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RECENT RESULTS AND OPEN PROBLEMS IN ANALYTIC COMPUTATIONAL COMPLEXITY

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There are many types of computational complexity depending on the model being studied. These include algebraic, combinatoric, analytic and parallel complexity. Recent progress in algebraic complexity is summarized by Borodin and Munro [1] and the state of the art in analytic complexity is covered in Traub [8]. Material on parallel complexity may be found in Traub [7] and Heller [4].

Although we will focus here on analytic complexity, we begin by reporting a recent algebraic result which was stimulated by a question in analytic complexity.

Given a power series for f we wish to calculate the first n terms of the reverse power series. The classical algorithms for doing this are at least $O(n^3)$. Brent and Kung [2] have shown that this problem can be solved with complexity $O\big((n\log n)^{3/2}\big)$. No non-trivial lower bound is known for this problem. They also show that the problem of reversion of power series is equivalent to the problem of calculating the first n terms of the polynomial resulting from the composition of two polynomials of degree n.

Finally they show (Brent and Kung [3]) that the polynomial consisting of the first n terms of the reverse power series can be evaluated in $O(n\log n)$. This result gives us an upper bound on the combinatory cost of certain iterations.

We turn to recent results and open problems in analytic computational complexity, or more specifically in iterative computational complexity.

Let f be a nonlinear operator, $f\colon D \subset B_1 \to B_2$ where B_1 and B_2 are two Banach spaces. Let α be a simple zero of f and let x_i be a sequence of approximations for α which are generated by an algorithm Φ. We shall consider here only implicit problems, that is problems where only certain functionals of f are available.

Let e_i represent some measure of the error of x_i. For example e_i might represent $\|x_i-\alpha\|$, absolute error,

$\dfrac{\|x_i-\alpha\|}{\|\alpha\|}$, the relative error,

$\|f(x_i)\|$, the norm of the residual.

Assume that the e_i satisfy the error equation

(1) $$e_i = Ae^p_{i-1}, \quad i = 1, 2, \ldots, k,\ p > 1,$$

where we call the p the non-asymptotic order and A the error coefficient. The model given by (1) is too simple to represent the sequence of errors of most iterations. We should use this simple model to derive some conclusions and then indicate below how the model should be modified so as to represent algorithms arising in practice.

Choose $\varepsilon > 0$. Let k be the number of iterations required so that $e_k = \varepsilon e_0$. Assume that the cost per step is a constant, c, and define the complexity, comp, by comp $= kc$. Let $w_p = \dfrac{1}{A^{1/(p-1)}e_0}$, $t = \log(1/\varepsilon)$, $z = c/\log p$. Then $e_k < e_0$ iff $w_p > 1$. (All logarithms are to base 2.)

Then it is easy to establish the following:

THEOREM. *Let* $2 \leqslant w_p \leqslant t$. *Then*

$$z(\log t - \log\log t) < \text{comp} \leqslant z\log(1+t).$$

Let $1 < w_p \leqslant 2$. *Then*

$$z(\log t - \log\log w_p) \leqslant \text{comp} \leqslant z\big(\log(1+t) - \log\log w_p\big). \ \blacksquare$$

Observe the following:

(1) These results are non-asymptotic.

(2) The lower and upper bounds are strict.

(3) We have decomposed the bounds on the complexity as the product of two factors and that one of these factors, the cost index z, is independent of ε and the error coefficient A.

(4) Provided only that $w_p \geqslant 2$ the bounds on the complexity are independent of A.

(5) If $1 < w_p \leqslant 2$, then the bounds on the complexity do depend on the error coefficient.

The theorem may be used to compare any two algorithms in a class of algorithms Φ but we shall not pursue that here.

We now examine the structure of the cost index z. Let $c = u(f, S) + d(\Phi)$, where S is the information set of the algorithm, u is the cost of new information, and d is the combinatory cost. An example may help to clarify these definitions.

EXAMPLE. Newton iteration; $f\colon D \subset R \to R$. Let $c(f)$, $c(f')$ denote the number of arithmetics to evaluate f, f'. Then $S = (f, f')$, $u = c(f)+c(f')$, $d = 2$. Then if $w = 1/(Ae_0) \geqslant 2$, comp $\leqslant \big(c(f)+c(f')+2\big)\log(1+t)$. $\blacksquare$

We are interested in obtaining lower bounds on a class of iterations.

