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WEIGHTED CONVOLUTION ALGEBRAS AND THEIR HOMOMORPHISMS

SANDY GRABINER

Department of Mathematics, Pomona College
Claremont, California 91711-6348, U.S.A.

1. Introduction. Weighted convolution algebras on $\mathbb{R}^+ = [0, \infty)$ arise naturally in the study of semigroups of operators and also as important examples of Banach algebras, as is already clear in the classical books by Hille and Phillips [HP, Chapters 4 and 15] and Gelfand, Raikov, and Shilov [GRS, Chapter 3]. Much of the theory of these algebras shows a strong interplay with the study of semigroups of operators.

Suppose that $U(t)$ is a nonnilpotent semigroup of bounded operators on a Banach space X, and let $\omega(t) = \|U(t)\|$. We let $L^1(\omega)$ be the Banach space of (equivalence classes of) measurable functions f on $\mathbb{R}^+$ for which $f\omega \in L^1(\mathbb{R}^+)$ with the inherited norm

$$\|f\|_\omega = \|f\omega\|_1 = \int_0^\infty |f(t)|\omega(t)\, dt\,. \tag{1.1}$$

We can then define the *semigroup operational calculus* by the continuous linear map $\phi : L^1(\omega) \to B(X)$ given by

$$\phi(f)x = \int_0^\infty f(t)U(t)x\, dt \tag{1.2}$$

for x in X. Of course we need to know that the above integral makes sense, at least as a Bochner integral. Actually, as soon as we assume that all $U(t)x$ are strongly measurable, it follows that $U(t)x$ is continuous for $t > 0$ [HP, Th. 10.2.3, p. 305] and hence $\omega(t) = \sup_{\|x\|=1} \|U(t)x\|$ is lower semicontinuous. Thus the operational calculus integral of formula (1.2) is well defined. In fact, when $f(t)$ is piecewise continuous, the integral is an improper Riemann integral.

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Since $U(t)$ is a semigroup, it follows that $\omega(t)$ is *submultiplicative*; that is, $\omega(s+t) \le \omega(s)\omega(t)$ for all s and t in $\mathbb{R}^+$. Hence $L^1(\omega)$ is a Banach algebra under the usual convolution multiplication

$$f * g(x) = \int_0^x f(x-t)g(t)\,dt\,. \tag{1.3}$$

In this case, the operational calculus map of formula (1.2) is a continuous homomorphism from $L^1(\omega)$ into $B(X)$, the algebra of bounded operators on X [HP, Th. 15.2.1, p. 436]. The proof of this is exactly the same as for the convolution property of the Laplace transform. The semigroup operational calculus has proved particularly useful in Banach algebra theory because Sinclair has shown that every Banach algebra A with approximate identity has a uniformly continuous semigroup a^t with $\lim_{t\to 0^+} a^t x = x$ for all x in A ([Si1], [Si2]). The resulting homomorphism, given by formula (1.2), often allows one to use the structure of the domain space $L^1(\omega)$ to study the semigroup a^t and the algebra A ([Si2], [Es]).

Recall that the most important semigroups $U(t)$ of operators are the *strongly continuous semigroups*; that is, $U(t)x$ is continuous for $t \ge 0$, for all x in X. We will often need a weaker property, so we say that the semigroup $U(t)$ is *almost continuous* if it is strongly continuous for $t > 0$ and has $\omega(t) = \|U(t)\|$ bounded as $t \to 0^+$. This means that the weight $\omega(t)$ is locally bounded on $\mathbb{R}^+$. The most useful conditions on the weights seem to be:

Definition (1.4). Suppose that $\omega(t)$ is a positive Borel function on $\mathbb{R}^+$ with both $\omega(t)$ and $1/\omega(t)$ bounded on compact sets. Then $\omega(t)$ is an *algebra weight* if it is submultiplicative, right continuous, and has $\omega(0) = 1$.

The conditions in the above definition are just a convenient normalization. Whenever $L^1(\omega)$ is an algebra under convolution and $\omega(t)$ is bounded as $t \to 0^+$, we can always find an algebra weight $\overline{\omega}(t)$ for which $L^1(\overline{\omega})$ is just $L^1(\omega)$ under an equivalent norm ([Gr2, Th. 3.1, p. 538], [Gr5, Th. 2.1, p. 591]). For weights coming from a semigroup $U(t)$ on X, we can also renorm X so that the weight $\|U(t)\|$ is strongly algebraic (for the details, see Lemma (2.1) and the proof of Th. 2.4 in [Gr4]). From now on, whenever $L^1(\omega)$ is an algebra, we will assume the weight to be normalized to satisfy the conditions in Definition (1.4).

The two most important cases of convolution algebras $L^1(\omega)$ are the classical case $L^1(\mathbb{R}^+)$, that is, $\omega(t) \equiv 1$, and the cases where $L^1(\omega)$ is a radical Banach algebra, which, as is well known, happens precisely when $\lim_{t\to\infty} \omega(t)^{1/t} = \inf_{t>0} \omega(t)^{1/t} = 0$ (see Theorem (2.5), below, or [HP, pp. 148–149] or [Da2, Th. 4.4, p. 189]). The simplest examples of algebra weights are $\omega(t) = e^{-\phi(t)}$, where $\phi(t)$ is a convex function on $\mathbb{R}^+$ with $\phi(0) = \phi(0^+) = 0$. For some general conditions on $\omega(t)$ that imply that $L^1(\omega)$ is an algebra see [Gr2, Section 2], and see [HP, Chapter 7], which also discusses the properties of submultiplicative functions.

