates, we denote derivatives with respect to the v^α's by subscripts and derivatives with respect to the z^α's by subscripts after a semicolon. For example:

$$G_{\alpha;\bar{\jmath}} = \frac{\partial^2 G}{\partial v^\alpha \partial \overline{z^j}}. \tag{1.8}$$

An important tool is provided by the holomorphic curvature of a complex (strictly pseudoconvex) Finsler metric F. Let $(p;v) \in T^{1,0}M \setminus \{\text{Zero section}\}$ and $\varphi \colon U \to M$ be a holomorphic map of the unit disk $U \subset \mathbb{C}$ into M with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$. Then $\varphi^* G = \varphi^* F^2$ defines a pseudohermitian metric on U and one may compute its Gaussian curvature $K(\varphi^* G)$ on $U \setminus \{\varphi' = 0\}$. We define the *holomorphic curvature* $K_F(p;v)$ of F at $(p;v)$ by:

$$K_F(p;v) = \sup\{ K(\varphi^* G)(0) \} \tag{1.9}$$

where the supremum is taken for all such φ. This is a very useful notion which is meaningful for just upper semicontinuous Finsler metrics and enjoys many properties. For instance it has a built-in Ahlfors' Lemma (cf. [AP2] for details) and coincides with the usual curvature for hermitian metrics (cf. [Wu]). Unfortunately in general it is not very manageable or computable in terms of F (or G). In [AP2] following an intuition of Royden [Ro] and recovering a definition coming from a different approach due to Kobayashi [K], it is shown that

$$K(p;v) = -\frac{2}{G(p;v)^2} G_\alpha(p;v) \Gamma^\alpha_{;i\bar{\jmath}}(p;v)\, v^i \overline{v^j}, \tag{1.10}$$

where we use the following notations:

$$\Gamma^\alpha_{;i} = G^{\alpha\bar{\mu}} G_{\bar{\mu};i} \qquad \Gamma^\alpha_{;i\bar{\jmath}} = \frac{\partial}{\partial \overline{z^j}} \Gamma^\alpha_{;i}. \tag{1.11}$$

In proving (1.10) it is also shown (cfr. Lemma 2.1 and Proposition 3.2 of [AP2]) that a holomorphic map φ in the competing family realizes the holomorphic curvature at 0 iff for all $\alpha=1,\dots,n$

$$(\varphi'')^\alpha(0)=-\Gamma^\alpha_{;i}(p;v)v^i+cv^\alpha, \tag{1.12}$$

where $c\in\mathbb{C}$ is an arbitrary constant.

Finally we must recall the notion of Kähler-Finsler metric. Following Rund [Ru] we say that a complete Finsler metric is *Kähler* if, defining the *torsion tensor* by

$$T_{\alpha i\bar\mu}=G_{\beta\bar\mu}(\Gamma^\beta_{i;\alpha}-\Gamma^\beta_{\alpha;i}), \tag{1.13}$$

where $\Gamma^\beta_{\alpha;i}=(\Gamma^\beta_{;i})_\alpha$, we have

$$T_{\alpha i\bar\mu}\,v^i\overline{v^\mu}=0. \tag{1.14}$$

If F is a hermitian metric, it is easily seen that (1.14) reduces to the usual Kähler condition.

2. Complex geodesic curves. We start by recalling results of [AP1] and [AP2]. Let M be a complex manifold of dimension n, and let F be a smooth strictly pseudoconvex Finsler metric on M. As usual, we set $G=F^2$, and we denote by $U(r)\subset\mathbb{C}$ the open disk of radius $r>0$ in $\mathbb{C}$.

A holomorphic map $\varphi\colon U(r)\to M$ is called a *segment of a hyperbolic complex geodesic curve* iff φ maps geodesics (parametrized by arc length) for the hyperbolic metric of the unit disk into geodesics (parametrized by arc length) of F. If $r=1$ then φ is called a *hyperbolic complex geodesic curve.*

If F is either the Kobayashi or the Carathéodory metric, any complex geodesic in the sense of Vesentini for the distance is a hyperbolic complex geodesic curve. The converse is in general not true: for instance, the universal covering of an annulus endowed with the Poincaré metric is a hyperbolic complex geodesic curve but it is not a complex geodesic in the sense of Vesentini.

Using (1.3) and the equation of geodesics for the hyperbolic metric, in [AP1] it was shown that $\varphi\colon U(r)\to M$ is a segment of a hyperbolic complex geodesic curve iff the following system of PDE is satisfied:

$$\begin{cases} D^\alpha_{-1}(\varphi)=(\varphi'')^\alpha+A(\varphi')^\alpha+\Gamma^\alpha_{;i}(\varphi;\varphi')(\varphi')^i=0,\\ N_\alpha(\varphi)=G_{\alpha\beta}(\varphi;\varphi')(\varphi''+A\varphi')^\beta-\big[G_{i;\alpha}(\varphi;\varphi')-G_{\alpha;i}(\varphi;\varphi')\big](\varphi')^i=0,\end{cases} \tag{2.1}$$

for $\alpha=1,\dots,n$, where the prime stands for $\partial/\partial\zeta$, and $A\colon U\to\mathbb{C}$ is the function

$$A(\zeta)=-\frac{2\bar\zeta}{1-|\zeta|^2}.$$

In [AP2] it was shown that $N(\varphi)=\big(N_1(\varphi),\dots,N_n(\varphi)\big)=0$ holds iff F is Kähler along φ, i.e., iff the torsion tensor (1.14) vanishes on the tangent bundle of the image of φ. Explicitly this is the case iff for $\alpha=1,\dots,n$

$$T_\alpha(\varphi;\varphi')=T_{\alpha i\bar\mu}(\varphi')^i\overline{(\varphi')^\mu}=[G_{\beta\bar\mu}(\Gamma^\beta_{i;\alpha}-\Gamma^\beta_{\alpha;i})](\varphi')^i\overline{(\varphi')^\mu}=0. \tag{2.2}$$

