предположения функции \( q_j(x) \) из (14), удовлетворяющие условиям леммы 3, а \( w_j(x,t) \) удовлетворяет уравнению (12), однородному условию (13) и граничным условиям

\[
\frac{\partial^{j+1} w_j(t,0)}{\partial x^{j+1}} = f_j(t) - \frac{\partial^{j+1} w_j(t,0)}{\partial x^{j+1}}, \quad j = 1, \ldots, k+1.
\]

В этом случае имеет место поглощенное в смысле поведения по времени такой результат.

Лемма 8. Пусть функции \( q_j(x) \) и \( f_j(t) \), \( j = 1, \ldots, k \), удовлетворяют условиям леммы 7, \( h_j > \frac{h_1 + i - 1}{2k+1}, \quad i = 2, \ldots, k+1 \), \( f_j^{(h_j)}(t) \in L_1(0, \infty) \), \( f_j^{(h_j)}(t) \in L_1(0, \infty) \). Тогда справедливо предельное равенство

\[
\lim_{t \to 1} (\theta-1) u(t,x) = \frac{f_j^{(h_j-1)}(0) - f_j^{(h_j)}(0)}{(h_j-1)}.
\]

Замечание. Построенные нами выше решения входят в класс единственности решения соответствующих задач.

Литература


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prove the convergence of Galerkin's method and of a projection-iteration method, which combines Galerkin's method and the iteration method mentioned before. The projection-iteration method is useful especially if the maximal monotone operator \( A \) is linear, because in this case our problem is reduced to relatively simple linear problems. In Section 3 we consider two examples. We show that the results of Sections 1 and 2 are applicable to certain pseudo-parabolic and evolution equations. Let us remark that an existence result for initial value problems of the form

\[
Au + B \frac{du}{dt} = 0, \quad u(0) = a,
\]

where \( A \) and \( B \) are nonlinear operators was proved already by Barbu [2]. His assumptions on \( A \) and \( B \) are quite different from those used in this paper.

1. Iteration

Let \( X \) be a real Hilbert space with the scalar product \( \langle \cdot, \cdot \rangle_X \) and let \( X^* \) be the dual of \( X \). By \( \langle \cdot, \cdot \rangle \) we denote the pairing between \( X^* \) and \( X \). The norm in the Cartesian product \( X \times X \) we define by

\[
\|[u,v]\|_{X \times X} = \|(u,v)\|_X = \sqrt{\|u\|_X^2 + \|v\|_X^2} \quad \forall [u,v] \in X \times X.
\]

As usual (see e.g. Brezis [3]) we consider every set \( A \subset X \times X \) as a multivalued mapping from \( X \) to \( X \). We use the following notation:

\[
Au = \{v : [u,v] \in A\} \quad \forall u \in X,
\]

\[
D(A) = \{u : u \in X, Au \neq \emptyset\}.
\]

**Definition 1.** A set \( A \subset X \times X \) is said to be monotone if

\[
\langle u_1 - u_2, v_1 - v_2 \rangle_X \geq 0 \quad \forall [u_1,v_1], [u_2,v_2] \in A.
\]

A monotone set \( A \subset X \times X \) is said to be maximal monotone if it is maximal with respect to inclusion among the monotone subsets of \( X \times X \).

**Definition 2.** If \( A \in X \to X^* \), we define \( \text{Mon}(A) \) (the so-called monotonicity constant of \( A \)) by

\[
\text{Mon}(A) = \inf_{u,v \in X} \frac{\langle Au-Av, u-v \rangle}{\|u-v\|_X^2}.
\]

\( A \) is said to be (strongly) monotone if \( \text{Mon}(A) \) is (strictly) positive.

**Definition 3.** Let \( Y \) and \( Z \) be Banach spaces. If \( A \in (Y \to Z) \), we define \( \text{Lip}(A) \) (the Lipschitz constant of \( A \)) by

\[
\text{Lip}(A) = \sup_{u,v \in Y} \frac{\|Au-Av\|_Z}{\|u-v\|_Y}.
\]

If \( \text{Lip}(A) < \infty \), then \( A \) is Lipschitzian.