Together with Banach algebras of power series and the *Volterra algebra* (that is, $L^1[0,1]$ under convolution), the radical $L^1(\omega)$ have been the standard examples of radical Banach algebras. They provide key examples in Esterle's classification of commutative radical Banach algebras [Es], and they play a key role in automatic continuity questions ([LB], [Da1, Section 7]). Results first proved for the Volterra algebra or radical Banach algebras of power series often served as a source of fruitful conjectures about the more complicated $L^1(\omega)$ algebras (see [Gr3] for an explicit discussion of some of these analogies). This was true in particular in the first paper [Al] to deeply study the structure of radical $L^1(\omega)$ (this paper, which heavily influenced all subsequent work, circulated privately for many years before it was published). For a locally integrable function $f(t)$ on $\mathbb{R}^+$, we let

$$\alpha(f) = \inf(\operatorname{support} f), \qquad \text{with } \alpha(0) = \infty\,; \tag{1.5}$$

and for $a \geq 0$, we let

$$\begin{aligned} L^1(\omega)_a &= \{f \in L^1(\omega) : \alpha(f) \geq a\} \\ &= \{f \in L^1(\omega) : f = 0 \text{ a.e. on } [0,a]\}\,. \end{aligned} \tag{1.6}$$

The ideals $L^1(\omega)_a$ for $a \geq 0$, and $\{0\}$ are now called *standard ideals*, and the function f in $L^1(\omega)$ is *standard* in $L^1(\omega)$ if the closed ideal it generates is a standard ideal. Allan [Al] identified the key question about the ideals of radical $L^1(\omega)$, namely, determining when all ideals are standard and determining which f are standard in $L^1(\omega)$. He found certain f, in particular all f in $L^1(\mathbb{R}^+)$, which are standard in all radical $L^1(\omega)$. Subsequently, in what is still the deepest result in the subject, Domar [Do] found a large class of $L^1(\omega)$ in which all ideals are standard. Dales and McClure [DM] constructed a radical $L^1(\omega)$ with nonstandard ideals, by building on a complicated example due to Thomas [Th] of a radical Banach algebra of power series with nonstandard ideals.

The study of homomorphisms between convolution algebras on $\mathbb{R}^+$ begins with the fundamental paper of Ghahramani [Gh1], where the emphasis is on isomorphisms. He was motivated in part by earlier work on the Volterra algebra [KS] and on Banach algebras of power series [Gr1]. In the present survey we will concentrate on homomorphisms which are not isomorphisms. The study of such homomorphisms will force us to discuss many other questions about the $L^1(\omega)$ algebras, even though we will not systematically survey work on these other questions. The paper by Bade and Dales [BD], in particular, is a gold mine of useful results and techniques, and we will refer to it as needed. We also strongly recommend the survey by Dales [Da2]. The conference proceedings [LB] and [CN] contain many papers on weighted convolution algebras and their homomorphisms.

Now let $\phi : L^1(\omega_1) \to L^1(\omega_2)$ be a continuous nonzero homomorphism. We will concentrate on describing progress on the following two questions which essentially ask how large the range and how small the nullspace of ϕ must be.

Question 1. If $L^1(\omega_1) * f$ is dense in $L^1(\omega)$, must $L^1(\omega_2) * \phi(f)$ be dense in $L^1(\omega_2)$? (If so, we call ϕ a *standard homomorphism*.)

Question 2. Must ϕ be one-one?

For radical algebras, any negative answers to either Question 1 or Question 2 would provide solutions to the main open questions about ideals in radical $L^1(\omega)$. In Question 1, we always have $\alpha(\phi(f)) = 0$ [Gr5, Lemma 4.5, p. 605], so a negative answer would give a nonstandard g with $\alpha(g) = 0$. If the answer to Question 2 is no, then the kernel of ϕ is a nonstandard closed prime ideal.

In Section 2, we develop basic algebraic facts about convolution and convergence of functions and measures on $\mathbb{R}^+$. In Section 3, we indicate what is known about the standard homomorphism problem and sketch the proof that a number of properties of ϕ are equivalent to that given in Question 1. The key property is whether a particular semigroup is strongly continuous. In Section 4, we describe the results on convergence of sequences and on strong continuity of semigroups relevant to the standard homomorphism problem.

In Section 5, we will describe what is known about Question 2. We will see in Theorem (3.2) that the homomorphism ϕ is an operational calculus map for a particular semigroup. So, we consider Question 1 together with the question of when the operational calculus map of formula (1.2) is one-one. This is the only section on which we include new results, since our previous work on this question applied only to radical convolution algebras and quasinilpotent semigroups.

Much of the work on homomorphisms per se is joint work with Fereidoun Ghahramani, often based on separate work of one or the other of us. I would like to thank Ghahramani for his help and friendship over many years. Many of my other friends in the Banach algebra community have studied convolution algebras and their analogues, and much of this work will be cited. I particularly want to thank Graham Allan, Bill Bade, Garth Dales, Peter McClure, Allan Sinclair, and Michel Solovej for many useful conversations about the $L^1(\omega)$ algebras.

2. Algebra in $M_{\text{loc}}(\mathbb{R}^+)$. We let $M_{\text{loc}}(\mathbb{R}^+)$ be the space of locally finite Borel measures on $\mathbb{R}^+$; that is, the complex linear combinations of σ-finite regular Borel measures on $\mathbb{R}^+$. We define convolution and support of measures in the usual way so that we can identify the locally integrable function $f(t)$ with the measure $f(t)dt$ and have $(f(t)dt) * (g(t)dt) = (f * g)dt$ and $\alpha(f(t)dt) = \alpha(f)$ (see definitions (1.2) and (1.5)). Then $M_{\text{loc}}(\mathbb{R}^+)$ is an algebra with $L^1_{\text{loc}}(\mathbb{R}^+)$ as an ideal. In contrast to other measure algebras, like $M_{\text{loc}}(\mathbb{R})$ for instance, $M_{\text{loc}}(\mathbb{R}^+)$ is an integral domain, as follows from the following classical result.

THEOREM (2.1) (Titchmarsh Convolution Theorem). *If μ and ν are nonzero measures in $M_{\text{loc}}(\mathbb{R}^+)$, then $\alpha(\mu * \nu) = \alpha(\mu) + \alpha(\nu)$. In particular, $\mu * \nu \neq 0$.*

Proofs of the Titchmarsh Convolution Theorem for functions can be found in many places, such as [Al] or [Da2, Th. 3.10, p. 188]. There are several proofs in [Mi]. The extension for measures then follows from noticing that if $u(t) \equiv 1$, then $\alpha(u * \lambda) = \alpha(\lambda)$ for all λ in $M_{\text{loc}}(\mathbb{R}^+)$.