A couple of geometric properties (obtained in [AP2]) of the solutions of the system $D_{-1}(\varphi) = \big(D^1_{-1}(\varphi), \dots, D^n_{-1}(\varphi)\big) = 0$ are recalled in the following:

PROPOSITION 2.1. *Let* $\varphi: U(r) \to M$ *be a holomorphic solution of* $D_{-1}(\varphi) = 0$. *Then*:

(i) φ *realizes the holomorphic curvature of* F *at every point of* $U(r)$;

(ii) *if* $F\big(\varphi(0); \varphi'(0)\big) = 1$ *then* φ *is an isometry at every point of* $U(r)$ *with respect to the hyperbolic metric of the unit disk restricted to* $U(r)$ *and the metric* F *on* M.

This result shows that holomorphic solutions of $D_{-1}(\varphi) = 0$ are interesting on their own. In [AP2] a complete result on existence and uniqueness of the solutions was provided; in order to state it we need to introduce a further tensor H of type $(4, 0)$ on $T^{1,0}M$. Its components in local coordinates are given by

$$H_{\alpha i \bar\mu \bar\jmath} = (G_{\tau\bar\mu}\Gamma^\tau_{;\alpha\bar\jmath})_i - (G_{\tau\bar\mu}\Gamma^\tau_{;i\bar\jmath})_\alpha; \tag{2.3}$$

if F is a hermitian metric, the vanishing of H is equivalent to a symmetry condition (satisfied if F is Kähler) on the curvature tensor of the Chern connection associated to the metric F.

We have the following (cf. [AP2]):

THEOREM 2.2. *For any* $p \in M$ *and* $v \in T^{1,0}_p M$ *with* $F(p; v) = 1$ *the Cauchy problem*

$$\begin{cases} D_{-1}(\varphi) = 0, \\ \varphi(0) = p, \ \varphi'(0) = v, \end{cases} \tag{2.4}$$

has a unique holomorphic solution $\varphi: U(r) \to M$ *iff one of the following equivalent conditions holds*:

$$[\Gamma^\alpha_{;i\bar\jmath} - \Gamma^\alpha_{\bar\beta;i}\Gamma^{\bar\beta}_{;\bar\jmath}]v^i\overline{v^j} = 2Gv^\alpha \tag{2.5}$$

for all $\alpha = 1, \dots, n$ *and* $v \in T^{1,0}M$ (*where* $\Gamma^{\bar\beta}_{;\bar\jmath} = \overline{\Gamma^\beta_{;j}}$), *or*

$$K_F \equiv -4 \quad \text{and} \quad H_\alpha(v) = H_{\alpha i\bar\mu\bar\jmath}\overline{v^\mu}v^i\overline{v^j} = 0 \tag{2.6}$$

for all $\alpha = 1, \dots, n$ *and* $v \in T^{1,0}M$. *Furthermore*, *if* F *is complete and either* (2.5) *or* (2.6) *holds*, *then* φ *is defined on the whole unit disk* U.

It is then clear that adding the Kähler condition one obtains necessary and sufficient conditions for the existence of hyperbolic complex geodesics curves:

THEOREM 2.3. *Let* F *be a smooth strictly pseudoconvex complex Finsler metric on a complex manifold* M. *Then there exists a unique segment of a hyperbolic complex geodesic curve* $\varphi: U(r) \to M$ *with* $\varphi(0) = p$ *and* $\varphi'(0) = v$ *for any* $p \in M$ *and* $v \in T^{1,0}_p M$ *with* $F(p; v) = 1$ *iff* F *is Kähler-Finsler with* $K_F \equiv -4$ *and such that* $H_\alpha \equiv 0$ *on* $T^{1,0}M$ *for all* $\alpha = 1, \dots, n$. *Furthermore if* F *is complete then all segments extend to hyperbolic complex geodesic curves.*

Here we would like to describe an extension of these results to maps from $\mathbb{C}$ or $\mathbb{P}_1$ rather than from U. We need some definitions to start with.

The euclidean metric ε on $\mathbb{C}$ is defined by

$$\varepsilon = dz \otimes d\bar{z}, \tag{2.7}$$

so that its curvature is $K_\varepsilon \equiv 0$ and the equation of a (real) geodesic σ is given by

$$\ddot{\sigma} = 0. \tag{2.8}$$

The Fubini-Study metric on $\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}$ is given with respect to the standard coordinate on $\mathbb{C} \subset \mathbb{P}_1$ by

$$\nu = \frac{1}{(1+|z|^2)^2}\, dz \otimes d\bar{z}, \tag{2.9}$$

so that its curvature is $K_\nu \equiv 4$ and the equation of a (real) geodesic σ is given by

$$\ddot{\sigma} - \frac{2\dot{\sigma}^2\bar{\sigma}}{1+|\sigma|^2} = 0. \tag{2.10}$$

