Соотношение (2.14) в нашем случае, как не трудно видеть, есть

$$Y_{i-1} = -D_i^{(2)} y_i + d_i, \quad i = N(-1)1. \tag{3.7}$$

Сравнение с соотношениями матричной прогонки в [3], уравнения (3.4)–(3.7) есть не что инное, как эта прогонка, при определении $x_i = -D_i^{(2)}$.

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THE RECURRENT CALCULATION OF THE INVERSE OF A SPECIAL CLASS OF MATRICES

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1. Introduction

In this paper the notion of an F-matrix is introduced and two new algorithms to calculate its inverse are suggested.

The first calculates the inverse of an F-matrix with the aid of the inverse of a triangular matrix arising by an adequate adjustment of the original matrix. The second settles the same problem with the aid of recurrent relations.

Both algorithms are appraised from the viewpoint of the number of their arithmetical operations, being thereby compared with the Gauss method of inverse of that type of matrices.

In conclusion it is shown that the algorithms suggested may be usefully utilized to calculate the inverse of an F-matrix if this is of large dimension and sparse.

The notion of an F-matrix is introduced by the following

DEFINITION 1. A real square regular matrix of nth order $A = (a_{ij})$ is called an *F-matrix* if

(a) $a_{ij} = 0$ for $j-i > w$,

(b) $a_{ij} \neq 0$ for $j-i = w$,

$i = 1, 2, \ldots, n, j = i+w, i+w+1, \ldots, n,$ where $1 \leqslant w \leqslant n-1$ is a given natural number.

Remark. The presuppositions of Definition 1 being satisfied, an upper Hessenberg matrix is an F-matrix with $w = 1$. A diagonal $(2w+1)$-matrix is also an F-matrix.

2. The algorithm M1

Let A be an F-matrix. Border it successively from above with n-dimensional vectors e_j^T, where $e_j^T = (0, 0, \ldots, 1, \ldots, 0)$ and from right with $(n+w)$-dimensional basic vectors $f_s, f_s^T = (0, 0, \ldots, 1, \ldots, 0)$ in which there is 1 at the jth or sth point, respectively. We get thereby a lower triangular matrix of $(n+w)$th order, denoted T.

The calculation of the inverse of matrix A is derived under the assumption that the lower triangular matrix T^{-1} is known by suitably shattering the T^{-1} and T matrices in the relation

$$T^{-1}T = \begin{bmatrix} C_1 & C_2 \\ C & C_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A & A_3 \end{bmatrix} = I_{n+w}, \tag{1}$$

where I_{n+w} is the unit matrix of $(n+w)$th order, C_2 and C, A_2 are square matrices of nth order and wth order, respectively.

We have

LEMMA 1. *Let T^{-1} and T be the above-mentioned matrices shattered into blocks according to* (1). *Then C is regular.*

Proof. It follows from (1) that

$$C_1A_1+C_2A = I_n, \tag{2}$$

$$CA_1+C_3A = O_1, \tag{3}$$

$$C_3A_3 = I_w, \tag{4}$$

where O_1 is the zero $(w\times n)$-matrix.

Since A is regular, we get from (3)

$$C_3 = -CA_1A^{-1}.$$

By substituting this relation into (4) and by arrangement we get

$$C(-A_1A^{-1}A_3) = I_w,$$

and hence C is regular.

THEOREM 1. *Let A be an F-matrix and T^{-1} and T known matrices shattered into blocks according to* (1). *Then*

$$A^{-1} = C_2-C_1C^{-1}C_3. \tag{5}$$

Proof. From (3) it follows that

$$A_1 = -C^{-1}C_3A.$$

By substituting into (2) we get

$$(C_2-C_1C^{-1}C_3)A = I_n,$$

and hence (5) holds.

The calculation of the inverse of a matrix A of nth order has been reduced to calculating a matrix T^{-1} of $(n+w)$th order, a full matrix C^{-1} of wth order and to carrying out relation (5). The inverse of an F-matrix thus calculated is denoted as algorithm M1.

From the viewpoint of the number of arithmetical operations it is more advantageous to calculate the matrix T^{-1} in the following way: Shatter the matrices T and T^{-1} in the relation $TT^{-1} = I_{n+w}$ as follows:

$$TT^{-1} = \begin{bmatrix} E_n & O_2 \\ G_w & I_w \end{bmatrix} \begin{bmatrix} F_n & O_2 \\ J_w & I_w \end{bmatrix} = I_{n+w}, \tag{6}$$

where E_n, F_n are lower triangular matrices of nth order and O_2 is the zero $(n\times w)$-matrix. According to [2] we have

THEOREM 2. *Let T and T^{-1} be matrices broken up into blocks according to* (6). *Then the individual blocks of the matrix T^{-1} satisfy:*

$$F_n = E_n^{-1}, \quad J_w = -G_wF_n.$$

The algorithm M1 thus modified is now denoted as M1*.

Remark. The idea of bordering a matrix A according to (1) is applied in [1] to calculate the inverse of isoclinal matrices.