In this section we assume that we are given operators \( A, B \) and \( \Lambda \) such that

\[
A \in (X \to X^*), \quad m_A := \text{Mon}(A) > 0, \quad M_A := \text{Lip}(\Lambda < \infty),
\]

\[
B \in (X \to X^*), \quad m_B := \text{Mon}(B) > 0, \quad M_B := \text{Lip}(B) < \infty,
\]

\( \Lambda \subset X \times X \) is maximal monotone.

We consider the problem

\[
Au + Bu = 0, \quad u \in D(A).
\]

This problem can be written as follows:

\[
Au + Bu = 0, \quad [u,v] \in A.
\]

We are going to formulate (4) as a fixed point problem. Let \( L \) be the (linear) duality map from \( X \) onto \( X^* \) characterized by

\[
\langle Lu,v \rangle = \langle u,v \rangle \quad \forall u,v \in X
\]

and let \( p \in X \), \( q \in X \) be given real numbers. It is well known (cf. Brezis [3]) that the problem

\[
L(\langle u,v \rangle) = \langle u,v \rangle, \quad \forall [u,v] \in A
\]

has a unique solution \( [u,v] \in X \times X \) for arbitrary \( [u,v] \in X \times X \). Therefore, it makes sense to define an operator \( U_{p,q} \) of \( (X \times X) \to (X \times X) \) by

\[
U_{p,q}(\langle u,v \rangle) = \langle [u,v] \rangle, \quad \forall [u,v] \in A
\]

and \( \langle u,v \rangle \) is a fixed point of \( U_{p,q} \) if and only if \( [u,v] \) is a solution of (4).

**Lemma 1.** Let (1)-(3) be satisfied. If \( U_{p,q} \) is defined by (6) then

\[
\text{Lip}(U_{p,q}) \leq \text{Lip}(L-pA) + \text{Lip}(L-qB).
\]

**Proof.** Let \( \langle [u,v] \rangle \in X \times X \) and \( \langle [u,v] \rangle = U_{p,q}(\langle [u,v] \rangle) \), \( i = 1, 2 \). Then

\[
\|U_{p,q}(\langle [u_1,v_1] \rangle) - U_{p,q}(\langle [u_2,v_2] \rangle)\|_{X \times X} = \left\| [\langle u_1-v_1 \rangle, \langle v_1-v_2 \rangle] \right\|_X
\]

\[
\leq \|\langle u_1-u_2 \rangle\|_X + \|\langle v_1-v_2 \rangle\|_X
\]

\[
\leq \text{Lip}(L-pA) \|\langle u_1-u_2 \rangle\|_X + \text{Lip}(L-qB) \|\langle v_1-v_2 \rangle\|_X
\]

\[
\leq [\text{Lip}(L-pA) + \text{Lip}(L-qB)] \|\langle u_1-u_2 \rangle\|_X + \|\langle v_1-v_2 \rangle\|_X
\]

\[
\leq \left[ \text{Lip}(L-pA) + \text{Lip}(L-qB) \right] \|\langle [u_1,v_1] \rangle \|_{X \times X}.
\]

This proves the lemma.

**Remark 2.** Lemma 1 shows that \( U_{p,q} \) is strictly contractive if

\[
k := \text{Lip}(L-pA) + \text{Lip}(L-qB) < 1.
\]

In view of (1) and (2) we have (see e.g. Browder-Petryshyn [4])

\[
\text{Lip}(L-pA) \leq 1 - 2m_A + M_A^2, \quad \text{Lip}(L-qB) \leq 1 - 2m_B + M_B^2.
\]
Using (8) it is easy to see that one can satisfy (7) by a suitable choice of \( p \) and \( q \) if
\[
\left( \frac{m_1}{M_2} \right)^2 + \left( \frac{m_2}{M_2} \right)^2 > 1.
\]
This relation holds e.g. if we have \( A = L/2 \) or \( B = L/2 \).