Let $\omega(t)$ be a positive Borel function on $\mathbb{R}^+$. For all $1 \le p \le \infty$, we define $L^p(\omega)$ analogously to $L^1(\omega)$, using the obvious analogue of formula (1.1). We let

$M(\omega)$ be the Banach space of locally finite measures with norm

$$\|\mu\| = \|\mu\|_\omega = \int_{\mathbb{R}^+} \omega(t)\, d|\mu|(t) < \infty\,.$$

When $\omega(t)$ is submultiplicative, $M(\omega)$ is a Banach algebra with closed ideal $L^1(\omega)$; and each $L^p(\omega)$ is a Banach module for $M(\omega)$ and $L^1(\omega)$. We will often identify the measure μ with the operator of convolution by μ. Thus we can speak of the strong operator topology of $M(\omega)$ on $L^p(\omega)$. In particular, we identify the convolution semigroup $\{\delta_t\}_{t\geq 0}$ with the semigroup of right translation operators, since $\delta_a * \mu(E) = \mu(E+a)$. It is clear that for $1 \leq p < \infty$ and $1/p + 1/q = 1$ we have $L^p(\omega)^* = L^q(1/\omega)$ under the duality

$$\langle f, h\rangle = \int_{\mathbb{R}^+} f(t)h(t)\, dt\,. \tag{2.2}$$

We let $C_0(1/\omega)$ be the closed subspace of $L^\infty(1/\omega)$ composed of those continuous functions $h(t)$ with $\lim_{t\to\infty} h(t)/\omega(t) = 0$, and we define $\langle \mu, h\rangle$ for μ in $M(\omega)$ and h in $C_0(1/\omega)$ analogously. We then have, from [Gr5, Th. 2.2, p. 592],

THEOREM (2.3). *When $\omega(t)$ is an algebra weight, then $M(\omega)$ is (isometrically isomorphic to) the dual space of $C_0(1/\omega)$ and the multiplier algebra of $L^1(\omega)$.*

It is precisely because of the above theorem that it is useful to normalize the weight $\omega(t)$ to be an algebra weight in the sense of Definition (1.4).

Suppose that the bounded net $\{\lambda_n\}$ converges weak* in the algebra $M(\omega)$ to λ. It then follows easily from the separate weak*-continuity of convolution that weak*-lim $\lambda_n * \nu = \lambda * \nu$ for all ν in $M(\omega)$ ([Gh1, Lemma 1.1, p. 151], [Gr5, Lemma 3.1, p. 595]), and it follows from the fact that the continuous functions with compact support are dense in $C_0(1/\omega)$ and in all $L^q(1/\omega)$, $1 < q < \infty$, that if f belongs to $L^p(\omega)$ with $1 < p < \infty$, then weak-lim $\lambda_n * f = \lambda * f$ [GG2, Lemma 2.1]. Because $M(\omega)$ is an integral domain, much more is true.

LEMMA (2.4). *Suppose that $\{\lambda_n\}$ is a bounded net in $M(\omega)$. If either*

(i) *there is a $\nu \neq 0$ in $M(\omega)$ with $\lambda_n * \nu \to \lambda * \nu$ weak* in $M(\omega)$, or*
(ii) *there is an $f \neq 0$ in some $L^p(\omega)$ with $1 < p < \infty$ for which $\lambda_n * f \to \lambda * f$ weakly in $L^p(\omega)$,*

then $\lambda_n * \mu \to \lambda * \mu$ weak for all μ in $M(\omega)$ and $\lambda_n * g \to \lambda * g$ weakly in $L^p(\omega)$ for all g in all $L^p(\omega)$ with $1 < p < \infty$.*

Since $\{\lambda_n\}$ is bounded, it follows from weak*-compactness that every subnet of $\{\lambda_n\}$ has a subsubnet $\{\lambda'_n\}$ converging weak* to some λ_0 in $M(\omega)$. One then uses the fact that $M(\omega)$ is an integral domain to show that if either (i) or (ii) holds, then $\lambda_0 = \lambda$, so that $\lambda_n \to \lambda$ weak*. As pointed out above, the lemma then follows easily (for the details, see [Gr5, Lemma 3.2, p. 595] for $p = 1$, and [GG2, Lemma 2.1] for $p > 1$).

Considering $L^1(\omega)$ as a subspace of $M(\omega)$ is useful because closed bounded subsets of $M(\omega)$ are weak*-compact and because the semigroup $\{\delta_t\}$ of point-masses (or right translation operators) belongs to $M(\omega)$. Here is a simple application of these ideas.

THEOREM (2.5). *Suppose that $\omega(t)$ is an algebra weight with $\lim_{t\to\infty}\omega(t)^{1/t} = 0$. Then $L^1(\omega)$ is a radical Banach algebra, and the radical of $M(\omega)$ is the set $M_0(\omega) = \{\mu \in M(\omega) : \mu\{0\} = 0\}$.*

Proof. The collection of measures with compact support in $(0,\infty)$ is dense in $M_0(\omega)$, and hence so is the larger space $\bigcup_{a>0}\delta_a * M(\omega)$. So we just need to observe that every measure $\mu = \delta_a * \nu$, with ν in $M(\omega)$ and $a > 0$, is quasinilpotent, which follows since

$$\begin{aligned}\lim_{n\to\infty}\|\mu^n\|^{1/n} &\le (\lim_{n\to\infty}\|\delta_{an}\|^{1/n})(\lim_{n\to\infty}\|\nu^n\|^{1/n})\\ &= (\lim_{n\to\infty}\omega(an)^{1/n})(\lim_{n\to\infty}\|\nu^n\|^{1/n}) = 0\,.\end{aligned}$$

When $\lim_{t\to\infty}\omega(t)^{1/t} > 0$, both $M(\omega)$ and $L^1(\omega)$ are semisimple, because pointwise evaluation of the Laplace transform provides a collection of characters that never vanish [HP, Theorem 4.18.4, p. 149].

3. Standardness of homomorphisms.

3. Standardness of homomorphisms. We start with an extension theorem and a representation theorem for arbitrary nonzero homomorphisms. In this section, we will always assume that the weights are algebra weights in the sense of Definition (1.4).

THEOREM (3.1). *If $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism, then it has a unique extension to a homomorphism $\overline{\phi} : M(\omega_1) \to M(\omega_2)$. Moreover, $\|\overline{\phi}\| = \|\phi\|$.*

The above theorem is part of [Gr5, Theorem 3.4, p. 596]. The idea of the proof is to choose a bounded approximate identity $\{e_n\}$ in $L^1(\omega_1)$ with $\|e_n\| \to 1$ and to define $\overline{\phi}(\mu)$ as the weak* limit of $\{\phi(\mu * e_n)\}$. This limit exists because the integral domain property of $M(\omega_2)$ shows that all weak* convergent subsequences of $\{\phi(\mu * e_n)\}$ have the same limit.