Let M be a complex manifold of dimension n, and let F be a complex Finsler metric on M. A holomorphic map $\varphi: U(r) \to M$ is called a *parabolic* (respectively, *elliptic*) *segment of a complex geodesic curve* iff φ maps geodesics parametrized by arc length with respect to the euclidean metric (respectively, the Fubini-Study metric) restricted to $U(r)$ into geodesics parametrized by arc length of F on M. If $r = +\infty$ (respectively, if φ is defined on $\mathbb{P}_1$), then we call φ a *parabolic* (respectively, *elliptic*) *complex geodesic curve.*

We have the following characterization:

PROPOSITION 2.4. *Let $\varphi: U(r) \to M$ be a holomorphic map. Then φ is a segment of parabolic complex geodesic curve iff*

$$\begin{cases} D_0^\alpha(\varphi) = (\varphi'')^\alpha + \Gamma^\alpha_{;i}(\varphi;\varphi')(\varphi')^i = 0, \\ N_\alpha(\varphi) = 0, \end{cases} \tag{2.11}$$

for $\alpha = 1, \dots, n$, and φ is a segment of elliptic complex geodesic curve iff

$$\begin{cases} D_{+1}^\alpha(\varphi) = (\varphi'')^\alpha + \Gamma^\alpha_{;i}(\varphi;\varphi')(\varphi')^i + B(\zeta)(\varphi')^\alpha = 0, \\ N_\alpha(\varphi) = 0, \end{cases} \tag{2.12}$$

for $\alpha = 1, \dots, n$, where

$$B(\zeta) = \frac{2\bar{\zeta}}{1+|\zeta|^2}. \tag{2.13}$$

The proof is just a word by word repetition of the proof of Proposition 1.5 of [AP1] (which provides the characterization (2.1) of hyperbolic complex geodesic curves) replacing of course the equation for the geodesics for the hyperbolic metric by (2.9) and (2.10).

As in the case of hyperbolic complex geodesics, in order to find conditions for the existence of parabolic or elliptic complex geodesics one must study the systems

$$D_0(\varphi) = \big(D_0^1(\varphi), \dots, D_0^n(\varphi)\big) = 0$$

and

$$D_{+1}(\varphi) = \big(D_{+1}^1(\varphi), \dots, D_{+1}^n(\varphi)\big) = 0,$$

since $N(\varphi) = 0$ is, as before, equivalent to the metric F being Kähler-Finsler along φ.

Following the ideas developed in [AP2], we then start with:

PROPOSITION 2.5. *Let $\varphi: U(r) \to M$ be a holomorphic map with $\varphi'(0) \neq 0$. If $D_0(\varphi) = 0$ we have*:

(i) *φ realizes the holomorphic curvature of F at $\varphi(0)$ in the direction $\varphi'(0)$;*
(ii) *if $F\big(\varphi(0); \varphi'(0)\big) = 1$, then φ is an isometry with respect to the euclidean metric on $U(r)$ and F.*

On the other hand, if $D_{+1}(\varphi) = 0$ we have:

(iii) *φ realizes the holomorphic curvature of F at $\varphi(0)$ in the direction $\varphi'(0)$;*
(iv) *if $F\big(\varphi(0); \varphi'(0)\big) = 1$, then φ is an isometry with respect to the Fubini-Study metric on $U(r)$ and F.*

Proof. As observed in section 1 (see (1.12)), φ realizes the holomorphic sectional curvature at $\varphi(0)$ in the direction $\varphi'(0)$ iff

$$(\varphi'')^\alpha(0) = -\Gamma^\alpha_{;i}\big(\varphi(0); \varphi'(0)\big)(\varphi')^i(0) + c\,(\varphi')^\alpha(0),$$

for some $c \in \mathbb{C}$ and all $\alpha = 1, \dots, n$. But this is exactly $D_0(\varphi) = 0$ or $D_{+1}(\varphi) = 0$ evaluated at $\zeta = 0$ (and with $c = 0$); hence (i) and (iii) follow at once.

Now suppose $F^2\big(\varphi(0); \varphi'(0)\big) = G\big(\varphi(0); \varphi'(0)\big) = 1$. If $D_0(\varphi) = 0$, then using the homogeneity of G we get

$$G_\alpha(\varphi; \varphi')(\varphi'')^\alpha = -G_\alpha(\varphi; \varphi')\Gamma^\alpha_{;i}(\varphi; \varphi')(\varphi')^i = -G_{;i}(\varphi; \varphi')(\varphi')^i,$$

so that

$$\frac{\partial}{\partial\zeta}\big[G(\varphi; \varphi')\big] = G_{;i}(\varphi; \varphi')(\varphi')^i + G_\alpha(\varphi; \varphi')(\varphi'')^\alpha = 0.$$

Since along the curve $t \mapsto e^{i\theta}t$, for a fixed $\theta \in \mathbb{R}$, we have

$$\frac{\partial}{\partial\zeta} = \frac{1}{2}e^{-i\theta}\frac{d}{dt},$$

it follows that the function $t \mapsto G\big(\varphi(e^{i\theta}t); \varphi'(e^{i\theta}t)\big)$ is a solution of the Cauchy problem

$$\begin{cases} u'(t) = 0, \\ u(0) = 1. \end{cases}$$

Hence

$$G\big(\varphi(\zeta); \varphi'(\zeta)\big) = G\big(\varphi(\zeta); d\varphi_\zeta(d/dt)\big) \equiv 1 = \varepsilon_\zeta(d/dt, d/dt)$$

for all $\zeta \in U(r)$, and (ii) is proved.

