for solving differential-algebraic systems using implicit backward differentiation formulas, Proc. of the IEEE 60 (1972), pp. 98–108.

[3] C. G. Broyden, *A new method of solving nonlinear simultaneous equations*, Comput. J. 12 (1969), pp. 94–99.

[4] F. Ficken, *The continuation method for functional equations*, Comm. Pure Appl. Math. 4 (1951), pp. 435–456.

[5] G. Hachtel and M. Mack, *A pseudo dynamic method for solving nonlinear algebraic equations*, in R. A. Willoughby (Ed.), *Stiff differential systems*, Plenum Press, New York 1974, pp. 135–150.

[6] E. Lahaye, *Une méthode de résolution d'une catégorie d'équations transcendantes*, C.R. Acad. Sci. Paris 198 (1934), pp. 1840–1842.

[7] —, *Solution of systems of transcendental equations*, Acad. Roy. Belg. Bull. Cl. Sci. 5 (1948), pp. 805–822.

[8] J. M. Ortega and W. C. Rheinboldt, *Iterative solution of nonlinear equations in several variables*, Academic Press, New York 1970.

[9] A. M. Ostrowski, *On error estimates a posteriori in iterative procedures*, in *Spline functions and approximation theory*, A. Meir, A. Sharma (Eds.), Birkhauser Verlag, Basel 1973.

[10] —, *A posteriori error estimates in iterative procedures*, SIAM J. Num. Anal. 10 (1973), pp. 290–298.

[11] L. B. Rall, *Davidenko's method for the solution of nonlinear operator equations*, The University of Wisconsin, Mathematics Research Center, MRC Tech. Summary Rept. 948, October 1968.

[12] —, *A note on the convergence of Newton's method*, SIAM J. Num. Anal. 11 (1974), pp. 34–36.

[13] W. C. Rheinboldt, *On the solution of some nonlinear equations arising in the application of finite element methods*, in J. Whiteman (Ed.), *Mathematics of Finite Elements and Applications*, Academic Press, London 1976, pp. 465–482.

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A FINITE ELEMENT METHOD FOR A TWO POINT BOUNDARY VALUE PROBLEM WITH A SMALL PARAMETER AFFECTING THE HIGHEST DERIVATIVE

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We consider the following two point boundary value problem on the open interval $\Omega =]0, 1[$:

(1) Given $f_0 \in L^2(\Omega)$ find $u \in H^2(\Omega)$ such that

$$-\varepsilon u'' + a_1 u' + a_0 u = f_0 \quad \text{in } \Omega,$$
$$u(0) = u(1) = 0.$$

Here the parameter ε is assumed to satisfy $0 < \varepsilon \ll 1$.

In the interests of clarity we restrict our attention in what follows to the (trivial) case where $a_0 \geqslant 0$ and $a_1 > 0$ are constants. However the ideas may be extended without difficulty to the (non-trivial) variable coefficient case.

It is known that under the above assumptions as $\varepsilon \to 0$ the solution of (1) converges weakly in $L^2(\Omega)$ to the solution of the initial value problem.

(2) Given $f_0 \in L^2(\Omega)$ find $u \in H^1(\Omega)$ such that

$$a_1 u' + a_0 u = f_0 \quad \text{in } \Omega,$$
$$u(0) = 0.$$

We put $V = H_0^1(\Omega)$ and we define the continuous bilinear and linear forms

$$a(v, w) = \int_\Omega (\varepsilon v'w' + a_1 v'w + a_0 vw) \quad \forall v, w \in V,$$

$$f(v) = \int_\Omega f_0 v \quad \forall v \in V.$$

The variational formulation of (1) is then:

(3) Find $u \in V$ such that

$$a(u, v) = f(v) \quad \forall v \in V.$$

We now construct a discretization of (3) by the finite element method. Let the parameter $h > 0$ and the natural number N be chosen such that $Nh = 1$. Let $\{x_j\}_0^N$ be the nodes $x_j = jh$ and let $\bar{\Omega} = \bigcup_1^N K_j$ where $K_j = [x_{j-1}, x_j]$. For each fixed $\varepsilon > 0$ we associate a unique $\theta = \theta(\varepsilon)$ satisfying:

(4) $$0 \leqslant \theta \leqslant 1, \quad \lim_{\varepsilon\to 0} \theta = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{a_1 \theta h}{2\varepsilon} = 1.$$

We then construct a finite element subspace $V^h = V^h(\varepsilon) \subset V$ in the following manner.