Because of the uniqueness of the extension, we will henceforth let ϕ denote both the original map and its extension. While it is not hard to construct a homomorphism from $M(\omega_1)$ to $M(\omega_2)$ which does not map $L^1(\omega_1)$ to $L^1(\omega_2)$, Ghahramani has shown that any isomorphism of measure algebras always maps the corresponding L^1 algebras onto each other ([Gh4, Remark 2, p. 153], [Gh5, Th. 1, p. 465]).

The case $p = 1$ of the following theorem is [Gr5, Th. 3.6, p. 599]. For $p > 1$, see [GG2, Section 3].

THEOREM (3.2). *Suppose that $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is a continuous nonzero homomorphism and let $\mu_t = \phi(\delta_t)$. Then we have*:

(a) $\{\mu_t\}$ *is a weak*-continuous semigroup in* $M(\omega_2)$ *for* $t \geq 0$.

(b) $\{\mu_t\}$ *is strongly continuous on* $L^1(\omega_2)$ *for* $t > 0$.

(c) *For each* $1 < p < \infty$, $\{\mu_t\}$ *is a strongly continuous semigroup for* $t \geq 0$.

(d) *For every* f *in* $L^1(\omega_1)$ *and every* g *in* $L^p(\omega_2)$, *where* $1 \leq p < \infty$, *we have*

$$\phi(f) * g = \int_{\mathbb{R}^+} f(t)\mu_t * g\,dt\,. \tag{3.3}$$

Sketch of proof. Let $h = \phi(f) \neq 0$ in $L^1(\omega_2)$. Then $\mu_t * h = \phi(\delta_t * f)$ is norm continuous in $L^1(\omega_2) \subseteq M(\omega_2)$, and hence weak*-continuous in $M(\omega_2)$. It follows from Lemma (2.4) that $\{\mu_t\}$ is weak*-continuous in $M(\omega_2)$ and acts as a weakly continuous, and hence strongly continuous [HP, Th. 10.6.5, p. 324], semigroup on all $L^p(\omega_2)$ with $1 < p < \infty$. The measure-theoretic arguments for strong continuity on $L^1(\omega_2)$ for $t > 0$ can be found in [Gr5, p. 599].

The integral formula (3.3) is the operational calculus map of formula (1.2) for the semigroup μ_t acting on $L^p(\omega_2)$, and hence defines a bounded operator on $L^p(\omega_2)$. For $p = 1$, the map is a multiplier on $L^1(\omega_2)$ and hence a measure in $M(\omega_2)$. One then uses the fact that $M(\omega_2)$ is an integral domain to show that this measure equals $\phi(f) \in L^1(\omega_2) \subseteq M(\omega_2)$ (see [Gr5, p. 601] or [GG2, Cor. (3.5)]) and then reduces the $L^p(\omega_2)$ case for $p > 1$ to the case of $L^1(\omega_2)$ [GG2, Cor. (3.5)].

The next two results describe most of what is known about Question 1 of the introduction. Much of the hard work is hidden in formula (3.3) above.

Theorem (3.4). *Suppose that* $\phi : L^1(\omega_1) \to L^1(\omega_2)$ *is a continuous nonzero homomorphism and let* $1 \leq p < \infty$. *The following properties are equivalent, and they all hold if* $p > 1$.

(a) *The semigroup* $\mu_t = \phi(\delta_t)$ *is strongly continuous on* $L^p(\omega_2)$.

(b) *Whenever* $L^1(\omega_1) * f$ *is dense in* $L^1(\omega_1)$, *then* $L^p(\omega_2) * \phi(f)$ *is dense in* $L^p(\omega_2)$.

(c) *For each* h *in* $L^p(\omega_2)$, *we can write* $h = \phi(f) * g$ *for some* f *in* $L^1(\omega_1)$ *and* g *in* $L^p(\omega_2)$.

(d) ϕ *is continuous from the strong operator topology of* $M(\omega_1)$ *acting on* $L^1(\omega_1)$ *to the strong operator topology of* $M(\omega_2)$ *acting on* $L^p(\omega_2)$. *That is, whenever* $\{\lambda_n\}$ *is a net in* $M(\omega_1)$ *for which* $\lim(\lambda_n * f) = \lambda * f$ *in norm for all* f *in* $L^1(\omega_1)$, *then* $\lim(\phi(\lambda_n) * h) = \phi(\lambda) * h$ *for all* h *in* $L^p(\omega_2)$.

For $p = 1$, property (b) above is just our definition of standard homomorphism in Question 1 in the introduction. Our best current result in this case is [GGM, Th. (3.4), p. 284]:

Theorem (3.5). *The homomorphism* $\phi : L^1(\omega_1) \to L^1(\omega_2)$ *is standard if there is some* $b > 0$ *for which* $\lim_{t\to\infty} \omega(t + b)/\omega(t) = 0$.

In the next section, we will sketch a proof of Theorem (3.5) (which is [GGM, Th. (3.5), p. 285]) and explain the condition on the weight (cf. [BD]). In [GGM, Th. (2.2), p. 280] we proved Theorem (3.4) above for $p = 1$, together with a number of other equivalent conditions which can help determine when ϕ is standard. More can be said in the special case $\omega_1(t) \equiv \omega_2(t) \equiv 1$ [GG1, Th. (5.8), p. 524], though we do not have a complete answer in this case. The case $p>1$ is the main result in [GG2].

We now sketch a proof of Theorem (3.4) above. That part (a) follows for $p>1$ is just Theorem (3.2)(c).

