I am reporting on joint work with Brian Cole and Keith Lewis.

Let $A$ be a uniform algebra on a compact Hausdorff space $X$, i.e., let $A$ be an algebra of continuous complex-valued functions on $X$, closed under uniform convergence on $X$, separating points and containing the constants. Let $\mathcal{M}$ denote the maximal ideal space of $A$. Gelfand’s theory gives that $X$ may be embedded in $\mathcal{M}$ as a closed subset and each $f$ in $A$ has a natural extension to $\mathcal{M}$ as a continuous function. Set $\|f\| = \max_X |f|$.

We consider the following interpolation problem: choose $n$ points $M_1, \ldots, M_n$ in $\mathcal{M}$. Put

$$ I = \{ g \in A \mid g(M_j) = 0, \ 1 \leq j \leq n \}, $$

and form the quotient-algebra $A/I$. $A/I$ is a commutative Banach algebra which is algebraically isomorphic to $\mathbb{C}^n$ under coordinatewise multiplication. For $f \in A$, $[f]$ denotes the coset of $f$ in $A/I$ and $\|[f]\|$ denotes the quotient norm. We put, for $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$,

$$ \mathcal{D} = \{ w \in \mathbb{C}^n \mid \exists f \in A \text{ such that } f(M_j) = w_j, \ 1 \leq j \leq n, \ \text{and } \|[f]\| \leq 1 \}. $$

Our problem is to describe $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is a closed subset of the closed unit polydisk $\Delta^n$ in $\mathbb{C}^n$ and has non-void interior. It turns out that $\mathcal{D}$ has the following property which we call hyperconvexity. We write $\| \|_{\Delta^k}$ for the supremum norm on $\Delta^k$. Let $P$ be a polynomial in $k$ variables and choose $k$ points $w', w'', \ldots, w^{(k)}$ in $\mathbb{C}^n$. We apply $P$ to this $k$-tuple of points, using the algebra structure in $\mathbb{C}^n$. Then

$$ P(w', w'', \ldots, w^{(k)}) = (P(w'_1, w''_1, \ldots, w^{(k)}_1), P(w'_2, w''_2, \ldots, w^{(k)}_2), \ldots) \in \mathbb{C}^n. $$

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The paper is in final form and no version of it will be published elsewhere.
Definition. A compact set $Y$ in $\mathbb{C}^n$ with non-void interior is hyperconvex if whenever $w', w'', \ldots, w^{(k)}$ is a set of points in $Y$, then for every polynomial $P$ in $k$ variables with $\|P\|_{\Delta^n} \leq 1$, $P(w', w'', \ldots, w^{(k)})$ again lies in $Y$.

Theorem 1. For each uniform algebra $A$ and points $M_1, \ldots, M_n$, the set $D$ is hyperconvex. Conversely, every hyperconvex set arises in this way from some $A, M_1, \ldots, M_n$.

Examples of hyperconvex sets occur in the 1916 work of G. Pick [6]. Pick was the first to consider interpolation problems of this type. He fixed an $n$-tuple of points $z_1, \ldots, z_n$ in the unit disk $|z| < 1$.

Let us denote by $D_z$ the set of all points $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$ such that there exists a function $f$ in $H^\infty$ with $\|f\|_{\infty} \leq 1$ and $f(z_j) = w_j$, $1 \leq j \leq n$.

Pick’s Theorem. Let $w \in \mathbb{C}^n$. Then $w \in D_z$ if and only if the matrix

$$
\begin{pmatrix}
1 - w_j z_k \\
1 - z_j w_k
\end{pmatrix}
$$

is positive semi-definite.

We call the set $D_z$ a Pick body. If $A$ is the disk algebra, then $M$ is the closed unit disk and we may take the points $M_j$ to be $z_j$, $1 \leq j \leq n$. It is easy to show that the associated set $D$ coincides with the Pick body $D_z$. In particular, $D_z$ is a hyperconvex set in $\mathbb{C}^n$.

We have not found a geometric condition describing the general hyperconvex set, but we have obtained information in two special cases.

Theorem 2. Each hyperconvex set $Y$ in $\mathbb{C}^2$ is either the bidisk $\Delta^2$ or is a Pick body $D_z$ for some $(z_1, z_2)$. In either case there exists $\lambda$, $0 < \lambda \leq 1$, such that

$$
Y = \left\{(w_1, w_2) \mid |w_1| \leq 1, |w_2| \leq 1, \left| \frac{w_1 - w_2}{1 - w_1 w_2} \right| \leq \lambda \right\}.
$$

Theorem 3. Fix $n$. A compact set $Y$ with non-void interior in $\mathbb{C}^n$ is a Pick body if and only if $Y$ is hyperconvex and $\exists z = (z_1, \ldots, z_n)$ on the boundary of $Y$ such that the powers $z, z^2, \ldots, z^{n-1}$ taken in the algebra $\mathbb{C}^n$ all lie on the boundary of $Y$, and $|z_j| < 1$ for each $j$.

Theorem 2 is proved in [1] and Theorem 3 is proved in [2].

In addition, we have generalized Pick’s theorem to an arbitrary uniform algebra $A$ and points $M_1, \ldots, M_n$, by giving a necessary and sufficient condition on a point $w$ to belong to $D$ in terms of the positive semi-definiteness of a certain family of $n \times n$ matrices. (In Pick’s case, where the algebra was the disk algebra, a single such condition sufficed.) (See [1], and also Nakazi [5] for related results.)

The interpolation problem we are considering is closely related to the so-called von Neumann inequality for operators on Hilbert space.
The first connection between Pick interpolation and operator theory was made in the pioneering paper of Sarason [7]. Recent work in this area is contained in [1], in Lotto [3], and in Lotto and Steger [4].

References