FUNCTIONAL ANALYSIS AND OPERATOR THEORY
BANACH CENTER PUBLICATIONS, VOLUME 30
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1994

THE BOUNDARY BEHAVIOUR OF $\mathfrak{S}_p$-VALUED FUNCTIONS ANALYTIC IN THE HALF-PLANE WITH NONNEGATIVE IMAGINARY PART

S. N. NABOKO

Department of Mathematical Physics, Faculty of Physics, St. Petersburg University
St. Petersgoff, Ulyanovskaya 1, 198904 St. Petersburg, Russia

1. Introduction. We consider the class of operator-valued functions T analytic in $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda > 0\}$, whose values $T(\lambda)$ are bounded operators on a separable Hilbert space H, for each $\lambda \in \mathbb{C}_+$. Moreover, we suppose that $\operatorname{Im} T(\lambda) := (T(\lambda) - T^*(\lambda))/(2i) \geq 0$. By analogy with the scalar case ($\dim H = 1$) such functions are called *operator-valued* (o.-v.) *R-functions* [2]. It is easy to check that the resolvent $(A-\lambda)^{-1}$ of a selfadjoint operator on H is an example of an o.-v. R-function:

$$((A-\lambda)^{-1} - [(A-\lambda)^{-1}]^*)/(2i) = [(A-\lambda)^{-1}]^*(\operatorname{Im}\lambda)[(A-\lambda)^{-1}] \geq 0$$

if $\operatorname{Im}\lambda > 0$. The same holds for the so-called "bordered" resolvent $T(\lambda) \equiv V^{1/2}(A-\lambda)^{-1}V^{1/2}$ with $V \geq 0$ selfadjoint ($V \in B(H)$, the Banach space of all linear bounded operators in H with the ordinary norm), which appears in a natural way in perturbation theory for a pair $\{A, A+V\}$ of selfadjoint operators. Namely, by the Hilbert identity, the resolvent of the perturbed operator $A+V$ satisfies

$$(I + V^{1/2}(A-\lambda)^{-1}V^{1/2})V^{1/2}(A+V-\lambda)^{-1} = V^{1/2}(A-\lambda)^{-1}.$$

So the "smoothed" resolvents $V^{1/2}(A+V-\lambda)^{-1}$ and $V^{1/2}(A-\lambda)^{-1}$ are proportional with coefficient $T(\lambda) := I + V^{1/2}(A-\lambda)^{-1}V^{1/2}$ which is an o.-v. R-function in $\mathbb{C}_+$. It is clear that the boundary behaviour of $T(\lambda)$ as λ tends to the real axis $\mathbb{R}$ determines the singularities of the perturbed resolvent $(A+V-\lambda)^{-1}$. In the framework of perturbation theory this leads to a connection between the spectral structure of $A+V$ and the boundary behaviour of $T(\lambda)$ on the real axis

1991 *Mathematics Subject Classification*: Primary 47B10.
The paper is in final form and no version of it will be published elsewhere.

[10, 14, 20, 22, 23]. For example, in scattering theory (the theory of perturbation of the continuous spectrum [14]) the existence and completeness of the so-called wave operators (establishing the unitary equivalence between the absolutely continuous parts of A and $A+V$) can be formulated in terms of the existence of the boundary values of some o.-v. R-functions (see for example [17], [18]). Analogous problems arise in the investigation of the structure of the singular spectrum of selfadjoint operators under smooth perturbation [10, 20, 23].