We next prove (a)⇒(b). Notice that it is enough to prove (b) for a single function f_0 with $L^1(\omega_1) * f_0$ dense. For if $L^1(\omega_1) * f$ is also dense, we can write $f_0 = \lim_{n\to\infty} f * h_n$ and hence $\phi(f_0) = \lim_{n\to\infty} \phi(f) * \phi(h_n)$; from which it is easy to show that $L^p(\omega_2) * \phi(f)$ contains the dense subspace $L^p(\omega_2) * \phi(f_0)$. Also, by replacing $L^1(\omega_1)$ by the isomorphically isometric algebra $L^1(e^{-rt}\omega_1(t))$ if necessary, we may assume $\lim_{t\to\infty} \omega(t)^{1/t} < 1$. Hence the function $u(t) \equiv 1$ belongs to $L^1(\omega_1)$ and $\lim_{t\to\infty} \|\mu_t\|^{1/t} < 1$. Now let $-A$ be the generator of the strongly continuous semigroup $\{\mu_t\}$. Then A is invertible and the Laplace transform formula for the resolvent [DS, pp. 620–622] yields

$$A^{-1}(g) = \int_0^\infty \mu_t * g \, dt\,.$$

But it follows from formula (3.3) that this integral is $\phi(u) * g$. Hence

$$L^p(\omega_2) * \phi(u) = \mathrm{Range}(A^{-1}) = \mathrm{Dom}(A)\,,$$

which is dense since $-A$ is the generator of a strongly continuous semigroup.

For (b)⇒(c) we turn $L^p(\omega_2)$ into a Banach module over $L^1(\omega_1)$ by defining the module multiplication $f \cdot g = \phi(f) * g$. Choose some f in $L^1(\omega_1)$ with $L^p(\omega_2) * \phi(f)$ dense and let $\{e_n\}$ be a bounded approximate identity in $L^1(\omega_1)$ (for instance, $e_n = n\chi_{[0,1/n)}$). Since $\lim_{n\to\infty} e_n \cdot (\phi(f) * g) = \phi(f) * g$ and $\phi(f) * L^p(\omega_2)$ is dense, it follows that e_n is a module approximate identity for $L^p(\omega_2)$. Part (c) is now just the Cohen factorization theorem for modules. The proof that (c)⇒(d) follows similarly, from writing $h = \phi(f) * g$. That (d)⇒(a) follows from $\phi(\delta_t) = \mu_t$ and the fact that $\{\delta_t\}$ is a strongly continuous semigroup.

Because of the usefulness of homomorphisms, it is natural to ask when homomorphisms, or isomorphisms, exist between convolution algebras. If $L^1(\omega_1)$ is semisimple, the homomorphism $\theta(f) = e^{-rt} f(t)$ will, for large enough r, map $L^1(\omega_1)$ into $L^1(\omega_2)$. Ghahramani [Gh5] has also determined when semisimple convolution algebras are isomorphic and has found all the isomorphisms. The map $\phi(f(t)) = af(at)$ is an isometric isomorphism from $L^1(\omega(t))$ onto $L^1(\omega(at))$, so one can always map $L^1(\omega_1)$ into $L^1(\omega_2)$ if there is an $a > 0$ for which $L^1(\omega_1(t)) \subseteq L^1(\omega_2(at))$. This, since our weights are right continuous, is equivalent to $\omega_2(at)/\omega_1(t)$ being bounded. Bade and Dales, as a consequence of some hard calculations, have shown [BD, Th. 4.1, p. 107] that, for radical algebras, this condition is equivalent to the existence of a continuous nonzero homomorphism from $L^1(\omega_1)$ to $L^1(\omega_2)$. Ghahramani [Gh3, Th. 1, p. 348] has found the analogous condition for the existence of an isomorphism between radical $L^1(\omega_1)$ and $L^1(\omega_2)$.

4. Types of convergence. Roughly speaking, the natural kinds of convergence weaker than norm convergence for a bounded sequence in the algebra $M(\omega)$ can be grouped into two types. The first type is equivalent to weak* convergence, and the second to strong convergence for $M(\omega)$ acting on $L^1(\omega)$. The best results, such as the positive answer to Question 1 given in Theorem (3.5), occur when these two types of convergence are equivalent. In Lemma (2.4) we gave a number of conditions easily shown to be equivalent to weak* convergence. The next result, extracted from [GG1, Section 3], gives some other conditions.

THEOREM (4.1). *Let $\{\lambda_n\}$ be a bounded sequence in the algebra $M(\omega)$, and let λ belong to $M(\omega)$. Then the following are equivalent.*

(a) *$\{\lambda_n\}$ converges weak* to λ.*

(b) *If f is a continuous function on $\mathbb{R}^+$ with $f(0) = 0$, then $\lim_{n\to\infty} \lambda_n * f(x) = \lambda * f(x)$ pointwise on $\mathbb{R}^+$.*

(c) *If g is a locally integrable function on $\mathbb{R}^+$, then every subsequence of $\{\lambda_n\}$ has a subsubsequence $\{\lambda'_n\}$ for which $\lambda'_n * g$ converges to $\lambda * g$ almost everywhere on $\mathbb{R}^+$.*

(d) *If $\eta \geq 0$ is (a fixed function) in $L^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$ and f belongs to $L^1(\omega)$, then $\lambda_n * f$ converges to $\lambda * f$ in the norm of $L^1(\omega\eta)$.*

(e) *Every subsequence of $\{\lambda_n\}$ has a subsubsequence $\{\lambda'_n\}$ for which $\lambda'_n[0, x]$ converges to $\lambda[0, x]$ for almost every x in $\mathbb{R}^+$.*

In [GG1], except for (e), we only showed that the above conditions were consequences of weak* convergence. We illustrate the converse argument by showing that (b) implies (a). It follows from weak* compactness that every subsequence of $\{\lambda_n\}$ has a subsubsequence $\{\lambda'_n\}$ which converges weak* to some μ in $M(\omega)$. We just need to show that this forces $\mu = \lambda$. Choose a nonzero continuous f with $f(0) = 0$; then, by (a)$\Rightarrow$(b) for the sequence $\{\lambda'_n\}$, we have that $\lambda'_n * f$ converges pointwise to $\mu * f$. Hence $\lambda * f = \mu * f$. Since $M(\omega)$ is an integral domain, this shows that $\lambda = \mu$ as required.

The next result, extracted from [GG1, Section 4], gives the types of convergence equivalent to strong convergence. The above theorem shows that weak* convergence of $\{\lambda_n\}$ is independent of the algebra $M(\omega)$, which contains $\{\lambda_n\}$ as a bounded sequence. This will not be true for normed convergence, because we will have examples where weak* convergence does not imply norm convergence (see [GG1, Th. (2.3), p. 509] or Theorem (4.4) below), but Theorem (4.1)(d) shows norm convergence always holds in some larger algebra.