3. The algorithm M2

Let A be an F-matrix. Note how in relation $AA^{-1} = I_n$ the relation $A\bar{a}_1^{-1} = \bar{e}_1$ looks, where respectively, $\bar{a}_1^{-1} = (a_{i1}^{-1})$ and $\bar{e}_1$ are the first columns of the matrices A^{-1} and I_n, i.e.,

$$\sum_{j=1}^{w+1} a_{1j}a_{j1}^{-1} = 1,$$

$$\sum_{j=1}^{w+2} a_{2j}a_{j1}^{-1} = 0, \tag{7}$$

$$\dots\dots\dots\dots\dots\dots$$

$$\sum_{j=1}^{n} a_{n-w,j}a_{j1}^{-1} = 0.$$

It can be seen that if in $\bar{a}_1^{-1}$ the first w elements are known, the remaining $n-w$ elements can be calculated from (7) recurrently.

In general, it holds that if the first w rows of the matrix A^{-1} are known, all columns A^{-1} can be independently recurrently calculated by the algorithmic formulae (7).

The task hence reads: find the first w rows of A^{-1}.

The solution renders relation (3), from which we get

$$A_1A^{-1} = -C^{-1}C_3. \tag{8}$$

Since A_1 is the first w rows of the unit matrix I_n, it follows from (8) that the first w rows of the matrix A^{-1} constitute the matrix $-C^{-1}C_3$. The matrices C and C_3 can be calculated recurrently from (1) without having to calculate other elements of T^{-1}.

The algorithm M2 then consist in the recurrent calculation of elements of the matrices C, C_3 from (1), in the transformation of the matrix $[C\,|\,C_3]$ into $[-I_w\,|\,-C^{-1}C_3]$ by elementary row adjustments and in the recurrent calculation of the other elements of the matrix A^{-1} by the algorithmic formulae (7).

Remark. Analogously to relation (8), it can be shown that the last w columns of the matrix A^{-1} form the matrix $-C_1C^{-1}$, where C_1, C are the matrices occurring in (1).

4. Appraisal of algorithms from the viewpoint of the number of arithmetic operations

The following table indicates the number of multiplications needed to calculate the inverse of a given F-matrix by the Gauss method (MG) and by the M1, M1* and M2 methods. The number of additions is almost in accordance with the number of multiplications in the individual methods. The number of divisions is equal to n in all these methods.

	Number of multiplications
MG	$\frac{1}{6}n^3+n^2w-\frac{1}{3}w^3+n^2+nw-w^2-\frac{1}{6}n-\frac{2}{3}w$
M1	$\frac{1}{6}n^3+\frac{3}{2}n^2w+\frac{3}{2}nw^2+\frac{5}{6}w^3+\frac{1}{2}n^2+nw-\frac{1}{2}w^2-\frac{2}{3}n+\frac{2}{3}w-1$
M1*	$\frac{1}{6}n^3+\frac{3}{2}n^2w+nw^2+\frac{5}{6}w^3+\frac{1}{2}n^2+\frac{1}{2}nw-\frac{1}{2}w^2-\frac{2}{3}n+\frac{2}{3}w-1$
M2	$\frac{1}{2}n^3+\frac{1}{2}n^2w+\frac{1}{2}nw^2-\frac{1}{2}w^3+\frac{1}{2}n^2-nw+\frac{1}{2}w^2-w$

5. Some special applications

In conclusion, let us indicate some special applications of the algorithms suggested.

Firstly, the calculation of the inverse of a given F-matrix if this is of a large dimension and sparse. Subsequently we shall point out the feasibility of implementing these algorithms on parallel computers.

Let A be an F-matrix of a large dimension and sparse. The calculation of A^{-1} is generally carried out by the Gauss or Gauss–Jordan method.

At the individual steps of these methods there always occurs the so-called fill-in, the change of certain zero elements of the matrix A into non-zero ones, enlarging the claims for the internal memory of the computer. There have been set up methods [3] which reduce the growth of fill-in. Their application always incurs the enlargement of the number of respectively arithmetic or logical operations. On these grounds it is of advantage to have an algorithm for solving the tasks with large and sparse matrices of a recurrent form in which the matrix components constitute the coefficients of these recurrent relations and their resulting values directly yield the solution of the problem (e.g. they are the elements of A^{-1}). In such a case no fill-in occurs. M2 is such type of an algorithm. After carrying out the calculation of the first w rows of the matrix A^{-1}, which is partly also recurrent, the relations of type (7) directly render the elements of A^{-1}.

In the methods M1 and M1* the calculation of T^{-1} and E_n^{-1} is also calculated recurrently, although the calculation of A^{-1} requires a completion of relation (5).

In algorithms M1, M1* and M2 the same type of recurrent relations shows up:

$$y_1 = c_0,$$

$$b_{i-1,i}y_i = \sum_{j=1}^{i-1} b_{i-1,j}y_j + c_{i-1}, \quad i = 2, 3, \ldots, m. \tag{9}$$

The calculation of y_i, $i = 1, 2, \ldots, m$, according to (9) is strictly serial and requires $O(m^2)$ multiplications and additions.

In paper [4] the implementation (9) was suggested to be carried out on an m-processor parallel computer, on which this process would be implemented on $O(m)$ multiplications and additions.

In paper [5] relations (9) are made parallel for a k-processor computer where $k \leqslant m$. In that case this process is implemented by $m^2/k+O(m)$ multiplications or additions.

The numerical stability of recurrent relations (9) is investigated in [6].

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