In what follows we say that the operator-valued function $T(\lambda)$, $\lambda \in \mathbb{C}_+$, *has boundary values* $T(\mathsf{k})$ on $\mathbb{R}$ if for almost all $\mathsf{k} \in \mathbb{R}$ the limit $\lim_{\lambda \to \mathsf{k}} T(\lambda) =: T(\mathsf{k})$ exists in some operator topology as λ tends to k staying in some angular sector in $\mathbb{C}_+$ that is not tangent to the real axis ($\equiv$ nontangential boundary values). The choice of the operator topology will essentially depend on the properties of the operator function. For example, if

$$T(\lambda) = V^{1/2}(A-\lambda)^{-1}V^{1/2}\,, \qquad A = A^*\,,\ V \geq 0\,,\ V \in B(H)\,,$$

then the membership of V in the Schatten–von Neumann class $\mathfrak{S}_p$ leads to the investigation of boundary values in the $\mathfrak{S}_q$-operator topology, $q \geq p$. Here the class $\mathfrak{S}_p$, $p > 0$, consists of all compact operators T on the Hilbert space H such that $\sum_{n=1}^{\infty} s_n(T)^p < \infty$, where the *s-numbers* of the operator T, $s_n(T)$, are the square roots of the eigenvalues $\lambda_n(T^*T)$ of the positive selfadjoint compact operator T^*T, $n = 1, 2, \dots$ [11]. For $p \geq 1$ the class $\mathfrak{S}_p$ is a Banach space with the standard norm

$$\|T\|_{\mathfrak{S}_p} := \Big(\sum_{n=1}^{\infty} s_n(T)^p\Big)^{1/p}.$$

We keep this notation for $p < 1$, when $\mathfrak{S}_p$ is not a normed space but a quasinormed one.

We consider the following general problem. Suppose that the value(s) of an operator-valued R-function $T(\lambda)$ belong(s) at some point (or at any point) of $\mathbb{C}_+$ to a given operator class $\mathfrak{S}$. Can one then conclude that the nontangential boundary values exist a.e. on $\mathbb{R}$, and in what operator topology do the nontangential limits exist? We discuss various classes of operator-valued R-functions and various representations for such functions.

Finally, we note that the problems under investigation are closely connected with other domains of analysis such as perturbation theory for selfadjoint and nonselfadjoint operators [10, 14, 17, 18, 20, 22, 23], scattering theory [3, 17, 18], trace formulas, determinants of o.-v. functions and the so-called characteristic functions of nonselfadjoint operators [28, 29], Volterra operators in Hilbert space [12], the Hilbert transform on different classes of vector-valued functions on $\mathbb{R}$ and $\mathbb{T}$ [5, 6, 7, 9, 13, 24, 26], martingale theory [8] etc. Let us also mention a few earlier papers including some results on the boundary behaviour of analytic Banach-space-valued functions [1, 3, 25, 27, 28]. The proofs of the theorems of the present paper can be found in [19, 21].

2. Basic facts on classes of operator-valued R-functions. First we mention the following formula for an arbitrary o.-v. function $T(\lambda)$ analytic in $\mathbb{C}_+$ with nonnegative imaginary part ($T(\lambda) : H \to H$, $\operatorname{Im}\lambda > 0$):

$$T(\lambda) = A + B\lambda + V^{1/2}(I + \lambda\mathcal{L})(\mathcal{L} - \lambda)^{-1}V^{1/2}|_H \,, \tag{1}$$

where $A = A^*$, $B \geq 0$, $A, B \in B(H)$ [20]. Here the selfadjoint operators $\mathcal{L}$ and V ($V \geq 0$) act in an auxiliary Hilbert space $\mathcal{H} \supset H$ and $V|_{\mathcal{H}\ominus H} = 0$. Moreover, $V \in B(\mathcal{H})$. The last formula is a generalization of both the well-known Riesz–Herglotz theorem [2] on the representation of an arbitrary scalar R-function and of the Sz.-Nagy theorem on dilation [28] of contractions (maximal dissipative operators).