If \( A \) is a potential operator, i.e., if
\[
\langle Au, v \rangle = \lim_{t \to 0} \frac{1}{t} \left( \langle F(u + tv), F(u) \rangle \right) \quad \forall u, v \in X,
\]
where \( F \in (X \to \mathbb{R}) \), then we have
\[
\text{Lip}(L-pA) = \max(1-m_2p, M_2p-1).
\]
(Lemma 4.14, Kap. III, in [11] shows Lip(L-pA) < \max(1-m_2p, M_2p-1), and the inverse inequality can be shown easily by elementary estimations.) Correspondingly, if \( B \) is a potential operator, we have
\[
\text{Lip}(L-qB) = \max(1-m_1q, M_1q-1).
\]
Therefore, in the case of potential operators \( A \) and \( B \) we can satisfy (7) by a suitable choice of \( p \) and \( q \) if
\[
\frac{m_1M_2}{(m_1+M_2)^2} + \frac{m_2M_2}{(m_2+M_2)^2} > \frac{1}{4}.
\]

**Theorem 1.** Let (1)–(3) be satisfied. Moreover, let \( A \) and \( B \) satisfy condition (7) for fixed real numbers \( p > 0 \) and \( q > 0 \). Then problem (9) has a unique solution \( \langle u, v \rangle \). If the sequence \( (u_t, v_t) \) is determined by
\[
L(u_{t+1}) = L(pA)u_t + p(L-qB)v_t, \quad v_t \in A u_t, \quad t = 1, 2, \ldots,
\]
then \( (u_t, v_t) \) is \( X \times X \) arbitrary.

**Proof.** In view of Remark 1 the theorem is an immediate consequence of Lemma 1 and Banach’s fixed point theorem.

2. Projection and projection-iteration

We assume that we are given \( A, B \) and \( A \) as in Section 1. Let \( (X_n) \) be a sequence of subspaces of \( X \) such that
\[
X_n \subset X_{n+1}, \quad n = 1, 2, \ldots, \quad \bigcup X_n \text{ is dense in } X.
\]

We denote by \( X^*_n \) the dual space of \( X_n \) and by \( \langle \cdot, \cdot \rangle_n \) the pairing between \( X^*_n \) and \( X_n \).

We define operators \( A_n \in (X_n \to X^*_n) \) and \( B_n \in (X_n \to X^*_n) \) by
\[
\langle A_n u, v \rangle_n = \langle Au, v \rangle, \quad \langle B_n u, v \rangle_n = \langle Bu, v \rangle \quad \forall u, v \in X_n.
\]

Furthermore, we assume that \( (A_n) \) is a sequence such that
\[
\text{Lip}(L-pA_n) = \max(1-m_2p, M_2p-1), \quad \text{and}
\]
\[
\text{Lip}(L-qB_n) = \max(1-m_1q, M_1q-1).
\]
The condition (18) means that \( A \) is approximated in a certain sense by the operators \( A_n \). Besides (4) we consider the corresponding “Galerkin problems”
\[
A_n u_n + B_n v_n = 0, \quad [u_n, v_n] \in A_n.
\]

**Theorem 2.** Let (1)–(3), (7) and (18) be satisfied. Then, for every \( n \) the problem (19) has a unique solution \( \langle u_n, v_n \rangle \) and we have
\[
[u_n, v_n] \to [u, v] \quad \text{in } X \times X,
\]
where \( [u, v] \) denotes the solution of (12).

**Proof.** Let \( L_n \) be the duality map from \( X_n \) onto \( X_n^* \) characterized by
\[
\langle L_n u, v \rangle_n = \langle u, v \rangle \quad \forall u, v \in X_n.
\]
Because \( \text{Lip}(L-pA_n) \leq \text{Lip}(L-pA) \) and \( \text{Lip}(L-qB_n) \leq \text{Lip}(L-qB) \) the existence and uniqueness of a pair \( [u_n, v_n] \) satisfying (19) follows from Theorem 1. In view of (19) there exists a sequence \( (u, v) \) such that
\[
[u_n, v_n] \to [u, v] \quad \text{in } X \times X.
\]

Using (4), (19) and (21), we obtain
\[
\langle q(u_n - u) + p(v_n - v) \rangle_n \to \langle q(u - u) + p(v - v) \rangle.
\]

Hence
\[
\langle q(u_n - u) + p(v_n - v) \rangle \to \langle q(u - u) + p(v - v) \rangle.
\]
and
\[ \left( \| q(u_r-u) \| + \| p(v_r-v) \| \right)^{1/2} \leq \left( \frac{(pM_k^2 + qM_k)^2}{1-k} + 1 \right) \left( \| q(u-u_0) \| + \| p(v-v_0) \| \right)^{1/2}. \]

From (21) and (22) follows assertion (20).