THEOREM (4.2). *Suppose that $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$. If f belongs to $L^1(\omega)$, then the following are equivalent*:

(a) *$\{\lambda_n * f\}$ converges to $\lambda * f$ in norm in $L^1(\omega)$.*

(b) *$\{\lambda_n * f\}$ converges weakly to $\lambda * f$ in $L^1(\omega)$.*

(c) $\lim_{n\to\infty}\langle\lambda_n * f, h\rangle = \langle\lambda * f, h\rangle$ *for every* h *in the space of weighted uniformly continuous functions* $UC(1/\omega)$ (for a precise definition of this space, see [GG1, p. 516]).

The class of weights for which weak* convergence implies strong convergence is the following important class introduced and studied by Bade and Dales [BD]:

Definition (4.3). The algebra weight $\omega(t)$ is *regulated* at $a \geq 0$ if

$$\lim_{t\to\infty} \omega(t+b)/\omega(t) = 0 \qquad \text{for all } b > a\,.$$

It is easy to see [BD, Lemma 1.2, p. 81] that if any $\limsup_{t\to\infty} \omega(t+b)/\omega(t) = 0$, then $\omega(t)$ is a radical weight, and that if $\omega(t)$ is a radical weight, then $\liminf_{t\to\infty} \omega(t+b)/\omega(t) = 0$ for all $b > 0$. Thus $\omega(t)$ is regulated at a precisely when $\omega(t)$ is a radical weight for which $\lim_{t\to\infty} \omega(t+b)/\omega(t)$ exists for all $b > a$.

Theorem (4.4). *For an algebra weight* $\omega(t)$ *and a number* $a \geq 0$, *the following are equivalent*:

(a) $\omega(t)$ *is regulated at* a.

(b) *There is a* g *in* $L^1(\omega)$ *with* $\alpha(g) = a$ *for which convolution by* g *is a weakly compact operator on* $L^1(\omega)$.

(c) *For all* f *in* $L^1(\omega)$ *with* $\alpha(f) \geq a$, *convolution by* f *is a compact operator from* $M(\omega)$ *to* $L^1(\omega)$.

(d) *Whenever the sequence* $\{\lambda_n\}$ *in* $M(\omega)$ *converges weak* to* λ, *then for all* f *in* $L^1(\omega)$ *with* $\alpha(f) \geq a$ *we have* $\lambda_n * f \to \lambda * f$ *in norm in* $L^1(\omega)$.

Sketch of proof. If we fix f in $L^1(\omega)$, then variants of the Dunford–Pettis theorem show [BD, Th. 2.9, p. 90] that f acts compactly on $L^1(\omega)$ if and only if it acts weakly compactly. A simple argument then shows [GGM, Lemma (3.1), p. 283] that f also acts compactly from $M(\omega)$ to $L^1(\omega)$. Since convolution by f is weak* continuous, a standard argument then shows [GGM, Th. (3.2), p. 284] that compact action on $M(\omega)$ is equivalent to the convergence-improving property of (d). The hard part is proving that the regulated condition on the weight is equivalent to the compact action of convolution operators on $L^1(\omega)$. This is done by Bade and Dales in [BD, Lemma and Th. 2.2].

If $\omega(t)$ is not regulated at a, it is actually possible to find one single fixed sequence $\{\lambda_n\}$ in $M(\omega)$ which converges weak* to 0, but for which $\lambda_n * f$ diverges in norm for all f in $L^1(\omega)$ with $\alpha(f) \leq a$ (see [GG1, Th. (2.3), p. 509]).

We now sketch the proof of Theorem (3.5), showing that every continuous nonzero homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is standard if ω_2 is regulated at any $a \geq 0$ [GGM, Th. (3.4), p. 284].

Let $\mu_t = \phi(\delta_t)$, which we already know by Theorem (3.2)(a) to be weak* continuous in $M(\omega_2)$. Set

$$I = \{g \in L^1(\omega_2) : \lim_{t\to 0^+} \mu_t * g = g \text{ in norm}\}\,. \tag{4.5}$$

Since $\|\mu_t\|$ is bounded at $t \to 0^+$, it follows that I is a closed ideal in $M(\omega_2)$. It is easy to see that I contains all elements $g = \phi(f)$ in the range of ϕ, for

$$\lim_{t\to 0^+} (\mu_t * g) = \lim_{t\to 0^+} \phi(\delta_t * f) = \phi(f) = g\,,$$

since the right translation semigroup $\{\delta_t\}$ is strongly continuous on $L^1(\omega_1)$. It follows from [Gr5, Lemma 4.5, p. 605] that the range of ϕ, and hence the ideal I, contains some g with $\alpha(g) = 0$.

When $\omega_2(t)$ is regulated at $a \geq 0$, it follows from Theorem (4.4)(d) above that I contains the standard ideal $L^1(\omega_2)_a$ of formula (1.6). The restriction map induces an isomorphism from $L^1(\omega_2)/L^1(\omega_2)_a$ onto $L^1[0,a)$. Except for scaling, $L^1[0,a)$ is the Volterra algebra and hence has all its closed ideals standard (see, for instance, [Da1, Th. 7.9, p. 158]). Thus any closed ideal in $L^1(\omega_2)$ which contains $L^1(\omega_2)_a$ is standard (cf. [Gr2, Lemma (6.2), p. 548]). Thus I is a standard ideal containing some g with $\alpha(g) = 0$. This forces $I = L^1(\omega_2)$, so that $\{\mu_t\}$ is a strongly continuous semigroup on $L^1(\omega_2)$ as required by Theorem (3.4)(a).

The results in this section lead to a natural question [GG1, Question 3, p. 507] about norm convergence of $\lambda_n * g$.

Question 3. Suppose that $\{\lambda_n\}$ is a bounded sequence in $M(\omega)$ for which $\lim_{n\to\infty} \lambda_n * g = \lambda * g$ for some g in $L^1(\omega)$ with $\alpha(g) = 0$. Does $\lim_{n\to\infty} \lambda_n * f = \lambda * f$ for all f in $L^1(\omega)$?

If the answer to Question 3 is "yes" for the weight $\omega_2(t)$, then the ideal I of formula (4.5) is all of $L^1(\omega_2)$, without assuming $\omega_2(t)$ is regulated, so that the homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ is standard. If the answer to Question 3 is "no", then g is a nonstandard element of $L^1(\omega)$ with $\alpha(g) = 0$, since $\lambda_n * f \to \lambda * f$ for all f in $\mathrm{cl}(L^1(\omega) * g)$.