Let us give some motivation. Denoting the spectral measure of the operator $\mathcal{L}$ by E_t, $t \in \mathbb{R}$, we have

$$T(\lambda) = A + B\lambda + \int_{\mathbb{R}} \frac{1 + \lambda t}{t - \lambda}\, d(V^{1/2}E_tV^{1/2})\,.$$

Assuming that $\dim H = 1$ ($H \equiv \mathbb{C}$) we can rewrite this in the form

$$T(\lambda) = a + b\lambda + \int_{\mathbb{R}} \frac{1 + \lambda t}{t - \lambda}\, d\mu(t) \tag{2}$$

with constants $a \in \mathbb{R}$, $b \geq 0$ and a scalar Borel measure $d\mu(t)$ because for $\dim H = 1$ the operator V has rank one. Of course, formula (2) coincides with the Riesz–Herglotz representation for functions analytic in $\mathbb{C}_+$ with positive imaginary part. On the other hand, let us consider a dissipative (for simplicity bounded) operator T on an arbitrary Hilbert space H. Then it is easy to check that the operator-valued function $T(\lambda) = -(T + \lambda)^{-1}$ is an R-function. Indeed,

$$\begin{aligned}\operatorname{Im} T(\lambda) &= [-(T + \lambda)^{-1} + (T^* + \overline{\lambda})^{-1}]/(2i)\\ &= [(T + \lambda)^{-1}]^*(\operatorname{Im} T + \operatorname{Im}\lambda I)(T + \lambda)^{-1} \geq 0\,,\end{aligned}$$

and hence $T(\lambda)$ has a representation of the form (1). The simple asymptotic investigation of $T(\lambda)$ as $\lambda \to +i\infty$ shows that $B = 0$, $A = V^{1/2}\mathcal{L}V^{1/2}$ and $V^{1/2}(I + \mathcal{L}^2)V^{1/2} = P_H$ where P_H is the orthogonal projector on H in $\mathcal{H}$. So we have $-(T + \lambda)^{-1} = P_H(\mathcal{L} - \lambda)^{-1}|_H$, which means that $-\mathcal{L}$ is the so-called selfadjoint dilation [28] of the dissipative operator T in the Hilbert space $\mathcal{H} \supset H$. Note that in this case the spectrum of $\mathcal{L}$ is the whole of $\mathbb{R}$ [28].

Let us rewrite (1) as

$$T(\lambda) = A + (B + V)\lambda + (1 + \lambda^2)V^{1/2}(\mathcal{L} - \lambda)^{-1}V^{1/2}\,.$$

We see that instead of the general case it is enough to investigate operator-valued R-functions of the special form

$$V^{1/2}(\mathcal{L} - \lambda)^{-1}V^{1/2}|_H\,, \quad \text{with } V \geq 0\,,\ V|_{\mathcal{H}\ominus H} = 0\,,\ \mathcal{L} = \mathcal{L}^*\,.$$

The class of all such o.-v. R-functions will be denoted by $R_0(\mathfrak{S})$. Here $\mathfrak{S}$ is an operator class to which V belongs. In any case we will assume that $\mathfrak{S}$ is a linear space invariant under taking the adjoint and has the monotonicity property (i.e. if $T \in \mathfrak{S}$, $T \geq 0$, $S \in B(H)$, $0 \leq S \leq T$, then $S \in \mathfrak{S}$ and $\|S\|_{\mathfrak{S}} \leq \|T\|_{\mathfrak{S}}$). Of course all the classes $\mathfrak{S}_p$, $p > 0$, have these properties. The class $R_0(\mathfrak{S})$ is a natural generalization of the class R_0 of scalar functions [2]. It is possible to give an equivalent description of $R_0(\mathfrak{S})$ without leaving the initial Hilbert space H. Namely, $R_0(\mathfrak{S})$ coincides with the set of all $\mathfrak{S}$-valued R-functions $T(\lambda)$ satisfying

1) $w\text{-}\lim_{\tau\to\infty} T(i\tau) = 0$,

together with one of the following equivalent conditions [20]:

$2)_1$ $\int_{\mathbb{R}} \operatorname{Im} T(\mathsf{k} + i\varepsilon)\, d\mathsf{k} \in \mathfrak{S}$, $\varepsilon > 0$;
$2)_2$ $\sup_{\tau>0} \tau \|\operatorname{Im} T(i\tau)\|_{\mathfrak{S}_p} < \infty$ for $\mathfrak{S} = \mathfrak{S}_p$, $0 < p < \infty$;
$2)_3$ $\operatorname{tr} T(\lambda) \in R_0$ for $\mathfrak{S} = \mathfrak{S}_1$;
$2)_4$ $\sup_{\tau>0} \tau \|T(i\tau)\|_{\mathfrak{S}} < \infty$ for $\mathfrak{S} = \mathfrak{S}_p$, $p > 0$;
$2)_5$ there exist a selfadjoint operator $\mathcal{L}$ on a Hilbert space $\mathcal{H}$ and a bounded operator $R : H \to \mathcal{H}$ such that $R^*R \in \mathfrak{S}$ and

$$T(\lambda) = R^*(\mathcal{L} - \lambda)^{-1} R\,, \qquad \operatorname{Im}\lambda > 0\,.$$

In what follows we restrict our attention to this more special class $R_0(\mathfrak{S})$. This restriction is most convenient for formulating the theorems in terms of the "perturbation" V (see §1). Naturally, all statements can be easily modified in the general case of an arbitrary $\mathfrak{S}$-valued R-function. In Section 1 it was shown that o.-v. R-functions are directly connected with perturbation theory for selfadjoint and dissipative operators. It is very easy to extend the results about the boundary behaviour of o.-v. R-functions to the more general class of o.-v. functions (not necessarily R-functions) of the form $R_2^*(\mathcal{L} - \lambda)^{-1} R_1$, where $R_{1,2}$ are bounded operators from H into $\mathcal{H}$ such that $R_1^*R_1, R_2^*R_2 \in \mathfrak{S}$. We can reduce this case to the preceding one by using the well-known connection between bilinear and quadratic forms [21].

3. Boundary behaviour of $\mathfrak{S}_p$-valued R-functions, $p \neq 1$. Let us consider the case $\mathfrak{S} = \mathfrak{S}_p$, $p \neq 1$. The more complicated case $\mathfrak{S} = \mathfrak{S}_1$ will be examined later. We first prove that for $\mathfrak{S}_p$-valued ($p > 1$) R-functions the boundary values (even radial boundary values) in general do not exist, in the sense defined below. Actually, we prove a sharper result than in [19].

THEOREM 1. *Let $V \geq 0$ be an arbitrary selfadjoint compact operator on H which does not belong to the nuclear class $\mathfrak{S}_1$. Then there exists a selfadjoint operator L on H such that for the operator-valued R-function $T(\lambda) := V^{1/2}(L - \lambda)^{-1}V^{1/2}$ its "weak boundary values" $T(\mathsf{k}+i0)$ are unbounded operators in H for a.e. $\mathsf{k} \in \mathbb{R}$. The "weak boundary values" are understood in the following sense. For a fixed dense set of vectors $\varphi \in H$ the limit*

$$(T(\mathsf{k} + i0)\varphi, \varphi) := \lim_{\varepsilon\to+0} (T(\mathsf{k} + i\varepsilon)\varphi, \varphi)$$

exists.

This theorem says that for every class $\mathfrak{S}$ such that $\mathfrak{S}\setminus\mathfrak{S}_1 \neq \emptyset$ we cannot assert the existence of the boundary (radial) values for a.e. $\mathsf{k} \in \mathbb{R}$ even in the weak sense. Of course this theorem is closely connected with the Weyl–von Neumann theorem on transformation of the spectrum under nonnuclear perturbation.

Proof of Theorem 1. Let $\{s_n\}_{n=1}^{\infty}$, $s_n \downarrow 0$, $\sum_n s_n = \infty$, be the sequence of s-numbers of V ($s_n \equiv s_n(V)$). Then obviously there exists a decreasing sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\gamma_n \to 0$ as $n \to \infty$ (very slowly) but $\sum_n \gamma_n s_n = \infty$. Consider an infinite-to-one (at any point) covering of the real axis by intervals Δ_n of length $\gamma_n s_n$, $n = 1, 2, \ldots$ In view of the divergence of the series $\sum_n \gamma_n s_n$ it is very easy to find such a covering. Define an o.-v. R-function by