Finally, assume now $D_{+1}(\varphi)=0$. Arguing as before, we get

$$\begin{aligned}G_\alpha(\varphi;\varphi')(\varphi'')^\alpha + BG(\varphi;\varphi') &= G_\alpha(\varphi;\varphi')\bigl[(\varphi'')^\alpha + B(\varphi')^\alpha\bigr]\\ &= -G_\alpha(\varphi;\varphi')\Gamma^\alpha_{;i}(\varphi;\varphi')(\varphi')^i = -G_{;i}(\varphi;\varphi')(\varphi')^i.\end{aligned}$$

Thus

$$\frac{\partial}{\partial\zeta}\bigl[G(\varphi;\varphi')\bigr] = G_{;i}(\varphi;\varphi')(\varphi')^i + G_\alpha(\varphi;\varphi')(\varphi'')^\alpha = -BG(\varphi;\varphi'),$$

so that the function $t\mapsto G\bigl(\varphi(e^{i\theta}t);\varphi'(e^{i\theta}t)\bigr)$ for all $\theta\in\mathbb{R}$ is a solution of the Cauchy problem

$$\begin{cases} u'(t) = -\dfrac{4t}{1+t^2}u(t),\\ u(0)=1.\end{cases}$$

But also $u(t)=(1+t^2)^{-2}$ is a solution of the same problem; therefore

$$G\bigl(\varphi(\zeta);\varphi'(\zeta)\bigr)\equiv\frac{1}{(1+|\zeta|^2)^2},$$

and we are done. ∎

Now we may deal with the existence and uniqueness of solutions of the Cauchy problem for $D_0(\varphi)=0$ and $D_{+1}(\varphi)=0$. Let $S^{1,0}M=\{(p;v)\in T^{1,0}M \mid F(p;v)=1\}$ be the unit sphere bundle. Then we can summarize our results as follows:

Theorem 2.6. *The Cauchy problem*

$$\begin{cases} D_0(\varphi)=0,\\ \varphi(0)=p, \quad \varphi'(0)=v_0,\end{cases}\tag{2.14}$$

has holomorphic solution for all $(p;v_0)\in S^{1,0}M$ *iff one of the following equivalent conditions holds*:

$$[\Gamma^\alpha_{;i\bar{\jmath}} - \Gamma^\alpha_{\bar\beta;i}\Gamma^{\bar\beta}_{;\bar{\jmath}}]v^i\overline{v^j}=0\tag{2.15}$$

for all $\alpha=1,\dots,n$ *and* $v\in T^{1,0}M$, *or*

$$K_F\equiv 0 \quad \textit{and} \quad H_\alpha(v)=0\tag{2.16}$$

for all $\alpha=1,\dots,n$ *and* $v\in T^{1,0}M$. *Furthermore, when a holomorphic solution of* (2.14) *exists, it is unique.*

Theorem 2.7. *The Cauchy problem*

$$\begin{cases} D_{+1}(\varphi)=0,\\ \varphi(0)=p, \quad \varphi'(0)=v_0,\end{cases}\tag{2.17}$$

has holomorphic solution for all $(p;v_0)\in S^{1,0}M$ *iff one of the following equivalent conditions holds*:

$$[\Gamma^\alpha_{;i\bar{\jmath}} - \Gamma^\alpha_{\bar\beta;i}\Gamma^{\bar\beta}_{;\bar{\jmath}}]v^i\overline{v^j}=-2Gv^\alpha\tag{2.18}$$

for all $\alpha = 1, \dots, n$ *and* $v \in T^{1,0}M$, *or*

$$K_F \equiv 4 \quad \textit{and} \quad H_\alpha(v) = 0 \tag{2.19}$$

for all $\alpha = 1, \dots, n$ *and* $v \in T^{1,0}M$. *Furthermore, when a holomorphic solution of* (2.14) *exists, it is unique.*

The proofs of the two theorems are very similar. Actually, the proof of Theorem 2.6 is a bit simpler since $D_0(\varphi) = 0$ together with $\partial\varphi/\partial\bar\zeta = 0$ is a problem of Frobenius-type, and (2.15) is exactly the compatibility condition. This is not the case in Theorem (2.7), because $D_{+1}(\varphi) = 0$ is not an autonomous system. For this reason we shall work out the proof of the latter theorem only, leaving to the reader the proof of the former, which goes along the same lines with the obvious changes — which are, in fact, simplifications.

First of all, we remark that if (2.17) has a holomorphic solution, this is unique. Indeed, this follows from the uniqueness of solutions of Cauchy problems for O.D.E., and from the fact that holomorphic maps defined on a disk are completely determined by their restriction to a diameter.

A direct (though not trivial) check shows that (2.18) and (2.19) are equivalent. First of all, (2.18) implies $K_F \equiv 4$; in fact, since standard computations using the homogeneity of G (cf. section 2 of [AP2]) yield $G_\alpha \Gamma^\alpha_{\bar\beta;i} = 0$, using (2.18) we get

$$\begin{aligned}
[G(p;v)]^2 K_F(p;v) &= -2G_\alpha(p;v)\Gamma^\alpha_{;i\bar\jmath}(p;v)v^i\overline{v^j} \\
&= 4G_\alpha(p;v)G(p;v)v^\alpha - 2G_\alpha(p;v)\Gamma^\alpha_{\bar\beta;i}(p;v)\Gamma^{\bar\beta}_{;\bar\jmath}(p;v)v^i\overline{v^j} \\
&= 4[G(p;v)]^2.
\end{aligned}$$

