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INVARIANT SUBSPACES AND SPECTRAL MAPPING THEOREMS

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We discuss some results and problems connected with estimation of spectra of operators (or elements of general Banach algebras) which are expressed as polynomials in several operators, noncommuting but satisfying weaker conditions of commutativity type (for example, generating a nilpotent Lie algebra). These results have applications in the theory of invariant subspaces; in fact, such applications were the motivation for consideration of spectral problems. More or less detailed proofs are given for results unpublished before or published in short communications; in some other cases we give a scheme of proof.

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Notations. "Banach algebra" means a unital Banach algebra, and "operator" a bounded linear operator between Banach spaces. The space of all linear operators from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X},\mathcal{Y})$; moreover, $\mathcal{B}(\mathcal{X})=\mathcal{B}(\mathcal{X},\mathcal{X})$, and $\sigma(T)$ is the spectrum of T in $\mathcal{B}(\mathcal{X})$. For a subset $\mathcal{E}$ of a Banach algebra, let $\mathcal{A}_0(\mathcal{E})$ and $L(\mathcal{E})$ denote respectively the subalgebra and the Lie subalgebra generated by $\mathcal{E}$, and $\mathcal{A}(\mathcal{E})$ the closure of $\mathcal{A}_0(\mathcal{E})$; in operator algebras we also consider $\mathfrak{A}(\mathcal{E})$, the closure of $\mathcal{A}_0(\mathcal{E})$ in the weak operator topology. The lattice of all (closed) subspaces of $\mathcal{X}$ which are invariant for operators from a subset $\mathcal{E}$ of $\mathcal{B}(\mathcal{X})$ is denoted by $\operatorname{lat}\mathcal{E}$; on the other hand, $\operatorname{alg}\mathcal{L}$ is the algebra of all operators leaving invariant all subspaces from the set $\mathcal{L}$ of subspaces of $\mathcal{X}$. The commutant of $\mathcal{E}$ is denoted by $\mathcal{E}'$, and the commutator $ab-ba$ of elements a, b by $[a,b]$. For any subsets $\mathcal{E}$, $\mathcal{F}$ of a Banach algebra $\mathcal{A}$ let $\mathcal{D}(\mathcal{E},\mathcal{F})=\{x\in\mathcal{A}:[x,\mathcal{E}]\subset\mathcal{F}\}$ and $\mathcal{D}(\mathcal{E})=\mathcal{D}(\mathcal{E},\mathcal{E})$.

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1. The root algebra. We recall that a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called *transitive* if $\mathcal{M}\xi$ is dense in $\mathcal{Y}$ for each $\xi \in \mathcal{X}$. If $\mathcal{M}$ is a unital subalgebra in $\mathcal{B}(\mathcal{X})$ then transitivity is equivalent to triviality of $\operatorname{lat} \mathcal{M}$.

V. I. Lomonosov [8] proved that any transitive subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{X}, \mathcal{Y})$ has the following property: if T is a nonzero compact operator in $\mathcal{B}(\mathcal{Y}, \mathcal{X})$ then AT has a nonzero eigenvalue for some $A \in \mathcal{M}$. This result implies nontransitivity of the commutant of any compact operator $K in \mathcal{B}(\mathcal{X})$. Indeed, we may suppose that K is quasinilpotent, and so for any $A \in K'$ we have

$$\sigma(AK) \subset \sigma(A)\sigma(K) = \{0\}\,. \tag{1.1}$$

A natural direction of extending the Lomonosov theorem is the search for subspaces invariant for larger algebras than K'. An example of such algebras was introduced in [13]. Let $K \in \mathcal{B}(\mathcal{X})$ and let $\Delta = \Delta_K$ be the inner derivation $X \mapsto [K, X]$ of $\mathcal{B}(\mathcal{X})$. Then $K' = \operatorname{Ker} \Delta$. The subspaces $C_n(K) = \operatorname{Ker} \Delta^n$ are K'-bimodules and more generally

$$C_n(K)C_m(K) \subset C_{n+m-1}(K)\,. \tag{1.2}$$

It follows that $\bigcup_{n=1}^{\infty} C_n(K)$ is an algebra; its weak operator closure $C_\infty(K)$ is called the *root algebra* of K.

The root algebra of a compact operator must have invariant subspaces or contain all compact operators. The second possibility occurs if, for example, K is nilpotent or is similar to an infinite direct sum of nilpotent operators. An example of a nontransitive root algebra is given by the Volterra operator [1]

$$V : f(t) \mapsto \int_0^t f(s)\, ds$$

in $L^2([0;1])$. As proved by Sarason [11], $V' = \mathfrak{A}(V)$. We now describe all $C_n(V)$. Let T be the multiplication operator $f(t) \mapsto tf(t)$ in $L^2([0;1])$. It is connected with V by the "Volterra equation"

$$[T, V] = V^2\,. \tag{1.3}$$

Let $\mathcal{F}_n = \{\sum_{k=0}^{n-1} A_k T^k : A_k \in \mathfrak{A}(V)\}$.