**Remark 3.** Relation (22) shows that
\[ \| u-u, v-v \|_{X \times X} \leq \text{const} \inf_{\tilde{u}, \tilde{v}} \| u-u_0, v-v_0 \|_{X \times X}. \]

Therefore, the Galerkin sequence \((u_n, v_n)\) gives a "quasi-optimal" approximation of \([u, v]\) by elements of \(A_n\).

The next theorem shows that under the assumptions of Theorem 2 it is possible to approximate the solution \([u, v]\) of (4) by means of a so-called projection-iteration method.

**Theorem 3.** Let (1)–(3), (7) and (15)–(18) be satisfied. If the sequence \((\tilde{u}_n, \tilde{v}_n)\) is determined by
\[ \begin{align*}
L_n(q\tilde{u}_n + p\tilde{v}_n) &= q(L_n - pA_n)\tilde{u}_{n-1} + p(L_n - qB_n)\tilde{v}_{n-1}, & \tilde{u}_n &\in A_n u_n, \\
n &= 1, 2, ..., \quad \tilde{u}_0, \tilde{v}_0 \in X_1 \times X_1 \text{ arbitrary},
\end{align*} \]
then we have
\[ \tilde{u}_n, \tilde{v}_n \rightharpoonup [u, v] \text{ in } X \times X, \]
where \([u, v]\) denotes the solution of (4).

**Proof.** We define \(U_{n+1} : X \rightarrow X_1\) by
\[ U_{n+1} \equiv \left( (u, v) \right) \quad U_{n+1}(qf + pf) = q(L_n - pA_n)u_{n-1} + p(L_n - qB_n)v_{n-1}, \quad u \in A_n u_n. \]

Then \(U_{n+1}\) is strictly contractive with the contraction constant \(k\) (cf. (7)). The fixed point of \(U_{n+1}\) is \((u_n, v_n)\), where \([u_n, v_n]\) denotes the solution of (19). We can write (23) as follows:
\[ \begin{align*}
q(u_n + p_n) &= q(L_n - pA_n)u_{n-1} + p(L_n - qB_n)v_{n-1}, \quad u_n \in A_n u_n.
\end{align*} \]

Therefore, Theorem 3 follows immediately from Theorem 2 and Lemma 3.2, Kap. III, in [11].

### 3. Applications

In this section we show that it is possible to find periodic solutions of certain pseudo-parabolic equations or evolution equations using the methods of the previous sections.

Let \(S = [0, T]\) be a finite interval of the real axis. If \(E\) is a Hilbert space, we denote by \(L^2(S; E)\) the Hilbert space of all square integrable functions defined on \(S\) with values in \(E\) (provided with the usual scalar product) and by \(C(S; E)\) the space of all continuous mappings from \(S\) into \(E\) with the supremum norm.

Let \(Y\) be a real Hilbert space. By \(Y^\ast\) we denote the dual of \(Y\) and by \((\cdot, \cdot)_Y\) the pairing between \(Y^\ast\) and \(Y\). We assume that \((Y_n)\) is a sequence of subspaces of \(Y\) such that
\[ \begin{align*}
Y_n &\subseteq Y_{n+1}, \\
n &= 1, 2, ..., \\
\bigcup_n Y_n &\text{ is dense in } Y.
\end{align*} \]

By \(Y_n^\ast\) we denote the dual of \(Y_n\) and by \((\cdot, \cdot)_n\) the pairing between \(Y_n^\ast\) and \(Y_n\). For the sake of brevity we introduce the following notations:
\[ \begin{align*}
X &= L^2(S; Y), \\
X_n &= L^2(S; Y_n), \\
X_n^\ast &= L^2(S; Y_n^\ast), \\
(f, u) &= \int_0^T (f(t), u(t)) dt, \quad \forall f \in X^\ast, \forall u \in X, \\
(f, u)_n &= \int_0^T (f(t), u(t)) dt, \quad \forall f \in X_n^\ast, \forall u \in X_n.
\end{align*} \]

As in the previous section we denote by \(L\) and \(L_n\) the duality maps of \(X\) and \(X_n\), respectively.