For each K_j, $j = 1, \ldots, N$, we write

$$K_j = K_j^- \cup K_j^+$$

where

$$K_j^- = [x_{j-1}, x_{j-1} + \theta h], \quad K_j^+ = [x_{j-1} + \theta h, x_j]$$

and we define

$$V^h = \{v^h \in C^0(\bar{\Omega}) |\ v^h|_{K_j^-} \in P_1, v^h|_{K_j^+} \in P_0,\ 1 \leqslant j \leqslant N,\ v^h(0) = v^h(1) = 0\}.$$

Here P_k denotes the space of polynomials in one variable of degree $\leqslant k$. It is easy to see that for each $\varepsilon > 0$, $V^h \subset V$ and $\dim V^h = N-1$. Putting $v_j = v^h(x_j)$, $j = 0, \ldots, N$, the degrees of freedom of any $v^h \in V^h$ may be taken as $\{v_j\}_1^{N-1}$ and it is not hard to see that

$$v^h(x) = \sum_{j=1}^{N-1} v_j \varphi_j(x) \qquad \forall x \in \bar{\Omega}$$

where the basis $\{\varphi_j\}_1^{N-1}$ is given by

$$\varphi_j(x) = \begin{cases} (x - x_{j-1})/\theta h & \text{for} \quad x \in K_j^-, \\ 1 & \text{for} \quad x \in K_j^+, \\ (x_j + \theta h - x)/\theta h & \text{for} \quad x \in K_{j+1}^-, \\ 0 & \text{otherwise.} \end{cases}$$

The discrete formulation of (3) is then taken as

(5) Find $u^h \in V^h$ such that

$$a(u^h, v^h) = f(v^h) \qquad \forall v^h \in V^h.$$

It is easy to see that (5) is equivalent to

(6) Find $(u_1, \ldots, u_{N-1}) \in \mathbf{R}^{N-1}$ such that

$$\sum_{k=1}^{N-1} a(\varphi_k, \varphi_j) u_k = f(\varphi_j), \quad j = 1, \ldots, N-1.$$

If we now use the trapezoidal rule in the subintervals K_j^- and K_j^+, $j = 1, \ldots, N$, for evaluating the integrals approximately, it is not hard to see that (6) then gives

(7) $$\begin{cases} \left(-\dfrac{\varepsilon}{\theta h} + \dfrac{a_1}{2}\right) u_{j+1} + \left(\dfrac{2\varepsilon}{\theta h} + a_0 h\right) u_j + \left(-\dfrac{\varepsilon}{\theta h} - \dfrac{a_1}{2}\right) u_{j-1} \\ \qquad = h\big(f(x_{j-1} + \theta h) + f(x_j)\big)/2, \quad 1 \leqslant j \leqslant N-1, \\ u_0 = u_N = 0. \end{cases}$$

In the limit when $\theta = 1$, (7) is the usual central finite difference scheme for (1)

(8) $$\begin{cases} -\varepsilon \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + a_1 \dfrac{u_{j+1} - u_{j-1}}{2h} + a_0 u_j = f(x_j), \quad 1 \leqslant j \leqslant N-1, \\ u_0 = u_N = 0. \end{cases}$$

On the other hand in the limit when $\varepsilon = 0$ and thus $\theta = 0$, because of (4), (7) becomes the upwind finite difference scheme for (2)

(9) $$\begin{cases} a_1 \dfrac{u_j - u_{j-1}}{h} + a_0 u_j = \dfrac{f(x_{j-1}) + f(x_j)}{2}, \quad 1 \leqslant j \leqslant N-1, \\ u_0 = 0. \end{cases}$$

A particularly interesting intermediate choice for θ is

(10) $$\theta = (\tanh a_1 h/2\varepsilon)/(a_1 h/2\varepsilon).$$

It is easy to check that the θ defined by (10) satisfies (4), and that for this θ (7) becomes

(11) $$\begin{cases} -\dfrac{a_1 h}{2} \coth \dfrac{a_1 h}{2\varepsilon} \, \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + a_1 \dfrac{u_{j+1} - u_{j-1}}{2h} + a_0 u_j \\ \qquad = \dfrac{f(x_{j-1} + \theta h) + f(x_j)}{2}, \quad 1 \leqslant j \leqslant N-1, \\ u_0 = u_N = 0 \end{cases}$$

which is the finite difference scheme introduced by A. M. Il'in in [1] (apart from the inhomogeneous term which is simply $f(x_j)$ in his paper).