So we are left to prove that, under the hypothesis $K_F \equiv 4$, (2.18) is equivalent to $H_\alpha(v) = 0$ for all $v \in T^{1,0}M$ and $\alpha = 1, \dots, n$. Assume then that for all $(p;v) \in T^{1,0}M$ with $v \neq 0$ one has $K_F(p;v) = 4$. This is equivalent to

$$G_\beta \Gamma^\beta_{;i\bar\jmath} v^i \overline{v^j} = -2G^2;$$

hence differentiating with respect to $\overline{v^\nu}$ we get

$$-4GG_{\bar\nu} = \big[G_{\beta\bar\nu}\Gamma^\beta_{;i\bar\jmath} + G_\beta \Gamma^\beta_{\bar\nu;i\bar\jmath}\big] v^i\overline{v^j} + G_\beta \Gamma^\beta_{;i\bar\nu}\, v^i.$$

Using the homogeneity of G, the same computations used in Theorem 3.5 of [AP2] yield

$$-4G\, v^\alpha = \big[\Gamma^\alpha_{;i\bar\jmath} - \Gamma^\alpha_{\bar\mu;i}\Gamma^{\bar\mu}_{;\bar\jmath}\big] v^i\overline{v^j} + G^{\alpha\bar\nu} G_\beta \Gamma^\beta_{;i\bar\nu}\, v^i.$$

A word by word repetition of the conclusion of the proof of Theorem 3.5 in [AP2] allows one to conclude from this equality that

$$-4G\, v^\alpha = 2\big[\Gamma^\alpha_{;i\bar\jmath} - \Gamma^\alpha_{\bar\mu;i}\Gamma^{\bar\mu}_{;\bar\jmath}\big] v^i\overline{v^j} + G^{\alpha\bar\nu} H_\alpha(v),$$

and thus (2.18) holds iff $H_\alpha(v) = 0$ for $\alpha = 1, \dots, n$.

So (2.18) is equivalent to (2.19). We shall now show that (2.18) is equivalent to the holomorphic solvability of (2.17) for any initial data. The necessity of (2.18) is obtained by simply assuming the existence of holomorphic solutions of

$D_{+1}\big(\varphi(\zeta)\big) = 0$, and then by differentiating with respect to $\bar\zeta$; the easy computations are left to the reader (but see also the computations before Theorem 3.3 in [AP2]).

Let us now construct a holomorphic solution of (2.17) assuming (2.18). There exists $\varepsilon > 0$ such that for every $e^{i\theta} \in \mathbf{S}^1$ the O.D.E. Cauchy problem

$$\begin{cases} \ddot g^\alpha(t) = -\Gamma^\alpha_{;i}\big(g(t);\dot g(t)\big)\dot g^i(t) - B(t)\dot g^\alpha(t), & \text{for } \alpha = 1,\dots,n,\\ g(0) = p, \quad \dot g(0) = e^{i\theta}v_0 \end{cases}$$

has a unique solution $g_{e^{i\theta}}\colon(-\varepsilon,\varepsilon) \to M$. Let us define $\varphi\colon U(\varepsilon) \to M$ by

$$\varphi(\zeta) = g_{\zeta/|\zeta|}(|\zeta|). \tag{2.20}$$

If φ is holomorphic then a straightforward verification, using the Cauchy-Riemann equations in polar coordinates, shows that φ solves (2.17). Thus we must show that φ is holomorphic. To this end, we shall work on $T^{1,0}(S^{1,0}M)$, which we identify with $T(S^{1,0}M)$ in the usual way:

$$T^{1,0}(S^{1,0}M) \to T(S^{1,0}M), \qquad Y \mapsto Y^o = Y + \overline{Y},$$

where $\overline{Y}$ is the conjugate of Y in the complexified tangent bundle

$$T^{\mathbb{C}}(S^{1,0}M) = T^{1,0}(S^{1,0}M) \oplus T^{0,1}(S^{1,0}M).$$

Define a vector field $X \in \Gamma\big(T^{1,0}(S^{1,0}M)\big)$ by setting

$$X_{\tilde v} = v^i \frac{\partial}{\partial z^i} - \Gamma^\alpha_{;i}\, v^i \frac{\partial}{\partial v^\alpha}, \tag{2.21}$$

where $(z^1,\dots,z^n;v^1,\dots,v^n)$ are local coordinates near $\tilde v \in (T^{1,0}M)$, with $\tilde v \neq 0$. It is not hard to check that (2.21) defines a global section of $T^{1,0}(S^{1,0}M)$.

To describe the integral curve of $X^o = X + \overline{X}$ through the point $(p;e^{i\theta}v_0) \in S^{1,0}M$ for $\theta \in \mathbb{R}$, first of all set

$$\sigma_\theta(t) = g_{e^{i\theta}}(\tan t).$$

Then σ_θ is the unique solution of the Cauchy problem

$$\begin{cases} \ddot\sigma^\alpha_\theta = -\Gamma^\alpha_{;i}(\sigma_\theta;\dot\sigma_\theta)\,\dot\sigma^i_\theta, & \text{for } \alpha = 1,\dots,n,\\ \sigma_\theta(0) = p, \quad \dot\sigma_\theta(0) = e^{i\theta}v_0. \end{cases}$$

Thus, if we define curves $\tilde\sigma_\theta\colon(-\varepsilon,\varepsilon) \to S^{1,0}M$ by setting $\tilde\sigma_\theta(t) = (\sigma_\theta;\dot\sigma_\theta)$, then

$$\tilde\sigma_\theta(t) = e^{tX^o}(e^{i\theta}\tilde v_0) \quad \text{and} \quad \sigma_\theta(t) = \pi\big(e^{tX^o}(e^{i\theta}\tilde v_0)\big),$$

where $\tilde v_0 = (p;v_0)$, $e^{i\theta}\tilde v_0 = (p;e^{i\theta}v_0)$, $\pi\colon S^{1,0}M \to M$ is the projection and e^{tX^o} is the local one-parameter group associated to X^o.

