FUNCTIONAL ANALYSIS AND OPERATOR THEORY
BANACH CENTER PUBLICATIONS, VOLUME 30
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1994

PICK INTERPOLATION FOR A UNIFORM ALGEBRA

J. WERMER

Department of Mathematics, Brown University
Providence, Rhode Island 02912-0001, U.S.A.

I am reporting on joint work with Brian Cole and Keith Lewis.

Let A be a uniform algebra on a compact Hausdorff space X, i.e. let A be an algebra of continuous complex-valued functions on X, closed under uniform convergence on X, separating points and containing the constants. Let $\mathcal{M}$ denote the maximal ideal space of A. Gelfand's theory gives that X may be embedded in $\mathcal{M}$ as a closed subset and each f in A has a natural extension to $\mathcal{M}$ as a continuous function. Set $\|f\| = \max_X |f|$.

We consider the following interpolation problem: choose n points $M_1, \ldots, M_n$ in $\mathcal{M}$. Put

$$I = \{g \in A \mid g(M_j) = 0,\ 1 \le j \le n\}\,,$$

and form the quotient-algebra A/I. A/I is a commutative Banach algebra which is algebraically isomorphic to $\mathbb{C}^n$ under coordinatewise multiplication. For $f \in A$, $[f]$ denotes the coset of f in A/I and $\|[f]\|$ denotes the quotient norm. We put, for $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$,

$$\mathcal{D} = \{w \in \mathbb{C}^n \mid \exists\, f \in A \text{ such that } f(M_j) = w_j,\ 1 \le j \le n,\ \text{and } \|[f]\| \le 1\}\,.$$

Our problem is to describe $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is a closed subset of the closed unit polydisk Δ^n in $\mathbb{C}^n$ and has non-void interior. It turns out that $\mathcal{D}$ has the following property which we call *hyperconvexity*. We write $\|\ \|_{\Delta^k}$ for the supremum norm on Δ^k. Let P be a polynomial in k variables and choose k points $w', w'', \ldots, w^{(k)}$ in $\mathbb{C}^n$. We apply P to this k-tuple of points, using the algebra structure in $\mathbb{C}^n$. Then

$$P(w', w'', \ldots, w^{(k)}) = (P(w_1', w_1'', \ldots, w_1^{(k)}), P(w_2', w_2'', \ldots, w_2^{(k)}), \ldots) \in \mathbb{C}^n\,.$$

1991 *Mathematics Subject Classification*: 32E30, 46J15.

The paper is in final form and no version of it will be published elsewhere.

Definition. A compact set Y in $\mathbb{C}^n$ with non-void interior is *hyperconvex* if whenever $w', w'', \ldots, w^{(k)}$ is a set of points in Y, then for every polynomial P in k variables with $\|P\|_{\Delta^n} \le 1$, $P(w', w'', \ldots, w^{(k)})$ again lies in Y.

Theorem 1. *For each uniform algebra A and points $M_1, \ldots, M_n$, the set $\mathcal{D}$ is hyperconvex. Conversely, every hyperconvex set arises in this way from some $A, M_1, \ldots, M_n$.*

Examples of hyperconvex sets occur in the 1916 work of G. Pick [6]. Pick was the first to consider interpolation problems of this type. He fixed an n-tuple of points $z_1, \ldots, z_n$ in the unit disk $|z| < 1$.

Let us denote by $\mathcal{D}_z$ the set of all points $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$ such that there exists a function f in H^∞ with $\|f\|_\infty \le 1$ and $f(z_j) = w_j$, $1 \le j \le n$.

Pick's Theorem. *Let $w \in \mathbb{C}^n$. Then $w \in \mathcal{D}_z$ if and only if the matrix*

$$\left(\frac{1 - w_j \overline{w}_k}{1 - z_j \overline{z}_k}\right)$$

is positive semi-definite.

We call the set $\mathcal{D}_z$ a *Pick body.* If A is the disk algebra, then $\mathcal{M}$ is the closed unit disk and we may take the points M_j to be z_j, $1 \le j \le n$. It is easy to show that the associated set $\mathcal{D}$ coincides with the Pick body $\mathcal{D}_z$. In particular, $\mathcal{D}_z$ is a hyperconvex set in $\mathbb{C}^n$.

We have not found a geometric condition describing the general hyperconvex set, but we have obtained information in two special cases.

Theorem 2. *Each hyperconvex set Y in $\mathbb{C}^2$ is either the bidisk Δ^2 or is a Pick body $\mathcal{D}_z$ for some (z_1, z_2). In either case there exists λ, $0 < \lambda \le 1$, such that*

$$Y = \left\{ (w_1, w_2) \;\middle|\; |w_1| \le 1,\ |w_2| \le 1,\ \left|\frac{w_1 - w_2}{1 - \overline{w}_1 w_2}\right| \le \lambda \right\}.$$

Theorem 3. *Fix n. A compact set Y with non-void interior in $\mathbb{C}^n$ is a Pick body if and only if Y is hyperconvex and $\exists z = (z_1, \ldots, z_n)$ on the boundary of Y such that the powers $z, z^2, \ldots, z^{n-1}$ taken in the algebra $\mathbb{C}^n$ all lie on the boundary of Y, and $|z_j| < 1$ for each j.*

Theorem 2 is proved in [1] and Theorem 3 is proved in [2].

In addition, we have generalized Pick's theorem to an arbitrary uniform algebra A and points $M_1, \ldots, M_n$, by giving a necessary and sufficient condition on a point w to belong to $\mathcal{D}$ in terms of the positive semi-definiteness of a certain family of $n \times n$ matrices. (In Pick's case, where the algebra was the disk algebra, a single such condition sufficed.) (See [1], and also Nakazi [5] for related results.)

The interpolation problem we are considering is closely related to the so-called von Neumann inequality for operators on Hilbert space.

The first connection between Pick interpolation and operator theory was made in the pioneering paper of Sarason [7]. Recent work in this area is contained in [1], in Lotto [3], and in Lotto and Steger [4].

References

[1] B. Cole, K. Lewis and J. Wermer, *Pick conditions on a uniform algebra and von Neumann inequalities*, J. Funct. Anal. 107 (1992), 235–254.

[2] —, —, —, *A characterisation of Pick bodies*, J. London Math. Soc. 48 (1993), 316–328.

[3] B. A. Lotto, *von Neumann's inequality for commuting diagonalizable projections*, preprint, 1991.

[4] B. A. Lotto and T. Steger, *von Neumann's inequality fails for three commuting diagonalizable contractions*, preprint, 1991.

[5] T. Nakazi, *Commuting dilations and uniform algebras*, Canad. J. Math. 42 (1990), 776–789.

[6] G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. 77 (1916), 7–23.

[7] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. 127 (1967), 179–203.