For each fixed $\varepsilon > 0$ the solutions of (8) and (11) have at the mesh points $O(h^2)$ convergence as $h \to 0$ to the solution of (1), which is not however uniform in ε; the smaller ε is the larger the error constant becomes. The importance of (11) is that its solution has at the mesh points $O(h)$ convergence as $h \to 0$ to the solution of (1) which is *uniform in* ε for all $\varepsilon > 0$. This also is true in the limiting case $\varepsilon = 0$, since the limit of (11) is (9) whose solution has at the mesh points $O(h)$ convergence as $h \to 0$ to the solution of (2).

We have thus constructed finite element subspaces of continuous functions that are piecewise polynomials of alternating degree zero and one. These lead to finite difference schemes, of which special cases are the central finite difference scheme and Il'in's finite difference scheme.

It is known that the upwind finite difference scheme (9) is obtained from finite element subspaces of completely discontinuous functions that are piecewise constant. The limit subspace when $\varepsilon = 0$ of the finite element subspaces V^h constructed above is also the space of completely discontinuous functions that are piecewise constants, with discontinuities at the nodes $\{x_j\}_0^{N-1}$. It is not hard to show correspondingly that a natural limit when $\varepsilon = 0$ of the discrete variational formulation (5) above is a special case of the formulation of Lesaint and Raviart [2] of completely discontinuous finite element methods for ordinary differential equations.

Completely analogous results hold if we assume that $a_0 \geqslant 0$ and $a_1 < 0$. The same idea also works for variable coefficients. There is an obvious extension to problems in two dimensions if the shapes K_j are rectangles.

In a subsequent paper we establish error estimates in the maximum norm for our finite element method, which hold at each point of $\bar{\Omega}$ and which predict correctly the superconvergence results for uniform rectangular shapes of Lesaint and Raviart in the limit when $\varepsilon = 0$ for this special case.

References

[1] A. M. Il'in, *Differencing scheme for a differential equation with a small parameter affecting the highest derivative*, Mat. Zametki 6.2 (1969), pp. 237–248.

[2] P. Lesaint and P. A. Raviart, *On a finite element method for solving the neutron transport equation*; in: *Mathematical aspects of finite elements in partial differential equations* (ed. C. de Boor), Academic Press, 1974, pp. 89–123.

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SOME EQUILIBRIUM AND MIXED MODELS IN THE FINITE ELEMENT METHOD

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1. Introduction

The variational formulation, used in the finite element method, is based mostly on the minimum of potential energy. As a model problem, let us consider the second order elliptic equation

$$-\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+a_0(x)u = f, \quad x \in \Omega \subset \boldsymbol{R}^n, \tag{1.1}$$

with the following mixed boundary conditions:

$$\begin{aligned} u &= u_0 \quad \text{on } \Gamma_u, \\ a_{ij}\frac{\partial u}{\partial x_j}\nu_i &= g \quad \text{on } \Gamma_g, \\ a_{ij}\frac{\partial u}{\partial x_j}\nu_i+\alpha u &= g \quad \text{on } \Gamma_v. \end{aligned} \tag{1.1'}$$

Here the repeated latin index implies summation over the range 1 till n and the boundary $\partial\Omega \equiv \Gamma$ of Ω consists of four mutually disjoint parts

$$\Gamma = \Gamma_u \cup \Gamma_g \cup \Gamma_v \cup \mathscr{R},$$

where each of Γ_u, Γ_g, Γ_v is either open in Γ or empty and the $(n-1)$-dimensional measure of $\mathscr{R}$ is zero. ν denotes the unit outward normal to Γ. Assume that the coefficients a_{ij}, a_0, α are bounded measurable functions,

$$a_0(x) \geqslant 0, \quad \alpha(x) > 0 \text{ (almost everywhere)}$$

and that a positive constant c_0 exists such that

$$a_{ij}(x)t_it_j \geqslant c_0 t_i t_i \quad \forall t \in \boldsymbol{R}^n$$

holds almost everywhere on Ω (a.e.).