1.1. THEOREM [13]. $C_n(V) = \overline{\mathcal{F}_n}^w$.

Proof. It is clear that $V\mathfrak{A}(V)$ contains all the convolution operators Z_p,

$$Z_p f(t) = \int_0^t f(t-s)p(s)\, ds$$

where p is a polynomial; moreover, $\|Z_p\| \le \|p\|_{L^2[0;1]}$. Two useful facts follow easily: (i) $\overline{V\mathfrak{A}(V)}^w = \mathfrak{A}(V)$ and hence $I = w\text{-}\lim_\lambda E_\lambda V$ for some net $E_\lambda \in \mathfrak{A}(V)$

(1) *Editorial note*: See also pp. 370 and 373 in this volume.

where I is the identity operator, (ii) $V(L^2([0;1])) \subset \mathfrak{A}(V)(1)$ where $1 \in L^2([0;1])$, $1(t) = 1$.

The inclusion $\overline{\mathcal{F}_n}^w \subset C_n(V)$ is a simple consequence of (1.3). To prove the inverse inclusion we use induction. If $X \in C_1(V)$ then $(VX - A)(1) = 0$ for some $A \in \mathfrak{A}(V)$ (due to (ii)) and $VX - A$ commutes with $\mathfrak{A}(V)$. Since 1 is cyclic for $\mathfrak{A}(V)$ it follows that $VX - A = 0$, $VX \in \mathfrak{A}(V)$, $X = w\text{-}\lim_\lambda E_\lambda VX \in \mathfrak{A}(V)$. We thus proved that $C_1(V) = \overline{\mathcal{F}_1}^w$ (this is Sarason's theorem, the proof is included for it is very short; another short proof is in [1]). Suppose that the inclusion $C_n(V) \subset \overline{\mathcal{F}_n}^w$ is proved and take $X \in C_{n+1}(V)$. Setting $A = \Delta^n(X)$ it is not difficult to prove, using (1.3), that

$$\begin{aligned} &\Delta^n(n!V^{2n}X - AT^n) = 0\,, \\ &n!V^{2n}X - AT^n \in \overline{\mathcal{F}_n}^w\,, \\ &V^{2n}X \in \overline{\mathcal{F}_{n+1}}^w\,. \end{aligned}$$

Again applying the net E_λ we get X.

QUESTION 1. It is true that $\overline{\mathcal{F}_n}^w = \mathcal{F}_n$?

It follows immediately from Theorem 1.1 that $C_\infty(V) = \mathfrak{A}(V,T)$. The algebra $\mathfrak{A}(V,T)$ may be described in terms of $\operatorname{lat} V$. It is known that $\operatorname{lat} V = \{L_a : a \in [0;1]\}$ where $L_a = \{f \in L^2 : f(t) = 0 \text{ for } t \le a\}$; hence $\operatorname{alg}\operatorname{lat} V$ is the algebra $\mathfrak{N}$ of all "upper triangular" operators in $L^2([0;1])$ (the Volterra nest algebra). It is not difficult to prove that $\mathfrak{A}(V,T) = \mathfrak{N}$.

1.2. COROLLARY. $C_\infty(V) = \mathfrak{N}$.

Let us consider the next (transfinite) step in the sequence $C_n(K)$, $C_\infty(K)$:

$$C_{\infty+1}(K) = \{X \in \mathcal{B}(\mathcal{X}) : \Delta_K(X) \in C_\infty(K)\}\,.$$

1.3. THEOREM. $C_{\infty+1}(V) = C_\infty(V)$.

Proof. Let $\mathcal{H}$ be the space of all Hilbert–Schmidt operators in $L^2([0;1])$. Assigning to any operator its kernel function we define a bijection $\lambda : \mathcal{H} \to L^2(\Pi)$ where Π is the unit square. Operators in $\mathcal{H} \cap \mathfrak{N}$ correspond to functions $k(t,s) \in L^2(\Pi)$ with $k(t,s) = 0$ for $t < s$. In particular,

$$\lambda(V) = \begin{cases} 1, & t \ge s, \\ 0, & t < s. \end{cases}$$

It follows that $\lambda(XV) = \int_s^1 k(t,\tau)\,d\tau$, $\lambda(VX) = \int_0^t k(\tau,s)\,d\tau$ where $k = \lambda(X)$. So if $X \in \mathcal{H} \cap C_{\infty+1}(V)$ then

$$\int\limits_s^1 k(t,\tau)\,d\tau - \int\limits_0^t k(\tau,s)\,d\tau = 0$$

for all t, s with $t < s$. This implies easily that $k(t,s) = 0$ for $t < s$; in other words, $X \in \mathfrak{N}$. We have proved that $\mathcal{H} \cap C_{\infty+1}(V) \subset \mathfrak{N}$. Now for any $X \in C_{\infty+1}$ we have $VX \in \mathcal{H} \cap C_{\infty+1}(V) \subset \mathfrak{N}$, and $X = w\text{-}\lim_\lambda E_\lambda VX \in \mathfrak{N}$.

The result may be reformulated in the following way:

$$\mathcal{D}(V,\mathfrak{N})=\mathfrak{N} \quad (\text{or } \mathcal{D}(\mathfrak{A}(V),\mathfrak{N})=\mathfrak{N})\,.$$

This strengthens the equality $\mathcal{D}(\mathfrak{N})=\mathfrak{N}$ which is true for any nest algebra.

QUESTION 2. Is every derivation from $\mathfrak{A}(V)$ to $\mathfrak{N}$ inner? What can be said about higher cohomologies?

QUESTION 3. Is it true that $C_{\infty+1}(K)=C_\infty(K)$ for any compact operator K?