EXAMPLE. $f\colon D \subset R^N \to R^N$, Φ a one-point iteration with $S = (f, \ldots, f^{(n-1)})$. If $w_p \leqslant t$, then

$$\text{comp} \geqslant \frac{3}{\log 3} \min_{i=0,\ldots,n-1} c(f^{(i)})\,(\log t - \log\log t). \ \blacksquare$$

We may obtain upper bounds on the complexity of a class of iterations.

EXAMPLE. $f\colon D \subset R \to R$, Φ a multipoint iteration with $S = (\overbrace{f, \ldots, f}^{n})$.

$\exists \Phi^*$ such that $p(\Phi^*) = 2^{n-1}$, $\quad d(\Phi^*) = a_0n^2 + a_1n + a_2 = q(n)$, $a_0 > 0$.

If $w_p \geqslant 2$, then for the class of multipoint iterations

$$\text{comp} \leqslant \frac{(nc(f)+q(n))}{n-1}\log(1+t). \ \blacksquare$$

Additional information on the material discussed above may be found in Traub and Woźniakowski [9].

The model defined by (1) is a simplified one that does not describe the errors of real algorithms. In order to model real algorithms we must modify (1) in various ways. These include

1. $e_i = A_ie^p_{i-1}$,

2. $e_i = A_ie^{p_1}_{i-1} \ldots e^{p_m}_{i-m}$,

3. Inclusion of round-off error,

4. Permit variable cost c_i.

The theory resulting from these extensions will be reported in a later paper by Traub and Woźniakowski.

In general, to obtain lower and upper bounds on z, one must know relationships between the information set S, the cost per step c, the combinatory cost d, and the order p. We limit ourselves here to mentioning two recent results.

Let $f\colon D \subset R \to R$. Then Kung and Traub [6] conjecture that if the information set S has n elements and if the iteration has no memory, then $p \leqslant 2^{n-1}$. Woźniakowski [10] shows that the general conjecture is related to Birkhoff interpolation which has been open for some 70 years. However, the conjecture has been settled for many important special cases.

Now let $f\colon D \subset R^N \to R^N$. Classical iterations of order 3 require the evaluation of f, f', f''. Thus if each component of the evaluation of a vector or matrix costs unity, the complexity is $O(N^3)$. Kacewicz [5] has shown that one can also construct iterations of order 3 using $f, f', \int f$. The complexity of Kacewicz's algorithm is $O(N^2)$. Kacewicz reports many other results of interest. Kacewicz's work marks an important beginning in the general study of what information is relevant to the solution of a mathematical problem.

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МЕТОД КОЛЛОКАЦИИ ПРИ РЕШЕНИИ ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ И СИСТЕМ

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Сформулируем наш метод на примере одного интегрального уравнения первого рода:

$$(1)\qquad A\varphi \equiv \int_{\Gamma} K(x,y)\varphi(y)\,dy = f(x), \qquad x\in\Gamma,$$

где Γ — гладкая замкнутая кривая на плоскости, x, y — точки на Γ.

Пусть ядро $K(x,y)$ имеет логарифмическую особенность, то есть представимо в виде $\ln|x-y|+K_1(x,y)$, где K_1 — гладкая функция. При этом оператор A действует непрерывно из пространства $C^{n,\alpha}$ в $C^{n+1,\alpha}$ для любого целого r; $0<\alpha<1$. Тогда к этому уравнению применим не только метод регуляризации, но и следующий прямой метод.

Параметризуем Γ: $x = x(t)$, $t\in[0,1]$ и, сохранив для функций прежние обозначения, перепишем уравнение (1) в виде

$$(2)\qquad A\varphi \equiv \int_0^1 K(t,\tau)\varphi(\tau)\,d\tau = f(t), \qquad t\in[0,1].$$

Все функции считаем периодически продолженными.

Пусть $\tau_i = i\times h$ $(i=1,\ldots,n)$; $h=1/n$; $b_i(\tau)=b(\tau-\tau_i)$ — базисные функции интерполяции, то есть $b_i(\tau_j)=\delta_{ij}$. Функцию $b(\tau)$ считаем чётной. Приближенное решение ищем в виде:

$$\tilde{\varphi}(\tau) = \sum_{i=1}^{n}\varphi_i b_i(\tau).$$

Неизвестные величины φ_i отыскиваем из условий коллокации:

$$(3)\qquad A\tilde{\varphi}(\tau_j) = f(\tau_j), \qquad j=1,\ldots,n.$$

Теорема. *Пусть уравнение* (1) *однозначно разрешимо, и существует такое целое $p\geqslant 2$, что для любой функции $g\in C^p$*