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GENERATION OF $B(X)$ BY TWO COMMUTATIVE SUBALGEBRAS—RESULTS AND OPEN PROBLEMS

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Let X be a real or complex Banach space and let τ be a topology on the algebra $B(X)$ of all endomorphisms of X. For a non-void subset $S \subset B(X)$, let $\mathrm{alg}(S)$ denote the smallest subalgebra of $B(X)$ which contains S, that is, the set of all linear combinations of finite products of elements of S, and let $\mathrm{alg}_\tau(S)$ denote its τ-closure. We shall also consider the uniform (norm) topology on $B(X)$, which we denote by u, and the strong topology, which we denote by s. We say that $B(X)$ is *algebraically generated* by S if $B(X) = \mathrm{alg}(S)$, *uniformly generated* by S if $B(X) = \mathrm{alg}_u(S)$, and *strongly generated* by S if $B(X) = \mathrm{alg}_s(S)$. Our general problem is whether $B(X)$ can be generated by two commutative subalgebras A_1 and A_2, i.e. by $S = A_1 \cup A_2$. More specifically, we consider

Problem 1. Is $B(X)$ algebraically generated by two commutative subalgebras?

and a weaker

Problem 2. Is $B(X)$ uniformly generated by two commutative subalgebras?

The still weaker problem concerning strong generation is already solved in the affirmative and will be considered in Section 2.

In Section 1 we consider results and problems concerning algebraic and uniform generation, in Section 2 we consider strong generation and in Section 3 we give some historical comments concerning these problems and results.

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1. Algebraic generation and uniform generation. We say that a Banach space X is an *n-th power*, and write $X = Y^n$, if X is the direct sum of n mutually isomorphic closed subspaces. Classical Banach spaces which are nth powers include L_p-spaces, the spaces $C(\Omega)$ for Ω metrizable compact, the spaces $C^{(k)}(0,1)$, the disc algebra and others. We say that a subalgebra $A \subset B(X)$ has *square zero* if $TU = 0$ for all T and U in A; such an algebra is clearly commutative. In [18] we proved

1.1. THEOREM. *If $X = Y^n$ for some $n > 1$, then $B(X)$ is algebraically generated by two subalgebras of square zero, one of them being of dimension $n-1$. For $n = 2$ the converse is true: If $B(X)$ is algebraically generated by a subalgebra of square zero and by an operator of square zero (which generates a one-dimensional subalgebra), then $X = Y^2$.*

We tried to prove the converse of the first part of the above result for any $n > 1$. We did not succeed because it is false, as shown by the following result proved by Šemrl [12].

1.2. THEOREM. *If $B(X)$ is algebraically generated by two subalgebras of square zero, then one of them can be chosen to be finite-dimensional, in fact, one-dimensional if X is a square and two-dimensional otherwise.*

Šemrl posed in [12] the following question:

PROBLEM 3. Suppose that $B(X)$ is algebraically generated by two subalgebras of square zero. Does it follow that $X = Y^n$ for some $n > 1$?

By [20], Theorem 1.1 is also true for $n = 2$ with algebraic generation replaced by uniform generation (of course, in this case only the second part of the statement is interesting).

This suggests the following

PROBLEM 4. Suppose that $B(X)$ is uniformly generated by two subalgebras of square zero. Does it follow that it is algebraically generated by two such subalgebras?

Also the answer to the following is not known.

PROBLEM 5. Suppose that $B(X)$ is uniformly generated by two commutative subalgebras. Does it follow that it is algebraically generated by two such subalgebras?

In the case of a Hilbert space H we can algebraically generate $B(H)$ by two commutative subalgebras, one of them being a C^*-algebra (see [18]). This suggests the following

PROBLEM 6. Can $B(H)$ be algebraically or uniformly generated by two commutative C^*-algebras?