$$T(\lambda) = \sum_{n=1}^{\infty} \frac{s_n}{\mu_n - \lambda} P_n \,,$$

where μ_n is the midpoint of the interval Δ_n and P_n is the spectral projector of the nonnegative operator V corresponding to the eigenvalue s_n, $n = 1, 2, \ldots$ We have the representation (1) with $\mathcal{H} = H$, and $\mathcal{L} \equiv L = \sum_n \mu_n P_n$ is a selfadjoint operator with pure point spectrum. Choosing for the dense set of vectors φ the set of all finite linear combinations of arbitrary vectors from the ranges of the operators P_n we obtain

$$T(\mathsf{k} + i0) = \sum_{n=1}^{\infty} s_n(\mu_n - \mathsf{k})^{-1} P_n$$

for every $\mathsf{k} \in \mathbb{R} \setminus \{\mu_1, \mu_2, \ldots\}$. The linear operator $T(\mathsf{k} + i0)$ is densely defined at least on the same set of vectors φ. Fix $\mathsf{k} \in \mathbb{R} \setminus \{\mu_1, \mu_2, \ldots\}$. Then there exists a sequence $n_i \to \infty$ as $i \to \infty$ such that $\mathsf{k} \in \Delta_{n_i}$, $i = 1, 2, \ldots$, because our covering is infinite-to-one. Then

$$\|T(\mathsf{k} + i0)\| \geq s_{n_i}/|\Delta_{n_i}| = 1/\gamma_{n_i} \to \infty \qquad \text{as } i \to \infty \,,$$

which finishes the proof.

On the other hand, if $p < 1$ then for every $\mathfrak{S}_p$-valued R-function the nontangential boundary values in the $\mathfrak{S}_p$-"norm" exist a.e. on $\mathbb{R}$. This fact is an operator analogue of Kolmogorov's theorem on the Hardy classes H^p for $p < 1$ [15]. We give the precise formulation [19].

Theorem 2. *Let $T(\lambda)$ be an arbitrary $\mathfrak{S}_p$-valued R-function in $\mathbb{C}_+$, $0 < p < 1$. Then for almost all $\mathsf{k} \in \mathbb{R}$ the nontangential boundary values $T(\mathsf{k})$ ($\equiv T(\mathsf{k} + i0)$) exist in $\mathfrak{S}_p$, with the nontangential limit understood in the $\mathfrak{S}_p$-"norm", and*

$$\begin{aligned} A_p \int_{\mathbb{R}} \|T(\mathsf{k})\|_{\mathfrak{S}_p}^p \frac{d\mathsf{k}}{\mathsf{k}^2 + 1} &\leq \int_{\mathbb{R}} \|T^p(\mathsf{k})\|_{\mathfrak{S}_1} \frac{d\mathsf{k}}{\mathsf{k}^2 + 1} \\ &\leq C_p \|T(i)\|_{\mathfrak{S}_p}^p \equiv C_p \|A + (B + V)i\|_{\mathfrak{S}_p}^p \end{aligned}$$

for some constants A_p, C_p depending only on p, $C_p = O(1/(1-p))$, $0 < p < 1$, and $A_p = a^{1/p}$ where a is an absolute constant.

This theorem not only coincides with Kolmogorov's theorem if $\dim H = 1$ but also, in the general case, its proof is essentially an operator analogue of Smirnov's well-known proof of Kolmogorov's theorem. One just has to add a few elementary facts concerning the s-numbers of a compact operator and the so-called Matsaev–Palant inequality for powers of a dissipative operator [16].

4. $\mathfrak{S}_1$-valued R-functions. Preliminary facts. In the preceding section we saw that there is a "jump" when the parameter p goes through 1. It turns out that this jump is very sharp. Namely, every $\mathfrak{S}_1$-valued R-function has boundary values a.e. on $\mathbb{R}$.