QUESTION 4. Does the inclusion $C_\infty(K)\subset \operatorname{alg}\operatorname{lat}K$ hold for any unicellular operator K (that is, such that $\operatorname{lat}K$ is totally ordered by inclusion)?

QUESTION 5. Is it true that $\mathcal{D}(K,\operatorname{alg}\operatorname{lat}K)=\operatorname{alg}\operatorname{lat}K$ for any unicellular (compact) operator K?

E. Kissin [6] showed that $\mathcal{D}(\mathcal{A})\subset\mathcal{D}(\operatorname{alg}\operatorname{lat}\mathcal{A})$ for any subalgebra $\mathcal{A}$ of $\mathfrak{B}(H)$ where H is a Hilbert space. As a corollary, the existence of $\mathcal{D}(\mathfrak{A}(K))$-invariant subspaces for a wide class of compact operators is obtained. For an arbitrary compact operator K the problem of existence of $\mathcal{D}(\mathfrak{A}(K))$-invariant subspaces is unsolved. Let us formulate a weaker problem:

QUESTION 6. Must $\mathcal{D}(\mathfrak{A}(K))$ be nontransitive for any compact operator $K\neq 0$?

Clearly, $\mathcal{D}(\mathfrak{A}(K))\subset C_2(K)$. Even in a finite-dimensional space, $\operatorname{lat}C_2(K)$ can be trivial. But $C_2(K)$ cannot be transitive in this case (see below).

QUESTION 7. Must $C_2(K)$ be nontransitive for any compact operator $K\neq 0$?

The proof of nontransitivity of $C_2(K)$ is a natural first step in the investigation of $\operatorname{lat}C_\infty(K)$. One may suppose that K is quasinilpotent (all spectral subspaces of K are $C_\infty(K)$-invariant). Lomonosov's technique reduces Question 7 to the following spectral problem:

QUESTION 8. Let elements x, a of a Banach algebra satisfy the conditions

$$(*) \qquad [x,[x,a]]=0$$

and $\sigma(x)=0$. Is it true that $\sigma(ax)=0$?

This question turned out to be very productive—it gave rise to a number of other questions and results. We discuss that below.

The condition $(*)$ may be called the "Kleinecke–Shirokov condition" because of their famous theorem [7], [12] stating that $(*)$ implies the quasinilpotence of $[x,a]$.

2. Algebraic approach. One possible way to prove the quasinilpotence of ax (for x, a satisfying $(*)$) is by direct estimation of $\|(ax)^n\|$. For this we may try to represent $(ax)^n$, using $(*)$, as a sum of elements bx^mc with sufficiently large m.

2.1. THEOREM [13]. *Let elements x, a of an arbitrary algebra $\mathcal{A}$ satisfy* $(*)$. *Then*

$$(ax)^{2n-2} \subset \mathcal{A}_0 x^n \mathcal{A}_0 \tag{2.1}$$

for any $n > 1$, where $\mathcal{A}_0 = \mathcal{A}_0(x, a)$.

Proof. Let $U_{n,m}$ be the set of all linear combinations of "monomials" $x^{n_1}a^{m_1}\dots x^{n_k}a^{m_k}$ with $\sum_i n_i = n$ and $\sum_i m_i = m$. We prove the following

LEMMA. *For any r, q with $r + q = 2n - m + 1$,*

$$U_{n,m} \subset x^r\mathcal{A}_0 + \mathcal{A}_0 x^q\,. \tag{2.2}$$

To deduce Theorem 2.1 from the Lemma it is enough to notice that $x(ax)^{2n-3} \in U_{2n-2,2n-3}$ and one may take $r = q = n$ in (2.2):

$$x(ax)^{2n-3} \in x^n\mathcal{A}_0 + \mathcal{A}_0 x^n\,,$$

whence

$$(ax)^{2n-2} \in ax^n\mathcal{A}_0 + \mathcal{A}_0 x^n \subset \mathcal{A}_0 x^n \mathcal{A}_0\,.$$

Proof of the Lemma. $(*)$ easily implies the equality

$$x^k a x^l = \frac{k}{k+l}x^{k+l}a + \frac{l}{k+l}ax^{k+l} \tag{2.3}$$

for any $k, l \in \mathbb{N}$. In particular,

$$x^{q+1}a = (q+1)xax^q - qax^{q+1}\,. \tag{2.4}$$

We use induction on m. For $m=0$ the assertion is obvious. Let $m > 0$ and let w be a monomial in $U_{n,m}$. Three cases are possible:

1) $w = va$. Then $v \in U_{n,m-1}$ and by assumption $v \in x^r\mathcal{A}_0 + \mathcal{A}_0 x^{q+1}$. But $x^{q+1}a \in \mathcal{A}_0 x^q$ by (2.4), so $w \in x^r\mathcal{A}_0 + \mathcal{A}_0 x^q$.

2) $w = vax$. Then $v \in U_{n-1,m-1}$ and by assumption $v = x^r u_1 + u_2 x^{q-1}$ with $u_1, u_2 \in \mathcal{A}_0$. Hence $w = x^r u_1 ax + u_2 x^{q-1}ax$ and, by (2.3),

$$w = x^r u_1 ax + \frac{1}{q}u_2 ax^q + \frac{q-1}{q}u_2 x^q a\,.$$

It remains to notice that $u_2 x^q a$ is a monomial considered in 1).