5. Injectivity of homomorphisms. In this section, we survey progress on the question of when a continuous nonzero homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ must be one-one. This is really a special case of the question on when the semigroup operational calculus of formula (1.2) is one-one (for nonnilpotent semigroups of operators), since formula (3.3) says that the homomorphism ϕ is the operational calculus map for the semigroup $\phi(\delta_t)$ acting on $L^1(\omega_2)$. The semigroups of operators we consider are those we called almost continuous in the introduction, since this is the best we can guarantee about $\phi(\delta_t)$. Often we can show that certain operational calculus maps must be one-one, and, as a consequence, conclude that some homomorphisms are one-one, but homomorphisms are more special than general operational calculus maps.

When $L^1(\omega)$ is semisimple, evaluation of the Laplace transform $\widehat{f}(z) = \int_0^\infty e^{-tz} f(t)\,dt$ at some fixed point z_0 is the operational calculus map for the semigroup (of multiplication by) e^{-tz_0} on the complex numbers. But the map $f \to \widehat{f}(z_0)$ is far from one-one; its nullspace has codimension 1. Nonetheless, we have the following remarkable theorem of Ghahramani [Gh2, Th. 1, p. 309] (his proof for $L^1(\mathbb{R}^+)$ goes through in the general case).

THEOREM (5.1). *Every continuous nonzero homomorphism* $\phi : L^1(\omega_1) \to L^1(\omega_2)$ *between semisimple convolution algebras is one-one.*

Ghahramani's proof involves showing that there is a nonconstant analytic function $w(z)$ for which $\phi(f)^\wedge(z) = \widehat{f}(w(z))$. If $\phi(f) = 0$, then $\widehat{f}(\omega(z)) \equiv 0$ so that $\widehat{f}$, and hence f, are 0.

The case when the domain is semisimple and the range is radical follows from the following operational calculus result of Sinclair [Si2, Th. 3.6, p. 39].

THEOREM (5.2). *Suppose that* $U(t)$ *is a quasinilpotent*, *but not nilpotent*, *almost continuous semigroup of operators. Then the restriction of the operational calculus map* ϕ *to* $L^1(\mathbb{R}^+)$ (*or to any semisimple subalgebra*) *is one-one.*

Proof. Let $\omega(t) = \|U(t)\|$, so that the domain of ϕ is the radical algebra $L^1(\omega)$. Any function f belonging to some simple convolution algebra has $|f(t)|e^{-Kt}$ integrable for some K. It then follows from Allan's theorem ([Al, Th. 3], [Si2, Th. A2.1, p. 131]) that if the kernel of ϕ contains f, then the kernel must be a standard ideal $L^1(\omega)_a$. It is then easy to show (see proof of Theorem (5.5) below) that $U(a) = 0$, which contradicts the assumption that the semigroup $U(t)$ is not nilpotent.

COROLLARY (5.3). *Let* $\phi : L^1(\omega_1) \to L^1(\omega_2)$ *be a continuous nonzero homomorphism. If* $L^1(\omega_1)$ *is semisimple and* $L^1(\omega_2)$ *is radical*, *then* ϕ *is one-one.*

Proof. Let $\mu_t = \phi(\delta_t)$ and $\omega(t) = \|\mu_t\|_{\omega_2}$. Since $L^1(\omega_2)$ is radical, $\omega(t)^{1/t} \to 0$, so that $L^1(\omega)$ is a radical algebra. The operational calculus formula for $\{\mu_t\}$, that is, formula (3.3), extends ϕ to $L^1(\omega)$ (cf. [Gr5, Th. 3.17, p. 603]). It then follows from Theorem (5.2) that the restriction of ϕ to $L^1(\omega_1)$ is one-one.

If the radical $L^1(\omega)$ has no nonstandard closed ideals, then any operational calculus or homomorphism from $L^1(\omega)$ will be one-one. For very nice weights, Domar's theorem [Do] shows that no nonstandard ideals exist, but Dales and McClure [DM] have constructed radical $L^1(\omega)$ with nonstandard ideals. While a nonstandard ideal need not be the kernel of an operational calculus map (see [Gr4, Th. (2.4), p. 133]), we have shown how to start with a nonstandard ideal and use it to construct a noninjective operational calculus for a quasinilpotent strongly continuous semigroup [Gr4, Th. (2.5), and Cor. (2.6), p. 134]. For endomorphisms, however, we have the following result [Gr5, Cor. 5.3, p. 611]:

THEOREM (5.4). *Every continuous nonzero endomorphism of a radical* $L^1(\omega)$ *is one-one.*

The theorem will follow from the following operational calculus theorem, where the hypothesis is not on the weight $\omega(t) = \|U(t)\|$, but on the ranges of the $U(t)$. Originally we proved the result [Gr5, Th. 5.1, p. 610] only for quasinilpotent

$U(t)$ (see [Gr5, Appendix, p. 613]). Michel Solovej's master's thesis [So], and some questions he asked me, led to the improved result.

THEOREM (5.5). *Suppose that $U(t)$ is an almost continuous nonnilpotent semigroup of bounded operators on the Banach space X. If the range spaces $X_t = \mathrm{cl}(U(t)X)$ are distinct, then the operational calculus map ϕ is one-one.*

Proof. We prove that if f belongs to $L^1(\omega)$, where $\omega(t) = \|U(t)\|$, then

$$\alpha(f) \geq b \;\Leftrightarrow\; \phi(f)(X) \subseteq X_b\,. \tag{5.6}$$

Then if $\phi(f) = 0$, we would have $\alpha(f) = \infty$, or $f = 0$, as required. It is clear from the integral formula (1.2) for the operational calculus map that $\phi(f)x \in X_b$ for all x in X when $\alpha(f) \geq b$. Hence the closed ideal $J = \{f \in L^1(\omega) : \phi(f) \in X_b\}$ contains the standard ideal $L^1(\omega)_b$ of formula (1.6); so that J is itself a standard ideal [Gr2, Lemma (6.2), p. 548]. Suppose now that $\phi(f) \in X_b$ and that $\alpha(f) \leq a$, so that $J \subseteq L^1(\omega)_a$, and let g_h be $1/h$ times the characteristic function of $[a, a+h)$. Then for all x in X we have, by strong continuity of $U(t)x$,

$$U(a)x = \lim_{h\to 0}(1/h)\int\limits_a^{a+h} U(t)x\,dt = \lim_{h\to 0}\phi(g_h)x \in X_b\,.$$

This implies that $X_a \subseteq X_b$. But if $a < b$, the semigroup property implies $X_b \subseteq X_a$. This would contradict the assumption that the X_t are distinct. This proves formula (5.6), and hence proves the theorem.