THEOREM 3. *Let $T(\lambda)$ be an arbitrary operator-valued function in $R_0(\mathfrak{S}_1)$. Then $T(\lambda)$ has nontangential boundary values a.e. on $\mathbb{R}$ in the $\mathfrak{S}_p$-norm for all $p > 1$, and the boundary values $T(\mathsf{k})$ satisfy the estimate*

$$\int\limits_{\mathbb{R}} \|T(\mathsf{k})\|_{\mathfrak{S}_p}^p |\eta(\mathsf{k})|^p \, d\mathsf{k} \le C_p \|V\|_{\mathfrak{S}_1} ,$$

where $T(\lambda) = V^{1/2}(\mathcal{L} - \lambda)^{-1} V^{1/2}|_H$, $V \in \mathfrak{S}_1$ and the a.e. nonzero weight function η is the boundary value of the scalar analytic contractive function $\eta(\lambda) := \det((I + S(\lambda))/2)$. Here

$$S(\lambda) := (I + iV^{1/2}(\mathcal{L} - iV/2 - \lambda)^{-1} V^{1/2})|_H$$

is the so-called characteristic function [28] *of the maximal dissipative operator $\mathcal{L} + iV/2$ in $\mathcal{H}$.*

On the other hand, the boundary values of a $\mathfrak{S}_1$-valued R-function in general do not belong to $\mathfrak{S}_1$. Moreover, the boundary values belong to no class better than $\mathfrak{S}_\Omega$, the so-called adjoint Matsaev class [11]. The symmetrically normed ideal $\mathfrak{S}_\Omega$ consists of all compact operators T whose s-numbers satisfy the condition

$$\|T\|_{\mathfrak{S}_\Omega} := \sup_{n \ge 1} \Big(\sum_{k=1}^{n} s_k(T) \Big) \Big/ \Big(\sum_{k=1}^{n} 1/k \Big) < \infty .$$

Namely, for every sequence of $\{c_n\}_{n=1}^{\infty}$ of positive numbers with $c_n \to 0$ as $n \to \infty$, it is possible [19] to construct an example of an operator function $T(\lambda) \in R_0(\mathfrak{S}_1)$ such that for a.e. $\mathsf{k} \in \mathbb{R}$ we have

$$\limsup_{n \to \infty} \sum_{k=1}^{n} s_k(T(\mathsf{k}))/(c_n \ln n) = \infty .$$

In particular, $T(\mathsf{k})$ does not belong to the class $\mathfrak{S}_1$ for almost all $\mathsf{k} \in \mathbb{R}$. The construction of the example uses similar ideas to the construction of $T(\lambda)$ in the proof of Theorem 1. The assertion of Theorem 3 is closely connected with scattering theory (perturbation theory for continuous spectrum). See [3] where the proof of the existence (in $\mathfrak{S}_2$) of the boundary values for an arbitrary o.-v.

function from $R_0(\mathfrak{S}_1)$ leads to the construction of a trace class version of abstract scattering theory.

5. Trace class valued R-functions: more details about the boundary behaviour. It was mentioned earlier that the boundary values of a $\mathfrak{S}_1$-valued R-function $T(\lambda)$ in general do not belong to $\mathfrak{S}_1$. But for almost all $\mathrm{k} \in \mathbb{R}$ the boundary values $T(\mathrm{k})$ are in $\bigcap_{p>1} \mathfrak{S}_p$. Here we consider some results making this fact more precise. In what follows we mainly consider the nontangential boundary limits, but all results are valid (in appropriate sense) for the boundary values.

THEOREM 4. *If $T(\lambda)$ is an arbitrary $\mathfrak{S}_1$-valued R-function in $\mathbb{C}_+$, then the series*

$$\sum_n s_n(T(\lambda))\beta(|\ln s_n(T(\lambda))|)$$

is nontangentially bounded a.e. on $\mathbb{R}$ for every decreasing positive function $\beta \in L_1(\mathbb{R}_+)$, $\beta(0) < \infty$.