3) $w = vax^k$, $k > 1$. Then, by (2.4), $w = kvx^{k-1}ax - (k-1)vx^k a$ so w is a sum of monomials considered in 1) and 2).

2.2. COROLLARY. *If $x^n = 0$ then $(ax)^{2n-2} = 0$.*

2.3. COROLLARY. *If $p(x) = 0$ for some polynomial p of degree n then $[a, x]^{2n-1} = 0$.*

Proof. If $x^n = 0$ the assertion follows from the Lemma. The same is true if $(x - \lambda)^n = 0$ for $[a, x] = [a, x - \lambda]$. The general case may be reduced to this one in a usual way (spectral idempotents of x commute with a).

This result may be considered as an algebraic variant of the Kleinecke–Shirokov theorem.

Let us rewrite (2.1) in the form

$$(ax)^{2n-2} = \sum_{i=1}^{N} \lambda_i b_i x^n d_i \tag{2.5}$$

where $\lambda_i \in \mathbb{C}$, and b_i and d_i are monomials. It is clear that

$$\|(ax)^{2n-2}\| \le \|x^n\| \cdot \|x\|^{n-2}\|a\|^{2n-2} \sum_{i=1}^{n} |\lambda_i| \, .$$

Let $\Lambda(n)$ be the infimum of $\sum_{i=1}^{N} |\lambda_i|$ in all variants of (2.5). If $\Lambda(n)$ grows not very fast (for example exponentially) then quasinilpotence of ax follows immediately.

Question 9. Find the asymptotics of $\Lambda(n)$.

It was shown above that the desirable estimate is $\log \Lambda(n) = o(n)$. Is it true at least that $\log \Lambda(n) = O(n \log n)$? The proof of Theorem 2.1 gives $\log \Lambda(n) = O(2^n)$.

It is obvious that $\sum_{i=1}^{N} \lambda_i = 1$ in (2.5). So the question arises whether one can take $\lambda_i \ge 0$. For $n \le 3$ the answer is affirmative, but in the general case it is negative (V. S. Guba [4]). In fact, questions of this type concern the universal algebra $\mathcal{A}(*)$ of the relation $(*)$ ("Kleinecke–Shirokov algebra") defined as the quotient of the free algebra $\mathcal{F}_2$ by the ideal generated by the "word" $[x,[x,a]]$. Let $\mathcal{A}_+(*)$ be the cone in $\mathcal{A}(*)$ generated by monomials.

2.4. Theorem [4]. *If*

$$(ax)^m \in \mathcal{A}_+(*)x^n\mathcal{A}_+(*) \tag{2.6}$$

then $m \ge (n^2-1)/3$.

The proof of this and some other results in [4] is based on the linear monomial (i.e. mapping monomials to monomials) bijection between $\mathcal{A}(*)$ and the algebra of all polynomials in commuting variables $\{x_j\}_{j=1}^{\infty}$ satisfying $(x_{j+1}-x_j)^2 = 0$. The problem of existence of $m = m(n)$ such that (2.6) holds is unsolved even for $n = 4$; it is connected, as proved by V. S. Guba, with rather deep number-theoretic problems.

Question 8 is a special case of a more general problem: for which "polynomials" $p(\alpha,\beta) \in \mathcal{F}_2$ the spectral mapping theorem in a weak form

$$\sigma(p(a,x)) \subset p(\sigma(a)\times\sigma(x)) \tag{2.7}$$

holds if x, a satisfy $(*)$? Indeed, for $p(\alpha,\beta) = \alpha\beta$, (2.7) would imply

$$\sigma(ax) \subset \sigma(a)\sigma(x)\, , \tag{2.8}$$

whence $\sigma(ax) = 0$ if $\sigma(x) = 0$.

The Kleinecke–Shirokov theorem gives (2.7) for $p(\alpha,\beta)=\alpha\beta-\beta\alpha$. But in general, $(*)$ does not imply (2.7). Even (2.8) need not be true, as the following example shows:

$$a=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\qquad x=\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}.$$

So (2.7) follows from $(*)$ only for particular polynomials, and for some wider classes it holds under some additional restrictions on $\sigma(x)$. We describe this phenomenon in more detail, assuming that the algebra $\mathcal{A}_0(a,x)$ is finite-dimensional or, more generally, that x is an algebraic element.

Let $V_{n,m}$ be a homogeneous component of $\mathcal{F}_2$ (the linear span of monomials in α, β with α-degree m and β-degree n), and

$$\mathcal{K}=\sum_{n>m}V_{n,m},\qquad \mathcal{S}=\sum_{2n>m}V_{n,m}$$

2.5. Theorem. *If a polynomial $p(\alpha,\beta)\in\mathcal{F}_2$ satisfies the condition $p(\alpha,\beta-\lambda\cdot 1)\in\mathcal{S}$ for any $\lambda\in\sigma(x)$, then* (2.7) *holds.*