We need another vector field $Z \in \Gamma\big(T^{1,0}(S^{1,0}M)\big)$ defined by

$$Z_{\tilde v} = iv^\alpha \frac{\partial}{\partial v^\alpha}. \tag{2.22}$$

We note that

$$\pi_* Z = 0 \quad \text{and} \quad e^{\theta Z^o}\tilde v_0 = e^{i\theta}\tilde v_0,$$

so that we may write

$$\tilde\sigma_\theta(t) = e^{tX^o} e^{\theta Z^o}(\tilde v_0) \quad \text{and} \quad \sigma_\theta(t) = \pi\big(e^{tX^o} e^{\theta Z^o}(\tilde v_0)\big).$$

Since in polar coordinates

$$\frac{\partial}{\partial\bar\zeta} = \frac{ie^{i\theta}}{2t}\left(\frac{\partial}{\partial\theta} - it\frac{\partial}{\partial t}\right),$$

the holomorphicity of φ is equivalent to

$$\frac{\partial}{\partial\theta}\pi(e^{(\arctan t)X^o} e^{\theta Z^o}(\tilde v_0))\bigg|_{\theta=0} = tJ\frac{\partial}{\partial t}\pi(e^{(\arctan t)X^o} e^{\theta Z^o}(\tilde v_0))\bigg|_{\theta=0}, \tag{2.23}$$

where one may choose $\theta = 0$ since $\tilde v_0$ is any vector in $S^{1,0}_p M$.

We shall prove (2.23) following again ideas of [AP2]. A simple computation using the definitions (2.21), (2.22) and (2.18) — it is at this point that we need this hypothesis — shows that

$$[X^o, JX^o] = 4Z^o \quad \text{and} \quad [X^o, Z^o] = -JX^o. \tag{2.24}$$

Now define $\tilde v_\tau = e^{\tau X^o}\tilde v_0$ for $\tau > 0$, and let u be the curve on $T_{\tilde v_\tau}(S^{1,0}M)$ defined by

$$u(t) = e_*^{tX^o} Z^o_{e^{-tX^o}\tilde v_\tau}.$$

Then

$$\begin{aligned}\dot u(t) &= \frac{d}{dt}\big(e_*^{tX^o} Z^o_{e^{-tX^o}\tilde v_\tau}\big) = -e_*^{tX^o}[X^o, Z^o]_{e^{-tX^o}\tilde v_\tau}\\ &= e_*^{tX^o}(JX^o)_{e^{-tX^o}\tilde v_\tau},\end{aligned}$$

and

$$\ddot u(t) = -e_*^{tX^o}[X^o, JX^o]_{e^{-tX^o}\tilde v_\tau} = -4e_*^{tX^o} Z^o_{e^{-tX^o}\tilde v_\tau}.$$

In other words, $u(t)$ solves the Cauchy problem

$$\begin{cases}\ddot u = -4u,\\ u(0) = Z^o_{\tilde v_\tau}, \quad \dot u(0) = (JX^o)_{\tilde v_\tau}.\end{cases}$$

Therefore it must be given by

$$u(t) = (\cos 2t)Z^o_{\tilde v_\tau} + \frac{\sin 2t}{2}(JX^o)_{\tilde v_\tau};$$

in particular,

$$\pi_* e_*^{\tau X^o} Z^o_{\tilde v_0} = \pi_* u(\tau) = \frac{\sin 2t}{2} J\pi_* X^o_{\tilde v_\tau}.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial\theta}\pi(e^{(\arctan t)X^o} e^{\theta Z^o}\tilde v_0)\bigg|_{\theta=0} &= \pi_* e_*^{(\arctan t)X^o} Z^o_{\tilde v_0}\\ &= \frac{t}{1+t^2} J\pi_* X^o_{\tilde v_{\arctan t}},\end{aligned}$$

and

$$\frac{\partial}{\partial t}\pi(e^{(\arctan t)X^o}e^{\theta Z^o}\tilde{v}_0)\bigg|_{\theta=0} = \frac{\partial}{\partial t}\pi(e^{(\arctan t)X^o}\tilde{v}_0)$$
$$= \frac{1}{1+t^2}\pi_* X^o_{\tilde{v}_{\arctan t}},$$

which shows that (2.23) holds. ∎

Since the Kähler condition implies $N(\varphi) = 0$, Theorems 2.6 and 2.7 have the following consequence:

COROLLARY 2.8. *Let F be a Kähler-Finsler metric with $H_\alpha(p;v) = 0$ for all $\alpha = 1,\dots,n$ and $(p;v) \in S^{1,0}M$. Then*:

(i) *If $K_F \equiv 0$ then for any $(p_0;v_0) \in S^{1,0}M$ there exists a unique segment of a parabolic complex geodesic curve through $(p_0;v_0)$;*

(ii) *If $K_F \equiv 4$ then for any $(p_0;v_0) \in S^{1,0}M$ there exists a unique segment of a elliptic complex geodesic curve through $(p_0;v_0)$.*

As one may expect, if F is complete then the segments of parabolic or elliptic complex geodesic curves extend to whole complex geodesic curves, exactly as in the hyperbolic case.