This result leads to a complete solution of our problem in the so-called "Lorentz classes" $\mathfrak{S}_\pi$. Here $\mathfrak{S}_\pi$ [11] is the Banach space of all compact operators T (in the Hilbert space H) whose s-numbers satisfy

$$\|T\|_{\mathfrak{S}_\pi} := \sum_{k=1}^{\infty} s_k(T)\pi_k < \infty$$

($\pi \equiv \{\pi_n\}_{n=1}^{\infty}$ is a sequence decreasing to 0).

THEOREM 5. *The condition $\sum_n \pi_n/n < \infty$ is necessary and sufficient for an arbitrary $\mathfrak{S}_1$-valued R-function to have nontangential boundary limits a.e. on $\mathbb{R}$ in the class $\mathfrak{S}_\pi$.*

Finally, we state an assertion concerning the existence of the boundary limits in the "Marcinkiewicz classes" $\mathfrak{S}_\Pi$. The Banach space $\mathfrak{S}_\Pi$ is the class of all compact operators T whose s-numbers satisfy

$$\|T\|_{\mathfrak{S}_\Pi} := \sup_n \Big(\sum_{k=1}^{n} s_k(T)\Big)\Big/\Big(\sum_{k=1}^{n} \pi_k\Big) < \infty\,.$$

As a Banach space, $\mathfrak{S}_\Pi$ is dual to the Banach space $\mathfrak{S}_\pi$ with the same sequence π [11].

THEOREM 6. *Let $T(\lambda)$ be an arbitrary $\mathfrak{S}_1$-valued R-function, and B_n an increasing sequence such that $B_n \to \infty$ as $n \to \infty$ and $\sum_n 1/(nB_n) < \infty$. Then*

1) *the nontangential boundary limits of $T(\lambda)$ exist for a.e. $\mathrm{k} \in \mathbb{R}$ in the norm* $\sup_n(\sum_{k=1}^{n} s_k(T))/B_n$;

2) *for every nonnegative sequence a_n with $\sum_n a_n < \infty$,*

$$\sum_{n\ge 2} a_n\Big(\sum_{k=1}^{n} s_k(T(\mathrm{k}))\Big)/\ln n < \infty \quad \textit{for a.e. } \mathrm{k} \in \mathbb{R}\,;$$

3) *for a.e.* $\mathrm{k} \in \mathbb{R}$,

$$\liminf_{n\to\infty} n s_n(T(\mathrm{k})) = \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} s_k(T(\mathrm{k}))}{\ln n} = 0\,.$$

This theorem should be compared with the results in Section 3 concerning the construction of examples of $\mathfrak{S}_1$-valued R-functions whose boundary values do not belong to the class $\mathfrak{S}_{\Pi}$, where $\pi_n = c_n/n$ and $c_n \downarrow 0$ arbitrarily slowly as $n \to \infty$.

References

[1] K. Asano, *Notes on Hilbert transforms of vector valued functions in the complex plane and their boundary values*, Publ. Res. Inst. Math. Sci. 13 (1967), 572–577.

[2] F. V. Atkinson, *Discrete and Continuous Boundary Value Problems*, Russian edition, Mir, Moscow, 1968 (additional chapters devoted to R-functions written by M. G. Kreĭn and I. Kats).

[3] M. S. Birman and S. B. Entina, *A stationary approach to abstract scattering theory*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 401–430 (in Russian); English transl. in Math. USSR-Izv. 1 (1967).

[4] M. S. Birman and D. R. Yafaev, *Spectral properties of the scattering matrix*, Algebra i Analiz 4 (6) (1992), 1–27 (in Russian).

[5] J. Bourgain, *Vector valued singular integrals and the H^1-BMO duality*, Israel Sem. Geom. Aspects Funct. Anal. (1983/84), XVI, Tel Aviv Univ., 1984, 23 pp.

[6] J. Bourgain and W. J. Davis, *Martingale transforms and complex uniform convexity*, Trans. Amer. Math. Soc. 294 (1986), 501–515.

[7] A. V. Bukhvalov, *Continuity of operators acting in spaces of vector-valued functions, applications to the theory of bases*, Zap. Nauchn. Sem. LOMI 157 (1987), 5–22 (in Russian).