Proof. Using spectral projections we can reduce the general case to a special one, when $\sigma(x)$ is a singleton; replacing x by $x-\lambda\cdot 1$ we may suppose that $\sigma(x)=0$. Since x is an algebraic element, $x^k=0$ for some $k\in\mathbb{N}$. Now $p(\alpha,\beta)\in\mathcal{S}$ implies $p(\alpha,\beta)^{2k}\in\sum_{2n-m\ge 2k}V_{n,m}$; by the Lemma, $p(\alpha,\beta)\in x^k\mathcal{A}_0+\mathcal{A}_0x^k=0$. It follows that

$$\sigma(p(a,x))=\{0\}=p(\sigma(a)\times\sigma(x)).$$

It is not difficult to deduce from $(*)$ that any $p\in\mathcal{F}_2$ can be represented in the form $p(a,x)=\sum_{k\ge 0}x^kp_k(a,[a,x])$ with some $p_k\in\mathcal{F}_2$. Hence it suffices to consider (2.7) for polynomials of the form

$$p(\alpha,\beta)=\sum_k\beta^kp_k(\alpha,[\alpha,\beta]).\tag{2.9}$$

Such a representation is essentially unique (in other words, a and $[a,x]$ generate a free subalgebra in $\mathcal{A}(*)$). Let $\mathcal{B}(*)$ be the set of all $p\in\mathcal{F}_2$ satisfying (2.9) with $p_k\in\mathcal{K}$. It is clear that the image of $\mathcal{B}(*)$ in $\mathcal{A}(*)$ is a subalgebra.

2.6. Corollary. *Formula* (2.7) *holds for all $p\in\mathcal{B}(*)$.*

Proof. Since $p(\alpha,\beta-\lambda\cdot 1)=\sum_k(\beta-\lambda\cdot 1)^kp_k(\alpha,[\alpha,\beta])$ it remains to notice that $p_k(\alpha,\beta)\in\mathcal{K}$ implies $p_k(\alpha,[\alpha,\beta])\in\mathcal{S}$.

It is clear that there exist polynomials $p\notin\mathcal{B}(*)$ for which (2.7) holds, for example $p(\alpha,\beta)=\alpha$.

Question 10. What is the set of all $p\in\mathcal{F}_2$ for which (2.7) holds?

Question 11. Is Corollary 2.6 true if one removes the restriction "x is an algebraic element"?

For example, does $(*)$ imply quasinilpotence of $a[a,x]^2$?

3. Other weakened commutativity conditions. The important special case of $(*)$ is the condition

$$[a,x]=q(x) \tag{3.1}$$

where $q\neq 0$ is a polynomial; it is clear that the Volterra relation (1.3) is a special case of (3.1). Yu. V. Turovskiĭ [19] has proved that (3.1) implies (2.7) for all $p\in\mathcal{F}_2$; in particular, $\sigma(x)=0\Rightarrow\sigma(ax)=0$. We discuss two remarkable generalizations of this result, also obtained in [19].

THEOREM 3.1. *Let $\sigma(x)=0$ and let $\mathcal{M}$ be the algebra generated by all elements $a\in\mathcal{A}$ satisfying* (3.1) *for various q (in other words, $\mathcal{M}=\mathcal{A}_0(\mathcal{D}(\mathcal{A}_0(x)))$). Then $\sigma(bx)=0$ for all $b\in\mathcal{M}$.*

Outline of the proof. It is easy to see that all possible q in (3.1) form an ideal and the minimal polynomial of this ideal is $q(t)=t^k$, $k>1$ (if $[x,a]=x^kq_1(x)$ with $q_1(0)\neq 0$, then $[aq_1(x)^{-1},x]=x^k$). Let $n=k-1$. Then $[a,x]=x^{n+1}$ for some $a\in\mathcal{A}$. We have (by induction) $\Delta_a^m(x^n)=m!n^mx^{(m+1)n}$ for any $m\in\mathbb{N}$. Hence $\|x^{(m+1)n}\|\le\|\Delta_a\|^m\|x^n\|n^{-m}(m!)^{-1}$ so $\|x^{mn}\|^{1/(mn)}\le Cm^{-1/n}$. Now standard arguments show that

$$\|x^m\|^{1/m}=O(m^{-1/n})\,. \tag{3.2}$$

An important tool for the proof is the possibility of realizing the boundary spectrum of any element as the point spectrum of its image under some representation of the algebra. Let $b\in\mathcal{M}$ and $\lambda\in\partial(\sigma(bx))$. Then $bx\xi=\lambda\xi$ for some element $\xi\neq 0$ of a Banach $\mathcal{A}$-module $\mathcal{X}$. So

$$[x^m,b]x\xi+bx^{m+1}\xi=\lambda x^m\xi \tag{3.3}$$

for all $m\in\mathbb{N}$. The numbers $\alpha_m=\|x^m\xi\|$ satisfy $\alpha_m^{1/m}\to 0$ and, consequently, $\alpha_{m_k+1}=o(\alpha_{m_k})$ for a subsequence $m_k\to\infty$. Analysis in [19] shows that (3.2) permits selecting m_k in such a way that also $\|[x^{m_k},b]x\xi\|=o(\alpha_{m_k})$. Hence (3.3) implies $\lambda=0$.