THEOREM 2.9. *Let F be a complete smooth strictly pseudoconvex complex Finsler metric with $K_F \equiv 0$ (respectively, $K_F \equiv 4$) and $H_\alpha \equiv 0$ for all $\alpha = 1,\dots,n$. Let $(p;v) \in S^{1,0}M$. Then*:

(i) *there exists a unique holomorphic map $\varphi\colon \mathbb{C} \to M$ (respectively, $\varphi\colon \mathbb{P}_1 \to M$) with $\varphi(0) = p$ and $\varphi'(0) = v$ such that $D_0(\varphi) = 0$ (respectively, $D_{+1}(\varphi) = 0$); in particular, φ is an isometry with respect to the euclidean metric on $\mathbb{C}$ (respectively, the Fubini-Study metric on $\mathbb{P}_1$) and F.*

(ii) *If F is Kähler-Finsler then the map φ is a parabolic (respectively, elliptic) complex geodesic curve.*

Proof. Part (ii) follows immediately from the previous considerations regarding the meaning of $N(\varphi) = 0$. It is therefore enough to prove (i). We shall work in the case $K_F \equiv 0$; the argument in the elliptic case is exactly the same.

Consider the distribution $\mathcal{D} = \mathbb{C}X^o \oplus \mathbb{C}Z^o \subset T(S^{1,0}M)$, where X^o and Z^o are the vector fields defined in the proof of Theorem 2.6. The distribution $\mathcal{D}$ is involutive, since we have (2.24) and the remaining brackets are easily computed from the definitions:

$$[X^o, JZ^o] = X^o = [JX^o, Z^o], \quad [JX^o, JZ^o] = JX^o, \quad [Z^o, JZ^o] = 0.$$

If $\widetilde{L}$ is the maximal integral manifold of $\mathcal{D}$ passing through $(p;v)$, then the proof of Theorem 2.6 shows that $L = \pi(\widetilde{L}) \subset M$ is a Riemann surface locally parametrized by the holomorphic solutions of $D_0(\varphi) = 0$. Since F restricted to L is a complete hermitian metric of constant Gaussian curvature 0, because of Proposition 2.5.(ii), there exists a unique isometric holomorphic covering map $\psi\colon \mathbb{C} \to L$ such that

$\psi(0) = p$ and $\psi'(0) = v$. But the holomorphic solution $\varphi\colon U(\varepsilon) \to L$ of $D_0(\varphi) = 0$ with $\varphi(0) = p$ and $\varphi'(0) = v$ is another holomorphic isometry from the euclidean metric restricted to $U(\varepsilon)$ and F restricted to N; it follows that $\varphi = \psi|_{U(\varepsilon)}$, and necessarily ψ is the unique solution of the Cauchy problem (2.14) — and it has the required properties. ∎

In [AP2] it is shown that in order to have a hyperbolic complex geodesic curve through $(p; v) \in S^{1,0}M$ one does not need to know *a priori* that the Kähler condition holds everywhere; it is enough to know that the torsion tensor T_α vanishes at $(p; v)$. The same result is true for parabolic and elliptic complex geodesics since the proof depends only on the vanishing of the tensor H_α (cf. Proposition 3.8 of [AP2]).

3. A characterization of $\mathbb{C}^n$. We shall see that under some additional hypothesis, flat Kähler-Finsler metrics may be constructed only on $\mathbb{C}^n$.

We consider a complex manifold M of dimension n and we assume that on M is defined a smooth strictly pseudoconvex Finsler metric F such that

(3.1) F is Kähler-Finsler;

(3.2) $$K_F \equiv 0;$$

(3.3) $$H_\alpha(v) = 0 \quad \text{for all } v \in T^{1,0}_p M \text{ and } \alpha = 1, \ldots, n.$$

We start with a general definition. Let N be a differentiable manifold with a Finsler metric defined on it. Assume that for every $p \in N$ and $v \in T_pN$ there exists a geodesic $\gamma_{p,v}\colon (-\epsilon, \epsilon) \to N$ of class C^2 with $\gamma_{p,v}(0) = p$ and $\gamma'_{p,v}(0) = v$. A function $f\colon N \to \mathbb{R}$ is said *geodesically convex* (resp. *strictly geodesically convex*) iff one has $(f \circ \gamma_{p,v})''(t) \geq 0$ (resp. $(f \circ \gamma_{p,v})''(t) > 0$) for every $(p; v) \in TN \setminus \{\text{Zero section}\}$.

It is obvious that this definition may not be meanigful as, in general, one needs some restrictions on the Finsler metric in order to have nice geodesics. But, thanks to the results of the previous section we do not have any problem if (3.1), (3.2) and (3.3) hold.

We shall need the following remark.

LEMMA 3.1. *Suppose that* (3.1), (3.2) *and* (3.3) *are verified for a complex Finsler metric on M. Then every geodesically convex* (*resp. strictly geodesically convex*) *function is strictly plurisubharmonic* (*resp. strictly plurisubharmonic*).

Proof. Let $p \in M$ and $S^{1,0}_p M$. We need to show that if f is geodesically convex (resp. strictly geodesically convex) and L_f is its Levi form, then $L_f(p; v) \geq 0$ (resp. $L_f(p; v) > 0$).