[8] D. L. Burkholder, *Martingale transforms and the geometry of Banach space*, in: Lecture Notes in Math. 860, Springer, 1981, 35–50.

[9] —, *A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions*, in: Conf. Harmonic Anal. in Honor of A. Zygmund (Chicago, 1981), Wadsworth, Belmont, Calif., 1983, 270–286.

[10] L. D. Faddeev, *On the Friedrichs model in the theory of perturbations of the continuous spectrum*, Trudy Mat. Inst. Steklov. 73 (1964), 292–313 (in Russian); English transl. in Amer. Math. Soc. Transl. (2) 62 (1967).

[11] I. Ts. Gokhberg and M. G. Kreĭn, *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space*, Amer. Math. Soc., Providence, R.I., 1969.

[12] —, *The Theory of Volterra Operators in a Hilbert Space*, Amer. Math. Soc., Providence, R.I., 1970.

[13] J. A. Gutierrez and H. E. Lacey, *On the Hilbert transform for Banach valued functions*, in: Lecture Notes in Math. 939, Springer, 1982, 73–80.

[14] T. Kato, *Perturbation Theory for Linear Operators*, Springer, 1966.

[15] P. Koosis, *Introduction to H^p-Spaces*, Cambridge Univ. Press, 1980.

[16] V. I. Matsaev and Yu. A. Palant, *On the powers of a bounded dissipative operator*, Ukrain. Mat. Zh. 14 (1962), 329–337 (in Russian).

[17] S. N. Naboko, *A functional model of perturbation theory and its applications in scattering theory*, Trudy Mat. Inst. Steklov. 147 (1980), 84–114 (in Russian); English transl. in Proc. Steklov Inst. Math. 1981, (2).

[18] —, *On conditions for the existence of the wave operators in the nonselfadjoint case*, in: Probl. Mat. Fiz. 12, Leningrad Univ., 1987, 132–155 (in Russian).

[19] S. N. Naboko, *On the boundary values of analytic operator-valued functions with positive imaginary part*, Zap. Nauchn. Sem. LOMI 157 (1987), 55–69 (in Russian).

[20] —, *Uniqueness theorems for operator-valued functions with positive imaginary part and the singular spectrum in the selfadjoint Friedrichs model*, Ark. Mat. 25 (1987), 115–140.

[21] —, *Nontangential boundary values of operator-valued R-functions in a half-plane*, Algebra i Analiz 1 (5) (1989), 197–222 (in Russian); English transl.: Leningrad Math. J. 1 (5) (1990), 1255–1278.

[22] B. S. Pavlov and L. D. Faddeev, *Zero sets of operator functions with a positive imaginary part*, in: Lecture Notes in Math. 1043, Springer, 1984, 124–128.

[23] B. S. Pavlov and S. V. Petras, *On the singular spectrum of a weakly perturbed multiplication operator*, Funktsional. Anal. i Prilozhen. 4 (2) (1970), 54–61 (in Russian); English transl. in Functional Anal. Appl. 4 (1970).

[24] J. L. Rubio de Francia, *Fourier series and Hilbert transforms with values in UMD Banach spaces*, Studia Math. 81 (1985), 95–105.

[25] R. Ryan, *Boundary values of analytic vector valued functions*, Proc. Nederl. Akad. Wetensch. Ser. A 65 (1962), 558–572.

[26] J. Schwartz, *A remark on inequalities of Calderón–Zygmund type for vector-valued functions*, Comm. Pure Appl. Math. 14 (1961), 785–799.

[27] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[28] B. Sz.-Nagy and C. Foiaş, *Analyse Harmonique des Opérateurs de l'Espace de Hilbert*, Masson, Paris, 1967.

[29] V. F. Veselov and S. N. Naboko, *The determinant of a characteristic function and the singular spectrum of a nonselfadjoint operator*, Mat. Sb. 129 (171) (1986), 20–29 (in Russian); English transl. in Math. USSR-Sb. 57 (1987).