The inclusion (2.7) is a special case of the general "weak" spectral mapping theorem

$$\sigma(p(x_1,\ldots,x_m))\subset p(\sigma(x_1)\times\ldots\times\sigma(x_m)) \tag{3.4}$$

where $p\in\mathcal{F}_m$. The "strong" spectral mapping theorem deals with a joint spectrum $\sigma(x_1,\ldots,x_m)$ of an m-tuple of elements of Banach algebra $\mathcal{A}$:

$$\sigma(p(x_1,\ldots,x_m))=p(\sigma(x_1,\ldots,x_m))\,, \tag{3.5}$$

where p is an n-tuple of "polynomials": $p\in\mathcal{F}_m^n$. We consider Harte's joint spectrum; for any (finite or infinite) family $\{x_i\}_{i\in I}$ it is defined as the set of all families $\{\lambda_i\}_{i\in I}$ of complex numbers such that $\{x_i-\lambda_i 1\}_{i\in I}$ is contained in a proper left or right ideal of $\mathcal{A}$.

3.2. THEOREM [19]. *If elements x, a satisfy* (3.1), *then* (3.5) *holds for all $x_i\in\mathcal{A}_0(x,a)$ and all $p\in\mathcal{F}_m^n$.*

The proof is close in spirit to the proof of Theorem 3.1. A new technical tool is the following fact: any unital Banach algebra has a representation on a Banach space such that the left spectrum of any family of elements coincides with the point spectrum of its image (one can take the adjoint of the regular representation).

As a corollary, $\sigma(a,x)\neq\emptyset$ for x, a satisfying (3.1); on the other hand, $(*)$ does not yield nonvoidness of $\sigma(a,x)$.

Another condition of weakened commutativity, close to $(*)$, is a symmetrized version of $(*)$, considered by R. Harte [5]:

$$[x,[x,a]]=[a,[x,a]]=0\,.$$

More generally, the elements $x_1,\ldots,x_m$ are called *quasicommuting* if $[x_i,[x_j,x_k]] = 0$ for all i, j, k. It was proved in [5] that (3.5) holds for any quasicommuting $x_1,\ldots,x_m$, and any $p\in\mathcal{F}_m^n$. It is clear that quasicommutativity is equivalent to the nilpotence of degree 2 of the generated Lie algebra. Yu. V. Turovskiĭ [18], [20] considered spectral mapping theorems for general nilpotent and finite-dimensional solvable Lie subalgebras of Banach algebras.

3.3. Theorem [18]. *If L is a nilpotent Lie subalgebra of a Banach algebra $\mathcal{A}$, then* (3.5) *is true for any tuple of elements in $\mathcal{A}(L)$.*

3.4. Theorem [20]. *If $L\subset\mathcal{A}$ is a solvable finite-dimensional Lie subalgebra, then the weak spectral mapping theorem* (3.4) *holds in $\mathcal{A}(L)$.*

Notice that (3.1) provides solvability of $L=L(a,x)$. But L can be infinite-dimensional and Theorem 3.4 is not applicable. On the other hand, Theorem 3.2 shows that the spectral properties of $\mathcal{A}_0(L)$ are stronger than for finite-dimensional L.

Question 12. Is (3.5) or at least (3.4) true in $\mathcal{A}(a,x)$ for a, x satisfying (3.1)?

To clarify the relations between Theorems 3.3 and 3.4 let us notice that in an arbitrary Banach algebra $\mathcal{B}$ the validity of (3.4) for all $x_i\in\mathcal{B}$, $p\in\mathcal{F}_m$ is equivalent to commutativity of $\mathcal{B}/\mathrm{rad}\,\mathcal{B}$ (it is sufficient to have (3.4) with $m=2$). Since the condition (3.4) is "almost" independent of extensions of $\mathcal{B}$, Theorem 3.4 actually says that a Banach algebra generated by a finite-dimensional solvable Lie subalgebra is commutative modulo the radical. Theorem 3.3 describes a more subtle phenomenon, for the spectrum of a family of elements essentially depends on the algebra under consideration (even for commutative families).

It was proved in [20] that Theorem 3.3 characterizes nilpotent Lie algebras among solvable ones: for any finite-dimensional solvable Lie algebra L there exist a Banach algebra $\mathcal{A}$ and a Lie homomorphism $\tau: L\to\mathcal{A}$ such that (3.5) does not hold for some m-tuples in $\tau(L)$.

Let us call the element x of a Lie algebra L (*quasi*)*nilpotent* if $\pi(x)$ is (quasi)nilpotent for any representation π of L, and *radical* if $\pi(x)\in\mathrm{rad}(\mathcal{A}(\pi(L)))$.

The following result is due to Yu. V. Turovskiĭ; the proof gives information on methods used in the proof of Theorems 3.3 and 3.4.

THEOREM 3.5. *Let N be an abelian ideal in a Lie algebra L. Then*

(i) $N \cap [L, L]$ *consists of quasinilpotent elements*;
(ii) *if* $\dim N < \infty$ *then* $[L, N]$ *consists of radical elements.*

Proof. (i) It suffices to prove that if L is a Lie subalgebra in a Banach algebra $\mathcal{A} = \mathcal{A}(L)$, then $[L, L] \cap N$ consists of quasinilpotent elements. We prove more: $[\mathcal{A}, \mathcal{A}] \cap N$ consists of quasinilpotent elements. Let $a, b \in \mathcal{A}$, $[a, b] \in N$, $\sigma([a, b]) \neq 0$. Take $\lambda \in \sigma^l([a, b])$, $\lambda \neq 0$. Applying the spectral mapping theorem for commuting families (see [15]) and the theorem on the point realization of left spactra we conclude that there exist a function $\mu : N \to \mathbb{C}$ with $\mu([a, b]) = \lambda$ and a Banach $\mathcal{A}$-module $\mathcal{X}$ such that the subspace $\mathcal{Y} = \{\xi \in \mathcal{X} : x\xi = \mu(x)\xi, x \in N\}$ is nonzero. In particular, $[a, b]\xi = \lambda\xi$ for any $\xi \in \mathcal{Y}$. If we proved that $\mathcal{Y}$ is an $\mathcal{A}$-submodule this would mean that the commutator of two multiplication operators is equal to $\lambda \cdot 1$, a contradiction.