Note that not all algebras of the form $B(X)$ can be algebraically or uniformly generated by two subalgebras of square zero. In particular, if $B(X)$ has a non-zero multiplicative functional, then no number of square zero subalgebras can generate $B(X)$. For instance, if $X = J$ is the James space (see [4] or [15]), or if $X = C(\Gamma_{\omega_1})$, where ω_1 is the compact space consisting of all ordinals not greater than ω_1, the smallest uncountable ordinal, then $B(X)$ has a non-trivial multiplicative linear functional (see [5] and [15]). We do not know the answers to Problems 1 and 2 for these particular spaces. In the hope that they may provide counterexamples, we pose the following

PROBLEM 7. Let $X = J$, or $X = C(\Gamma_{\omega_1})$. Can $B(X)$ be uniformly generated by two commutative subalgebras?

If we asked a more general question of whether any Banach algebra can be generated in norm by two commutative subalgebras, the answer would be negative. This follows from the following result obtained by Aniszczyk, Frankiewicz and Ryll-Nardzewski [1].

1.3. THEOREM. *There exists a non-separable Banach algebra such that every commutative subalgebra is separable.*

2. Strong generation. The fact that for any X the algebra $B(X)$ is strongly generated by two commutative subalgebras follows immediately from the following result proved in [19].

2.1. THEOREM. *Assume that* $\dim X > 1$. *Then* $B(X)$ *is strongly generated by two subalgebras of square zero.*

In the case when X is separable a better result is proved in [6].

2.2. THEOREM. *Let* X *be a separable Banach space. Then* $B(X)$ *is strongly generated by two operators.*

In the case when X is a Hilbert space there is a still better result (for references see Section 3).

2.3. THEOREM. *For* H *a separable Hilbert space there exists an operator* T *such that* $B(H)$ *is strongly generated by* T *and* T^*.

This implies that for H separable the algebra $B(H)$ is strongly generated by $T+T^*$ and $i(T-T^*)$, so it is strongly generated by two commutative C^*-algebras. We do not know whether that is true in general, so we pose the following weaker version of Problem 6:

PROBLEM 8. Is the algebra $B(H)$ always strongly generated by two commutative C^*-algebras?

3. Some historical remarks. The first result on strong generation of $B(H)$ seems to be given in [2]. In this paper Chandler Davis answers a question of J. Dieudonné concerning the number of generators of $B(H)$. The result is

3.1. Theorem. *For H separable, the algebra $B(H)$ is strongly generated by two unitary operators.*

Another result of this type is given in [7]:

3.2. Theorem. *For H separable, the algebra $B(H)$ is strongly generated by two hermitian operators.*

In the sixties much attention was devoted to strong generation of von Neumann algebras ([3], [7]–[11], [13], [14], [16]; [8] also gives some results on uniform generation of C^*-algebras). The basic problem here seems to be the question of whether every von Neumann algebra acting on a separable Hilbert space is singly generated, i.e. whether it is strongly generated by some element T and its conjugate T^*. The usual condition imposed on such an algebra A is

$$A \text{ is } *\text{-isomorphic to } A \otimes M_2, \tag{$*$}$$

where M_2 is the algebra of all 2×2 complex matrices (or a similar condition with respect to M_n; the latter is in a certain sense analogous to our condition $X = Y^n$). As a typical result we quote the main result of [11].

3.3. Theorem. *Suppose that a von Neumann algebra A acts on a separable Hilbert space. Suppose also that A satisfies condition* $(*)$. *Then the following are equivalent*:

(a) *A has a single generator*;
(b) *A is generated by two partial isometries*;
(c) *A is generated by two operators*;
(d) *A is generated by two unitary operators*;
(e) *A is generated by three projections.*

Conditions (a) and (d) in combination with Theorem 3.1 give the result of Theorem 2.3.

The first result concerning algebraic and uniform generation of $B(X)$ seems to be given in [17] where Problem 1 (and so Problem 2) is solved in the case where X is a Hilbert space. Other results concerning algebraic and uniform generation are described in Section 1.

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