Instead of assumptions on the ranges of $U(t)$, we could have assumed that the nullspaces $N(U(t))$ were distinct. Formula (5.6) would then be replaced by

$$\alpha(f) \geq b \;\Leftrightarrow\; N(\phi(f)) \subseteq N(U(b))\,,$$

which is proved in essentially the same way as formula (5.6).

To apply Theorem (5.5) to homomorphisms and, in particular, to prove Theorem (5.4), we need to look at the function $\alpha(\mu_t) = \alpha(\phi(\delta_t))$. The basic facts about the support of μ_t when ϕ is an isomorphism were used and applied by Ghahramani [Gh3]. His results were adapted to the case that ϕ is a homomorphism by the author ([Gr5], [Gr6]). It follows from the Titchmarsh convolution theorem (Theorem (2.1) above) that $\alpha(\mu_t)$ is an additive function of t. From this it is easy to show ([Gr5, Th. 4.3, p. 605], [Gh3, Lemma 1, p. 344]) that there is an $A \geq 0$ for which $\alpha(\mu_t) = \alpha(\phi(\delta_t)) = At$. We say that A is the *character* of ϕ. Ghahramani shows [Gh3, Lemma 1, p. 344] that every isomorphism of radical $L^1(\omega)$ algebras has positive character, and we extend this result to endomorphisms [Gr5, Th. 4.7, p. 606]. Both proofs depend on a very nice result of Bade and Dales [BD, Th. 3.6, p. 99] relating $\|f^n\|^{1/n}$ to $\alpha(f)$. Thus the following lemma [Gr5, Th. 5.2, p. 611] will yield a proof of Theorem (5.4) above.

LEMMA (5.7). *If the homomorphism $\phi : L^1(\omega_1) \to L^1(\omega_2)$ has positive character, then ϕ is one-one.*

P r o o f. Let $\mu_t = \phi(\delta_t)$ so that $\alpha(\mu_t) = At$ with $A > 0$. To apply Theorem (5.5), we just need to show that the closures of the spaces $\mu_t * L^1(\omega)$ are distinct. If $g = \mu_t * f$ then the Titchmarsh convolution theorem says that $\alpha(g) = \alpha(\mu_t) + \alpha(f) \geq At$. On the other hand, if $\alpha(f) = 0$, then $\alpha(g) = Ab$. Hence every h in $\mathrm{cl}(\mu_b * L^1(\omega))$ has $\alpha(h) \geq Ab$, and some h in this space have $\alpha(h) = Ab$. Thus the spaces are distinct, proving the lemma, and thus proving Theorem (5.4).

As shown by Ghahramani for isomorphisms [Gh3, Prop. 2, p. 345] and by the author for homomorphisms with positive character [Gr5, Th. 4.9, p. 607], one can also determine $\alpha(\phi(f))$ for arbitrary f. Our original proof was only valid for radical algebras, since we used a standard ideal theorem of Allan [Al] only valid in this case.

THEOREM (5.8). *If* $\phi : L^1(\omega_1) \to L^1(\omega_2)$ *has positive character* A, *then* $\alpha(\phi(\mu)) = A\alpha(\mu)$ *for all* μ *in* $M(\omega_1)$.

P r o o f. It will be enough to prove $\alpha(\phi(f)) = A\alpha(f)$ for all f in $L^1(\omega_1)$, since we can find h in $L^1(\omega_1)$ with $\alpha(h) = \alpha(\phi(h)) = 0$ [Gr5, Lemma 4.5, p. 605] and then replace μ with $f = \mu * h$. The theorem will follow when we prove the following analogue of formula (5.6):

$$\alpha(f) \geq b \;\Leftrightarrow\; \alpha(\phi(f)) \geq Ab\,. \tag{5.9}$$

If $\alpha(f) \geq b$, then f is a limit of elements of the form $\delta_b * g$. But

$$\alpha\phi(\delta_b * g) = \alpha(\mu_b * \phi(g)) = \alpha(\mu_b) + \alpha(\phi(g)) \geq Ab\,,$$

so that $\alpha(f) \geq Ab$. Thus the closed ideal $J = \{f \in L^1(\omega) : \alpha(\phi(f)) \geq Ab\}$ contains the standard ideal $L^1(\omega_1)_b$, and therefore J is a standard ideal $L^1(\omega_1)_c$. We complete the proof by showing $c \geq b$. Now let $g_n = \delta_c * e_n$, with $e_n = n\chi_{[0,1/n)}$. Then $\lim_{n\to\infty} \alpha(\phi(e_n)) = 0$ (see [Gr5, p. 606]), so that

$$\lim_{n\to\infty} \alpha(\phi(g_n)) = \alpha(\mu_c) = Ac\,.$$

But each g_n belongs to J, so that $Ac \geq Ab$ and hence $c \geq b$. This completes the proof of formula (5.9) and of the theorem.

Solovej [So] has pointed out that one can use the above theorem to show that ϕ is one-one, giving another proof of Theorem (5.4). For if $f \neq 0$, then $\alpha(\phi(f)) = A\alpha(f) \neq \infty$, so that $\phi(f) \neq 0$.

The results that certain homomorphisms or operational calculus maps ϕ are one-one always extend to the corresponding measure algebras. For if $\phi(\mu) = 0$, so is every $\phi(\mu * f) = \phi(\mu)\phi(f)$ and hence $\mu * f = 0$, which forces $\mu = 0$.

The fact that endomorphisms of radical $L^1(\omega_1)$ have positive character and hence are one-one, can be extended to homomorphisms $\phi : L^1(\omega_1) \to L^1(\omega_2)$ as long as $L^1(\omega_2)$ is not "substantially larger" than $L^1(\omega_1)$ (see [Gr5, Cor. 4.8, p. 607] for a precise statement and proof). But we do not know the full answer to Question 2 in the introduction.

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