Let $c \in L$. Then $\sigma([x, c]) = 0$ for any $x \in N$ (Kleinecke–Shirokov theorem); so $\mu([x, c]) = 0$. Hence $[x, c]\xi = 0$ for $\xi \in \mathcal{Y}$, $xc\xi = cx\xi = \mu(x)c\xi$, $c\xi \in \mathcal{Y}$, $L\mathcal{Y} \subset \mathcal{Y}$, and $\mathcal{A}\mathcal{Y} \subset \mathcal{Y}$.

(ii) Let us prove first of all that $N_0 \subset \operatorname{rad} \mathcal{A}$, where N_0 is the set of all nilpotent elements in N. Indeed, N_0 is a finite-dimensional subspace in N consisting of quasinilpotents (by commutativity) so $N_0^m = 0$ for some $m \in \mathbb{N}$. Corollary 2.3 implies that N_0 is an ideal in L. Hence $LN_0 \subset N_0 + N_0L$ and $\mathcal{A}_0N_0 = N_0\mathcal{A}_0$ where $\mathcal{A}_0 = \mathcal{A}_0(L)$. So $(\mathcal{A}_0N_0)^m \subset N_0^m\mathcal{A}_0^m = 0$, $(\mathcal{A}N_0)^m = 0$, and $N_0 \subset \operatorname{rad} \mathcal{A}$.

Now consider the adjoint action of L on N. If $[a, x] = \lambda x$ for some $\lambda \neq 0$, then x is nilpotent ($[a, x^n] = n\lambda x^n$ for all n), so $x \in N_0 \subset \operatorname{rad} \mathcal{A}$. Hence on the finite-dimensional space $N/N \cap \operatorname{rad} \mathcal{A}$ the algebra L acts as the Lie algebra of nilpotent operators and we apply Engel's theorem. It shows that $[a_n, [a_{n-1}, \ldots, [a_1, x] \ldots] \in \operatorname{rad} \mathcal{A}$ for some $n \in \mathbb{N}$ and all $a_j \in L$. Hence $[a, c] \in \operatorname{rad} \mathcal{A}$ for $c = [a_{n-1}, [a_{n-2}, \ldots \ldots, [a_1, x] \ldots]$ and all $a \in \mathcal{A}$; in other words, c belongs to the centre modulo the radical. By (i), c is quasinilpotent, so $c \in \operatorname{rad} \mathcal{A}$. Repeating this argument we get $[a_1, x] \in \operatorname{rad} \mathcal{A}$ for all $a_1 \in L$ and as a consequence, for all $a_1 \in \mathcal{A}$. We have proved that $[\mathcal{A}, N] \subset \operatorname{rad} \mathcal{A}$.

With a slightly more transparent argument one can generalize the preceding results to the case of solvable N.

3.6. COROLLARY. *Suppose a Lie subalgebra L of a Banach algebra $\mathcal{A}$ is quasi-solvable* (*that is, L is generated by its finite-dimensional solvable ideals*). *Then $\mathcal{A}(L)$ is commutative modulo radical.*

It should be mentioned that Ş. Frunză [3] proved (in other terms) the non-voidness of the spectrum for any n-tuple of elements generating a solvable Lie algebra.

4. Related results and problems

I. Let us call the family $(a_1, \dots, a_m) \in \mathcal{A}^m$ *Lie-nilpotent* if $L(a_1, \dots, a_n)$ is nilpotent. In the work of A. S. Faĭnshteĭn [2] the famous definition of the Taylor spectrum of a commutative family of operators [16] was extended to Lie-nilpotent families.

In [17] for any module $\mathcal{X}$ over a Lie algebra L the Koszul complex $K(L, \mathcal{X})$ was defined. The family $A_1, \dots, A_m$ of operators on a Banach space $\mathcal{X}$ is called *regular* if $K(L(A_1, \dots, A_m), \mathcal{X})$ is exact; by definition [2], $\sigma_{\mathrm{T}}(A_1, \dots, A_m)$ is the set of all $(\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$ such that $(A_1 - \lambda_1, \dots, A_m - \lambda_m)$ is a regular family. One of the remarkable results in [2] states that for L nilpotent, exactness of $K(L, \mathcal{X})$ is equivalent to triviality of the cohomologies of L with coefficients in $\mathcal{X}$.

4.1. THEOREM [2]. *If $A = (A_1, \dots, A_m)$ is a nilpotent family of operators then*

$$\sigma_{\mathrm{T}}(p_1(A), \dots, p_n(A)) = p(\sigma_{\mathrm{T}}(A)) \tag{4.1}$$

for any $p = (p_1, \dots, p_n) \in \mathcal{F}_m^n$ such that the family $(p_1(A), \dots, p_n(A))$ is Lie-nilpotent.