Let $\varphi\colon U(\epsilon) \to M$ be a segment of a parabolic complex geodesic curve through $(p; v)$ which we know to exist because of Corollary 2.8. Then

$$L_f(p; v) = \frac{\partial(f \circ \varphi)}{\partial\zeta\partial\bar\zeta}(0) = \frac{1}{4}\left(\frac{\partial(f \circ \varphi)}{\partial x^2} + \frac{\partial(f \circ \varphi)}{\partial y^2}(0)\right).$$

As φ maps euclidean geodesics of $U(\varepsilon)$ into geodesics of M, the conclusion is immediate. ∎

We shall prove the following

THEOREM 3.2. *Let M be a connected complex manifold of dimension n with a complete strictly pseudoconvex Finsler metric F which satisfies* (3.1), (3.2) *and* (3.3). *If there exists a point $p \in M$ such that the squared distance function τ from p is strictly geodesically convex, then there exists a biholomorphic map $E: \mathbb{C}^n \to M$. Furthermore $\tau \circ E$ is the squared Minkowski functional of a strictly pseudoconvex complete circular domain in $\mathbb{C}^n$.*

Proof. We start with the following remark:

LEMMA 3.3. *The function τ has the following properties*:

(i) *τ in an exhaustion of M with $\tau^{-1}(0) = \{p\}$ and* $\sup \tau = +\infty$.
(ii) $\tau \in \mathbb{C}^\infty(M \setminus \{p\}) \cap \mathbb{C}^0(M)$.

Part (i) is a consequence of the completeness of the metric. Part (ii) follows from the fact that geodesics through p do exist and depend smoothly on the initial direction as it is possible to apply the usual O.D.E. regular dependence on parameters. We omit the details and refer to [AP1] where similar arguments are carried out carefully.

As a consequence of Lemma 3.1 we have that τ is also strictly plurisubharmonic on $M \setminus \{p\}$. The conclusion of the Theorem will be a consequence of results of [P] and [B] provided one may show that $u = \log \tau$ satisfies the complex Monge-Ampère equation $(\partial\bar{\partial}u)^n = 0$ on $M \setminus \{p\}$.

To this end it is enough to show that for any point $q \in M \setminus \{p\}$ there exists a Riemann surface L_q such that $u_{|L_q}$ is harmonic. As F is complete, the geodesics through p fill the entire manifold M and thus the parabolic complex geodesic curves through p fill all M. For $q \in M \setminus \{p\}$ let L_q be the image of one (*a priori* there may be many) such entire curves through p and q: $L_q = \varphi_q(\mathbb{C})$ where φ_q is a parabolic complex geodesic curve through p. By construction $\tau \circ \varphi_q(\zeta) = |\zeta|^2$ and hence $u|_{L_q}$ is harmonic. ∎

We close this section with two remarks. The first is about the hypotheses of Theorem 3.2. Analogies with the usual Kähler geometry suggest that the assumption on the geodesic convexity of the squared distance τ is unnecessary. In fact the assumption that the holomorphic sectional curvature vanishes together with the kählerianity should allow one to control the real sectional curvature of the metric and thus in turn to obtain information about the convexity of the distance function (cf. [GW]). In the same spirit it should be observed that as the parabolic complex geodesic curves through p are exactly the leaves of the Monge-Ampère foliation associated to τ, and as these do not meet in $M \setminus \{p\}$, it follows *a posteriori* that geodesics starting from p have no conjugate points.

The second remark is about the conclusion of the Theorem which leaves an open problem: in fact we do not classify the Finsler metrics. On $\mathbb{C}^n$ one may easily define many Finsler metrics which satisfy the hypotheses of Theorem 3.2, proceeding as follows. Let $\mu\colon \mathbb{C}^n \to \mathbb{R}_+$ be the Minkowski functional squared of a bounded strictly pseudoconvex complete circular domain. Then a complete strictly pseudoconvex complex Finsler metric F_μ satisfying the hypothesis of Theorem 3.2 is defined by

$$F_\mu^2(p;v) = \mu(v)$$

for all $p \in \mathbb{C}^n$ and $v \in \mathbb{C}^n \simeq T_p^{1,0}M$.

Given two such squared Minkowski functionals μ and ν, there exists a biholomorphic map which transforms F_μ in F_ν iff $\nu = \mu \circ A$ for some $A \in GL(n,\mathbb{C})$. It is a natural conjecture that Finsler metrics of this type are the only possible ones which satisfy the hypotheses of Theorem 3.2.

References

[A] M. Abate, *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, Cosenza, 1989

[AP1] M. Abate and G. Patrizio, *Uniqueness of complex geodesics and characterization of circular domains*, Manuscripta Math. 74 (1992), 277–297.

[AP2] —, —, *Holomorphic curvature of Finsler metrics and complex geodesics*, Preprint MPI 92–41, 1992.

[B] D. Burns, *Curvature of Monge-Ampère foliations and parabolic manifolds*, Ann. of Math. 115 (1982), 349–373.

[GW] R. E. Greene and H. Wu, *Function Theory on Manifolds which Posess a Pole*, Springer Lecture Notes 699, Berlin, 1979.

[K] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Dekker, New York, 1970.

[L] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–474.

[P] G. Patrizio, *A characterization of complex manifolds biholomorphic to a circular domain*, Math. Z. 189 (1985), 343–363.

[R] H. L. Royden, *Complex Finsler metrics*, in: Contemporary Mathematics. Proceedings of Summer Research Conference, American Mathematical Society, Providence, 1984, 119–124.

[Ru1] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer, Berlin, 1959.

[V] E. Vesentini, *Complex geodesics*, Compositio Math. 44 (1981), 375–394.

[Wu] H. Wu, *A remark on holomorphic sectional curvature*, Indiana Univ. Math. J. 22 (1973), 1103–1108.