In particular, (4.1) is true for one polynomial:

4.2. COROLLARY. *The Taylor spectrum $\sigma_{\mathrm{T}}(\cdot)$ has the projection property for Lie-nilpotent families of operators.*

Notice also that if the Lie algebra $L(p_1(A), \dots, p_n(A))$ is finite-dimensional then it is nilpotent. It would be of interest to know whether the condition of nilpotence of this algebra is necessary for (4.1).

II. Among applications of the preceding results to invariant subspaces let us mention the following consequence of Theorem 2.4 (a partial answer to Question 6):

4.3. COROLLARY [20]. *For any compact operator $K \neq 0$ the family $\mathcal{D}(\mathcal{A}_0(K))$ has a nontrivial invariant subspace.*

Recall that the family $\mathcal{E}$ of operators in a Banach space $\mathcal{X}$ is called *triangularizable* if there exists a maximal linear chain of subspaces in $\mathcal{X}$ consisting of $\mathcal{E}$-invariant subspaces. It was proved in [9] that a family $\mathcal{E}$ of compact operators is triangularizable if and only if $\mathcal{A}(\mathcal{E})$ is commutative modulo the radical. Hence Theorems 3.3 and 3.4 imply that every nilpotent or finite-dimensional solvable Lie algebra of compact operators is triangularizable (these results were obtained in another "purely compact" way in [21]). It follows from Corollary 3.6 that every quasisolvable Lie algebra of compact operators is triangularizable. In [20] it was shown that any pair of compact operators satisfying (3.1) is triangularizable. More general results of this kind can also be found in [20].

III. For commuting a, x the inclusion $\sigma(ax) \subset \sigma(a)\sigma(x)$ is obvious. But similar questions for pairwise commuting sets may be difficult.

QUESTION 13. Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{R}$ be a closed subalgebra consisting of quasinilpotent elements. Is it true that $\sigma(a_1x_1+a_2x_2)=0$ if $a_i\in\mathcal{A}$, $x_i\in\mathcal{R}$, $[a_i,x_j]=0$?

Let us also formulate a related problem:

QUESTION 14. Let $\mathcal{R}$ be a radical Banach algebra. For what Banach algebras $\mathcal{M}$ is the projective tensor product $\mathcal{R}\otimes\mathcal{M}$ radical?

It is not difficult to check that Question 13 has a positive answer if $\mathcal{R}$ has the "finite quasinilpotence" property: $r(M)=0$ for any finite subset $M\subset\mathcal{R}$. The spectral radius $r(M)$ is defined as $\limsup_n\|M^n\|^{1/n}$, the norm of a set being the supremum of the norms of its elements.

4.4. THEOREM [14]. *Any radical algebra of compact operators is finitely quasinilpotent.*

One of the corollaries of this result is the existence of a nontrivial subspace invariant for a radical algebra of compact operators and simultaneously for its commutant [14]. Yu. V. Turovskiĭ extended this result in several directions.

4.5. THEOREM [19]. *The union of two triangularizable sets M_1, M_2 of compact operators is triangularizable if M_1, M_2 commute or, more generally, n-commute* ($[\ldots[a_1,a_2],b_1],b_2],\ldots,b_n]=0$ *for any* $a_1\in M_1$, $a_2\in M_2$, $b_j\in M_1\cup M_2$).

In [19] some important Banach-algebraic properties of the map $M\mapsto r(M)$ were proved (for example subharmonicity). One of the applications is the following generalization of the Kleinecke–Shirokov theorem: for any $a\in\mathcal{A}$ the algebra $\Delta_a(\mathcal{A})\cap a'$ is finitely quasinilpotent.

QUESTION 15 ([2]). Is every radical Banach algebra finitely quasinilpotent?

IV. *Two additional remarks on the Kleinecke–Shirokov theorem.*

(i) It is easy to check that $(*)$ implies

$$x^m(ax)=m[x,a]x^m+ax^{m+1}$$

for all $m\in\mathbb{N}$. If x is nilpotent and $\sigma(ax)\neq0$ then (by the point spectra representation theorem) there exists a module $\mathcal{X}$, $\xi\in\mathcal{X}$ and $m\in\mathbb{N}$ such that $ax\xi=\lambda\xi$ ($\lambda\neq0$), $x^{m+1}\xi=0$, $x^m\xi\neq0$. So $m[x,a]x^m\xi=\lambda x^m\xi$, contradicting the quasinilpotence of $[x,a]$. We see that the Kleinecke–Shirokov theorem implies a weakened form of Theorem 2.1 (if x is nilpotent then ax is quasinilpotent). This elegant argument belongs to Yu. V. Turovskiĭ (unpublished).

(ii) In [10] the following question was considered (among others): is it true that $\sigma(b)=0$ if $b=\sum_{i=1}^n[x_i,a_i]$ and $[b,x_i]=0$ for all i? In general the answer is negative (b may be equal to 1) but it is affirmative if $\dim\mathcal{A}<\infty$ and, more generally, if b is a "trace class element", for example a nuclear operator.

([2]) *Editorial note*: This problem is also cited by V. Müller, this volume, p. 262.

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Editorial note: The recent paper by M. Lambrou, W. E. Longstaff and H. Radjavi, *Spectral conditions and reducibility of operator semigroups*, Indiana Univ. Math. J. 41 (1992), 449–464, seems to be closely connected with the problems discussed here. See also the papers of M. Mathieu and G. Murphy in this volume.