#### 3.1. Pseudo-parabolic equations

If \(u \in X\) we denote by \(u'\) the derivative of \(u\) in the sense of distributions on \([0, T]\) with values in \(Y\). Let
\[ \begin{align*}
W &= \{ u \in X, u' \in X \}, \\
W_n &= \{ u \in X_n, u_0 = u(T) \}, \\
W_n^\ast &= \{ u \in X_n^\ast, u_0 = u(T) \},
\end{align*} \]
and
\[ \begin{align*}
W_n &= \{ u \in X_n, u_0 = u(T) \}.
\end{align*} \]

We assume that \(A \in (X \times X^\ast)\) and \(B \in (X \times X^\ast)\) are operators satisfying (1) and (2). We are interested in problems of the type
\[ \begin{align*}
A u + B u' &= 0, \quad u \in W, \\
A u + B u' &= 0, \quad u \in W_n.
\end{align*} \]

Such problems occur for instance in the theory of viscoelasticity, where \(A\) and \(B\) are given by elliptic differential operators.

**Theorem 4.** Let (1), (2), (9) and (16) be satisfied. Then the problem (25) has a unique solution \(u\). If we put \(p = m_1 M_1^2\) and \(q = m_2 M_2^2\) and determine the sequence \((u_n)\) by
\[ \begin{align*}
L_n(qu_n + p_n u_n) &= q(L_n - pA_n)u_{n-1} + p(L_n - qB_n)u_{n-1}, \quad u_n \in W_n, \\
u_n(0) &= u_n(T), \quad n = 1, 2, ..., \\
u_0 \in W_1 \text{ arbitrary},
\end{align*} \]
then we have \(u_n \rightharpoonup u\) in \(W\).

**Proof.** We define \(A = X \times X^\ast\) by
\[ A = \{ (u, u') \in W, u(0) = u(T) \}. \]

This set \(A\) is maximal monotone (see e.g. [11], Lemma 1.7, Kap. VI). Evidently, problem (25) can be written as
\[ A u + B u' = 0, \quad (u, u') \in A. \]
Therefore, the first part of Theorem 4 follows from Theorem 1. Let \( A_n \subset X_n \times X_n \) be defined by
\[
A_n \coloneqq \{ [u, u'] \mid u \in W_n, u(0) = u(T) \}.
\]
By \( P_n \), we denote the orthogonal projection of \( X \) onto \( X_n \). If \( [u, u'] \in A \) then
\[
[P_n u, (P_n u')^T] = [P_n u, P_n u'] \in A_n
\]
and
\[
[P_n u, (P_n u')^T] \rightarrow [u, u'] \quad \text{in } X \times X.
\]
Hence, the sequence \( (A_n) \) satisfies condition (8). Therefore, (26) is a formulation of method (23) in the special case considered here. Consequently, the second part of Theorem 4 follows from Theorem 3.

Remark 4. Let \( V_n \) be of finite dimension \( d_n \), and let \( h_1, ..., h_n \), be a basis of \( V_n \).
Then we can represent \( u_n \) in the form
\[
u_n = \sum_{i=1}^{d_n} c_i h_i, \quad c_i \in L^2(S),
\]
and we may regard (26) (with \( n \) fixed) as a system of linear ordinary differential equations with respect to the unknown functions \( c_i \). The coefficients of this system are the elements of Gram's matrix of the basis \( h_1, ..., h_n \), which are independent of \( t \in S \).

Remark 5. In the same way as (25) we could treat initial value problems of the type
\[
Au + Bu' = 0, \quad u \in W, \quad u(0) = a,
\]
where \( a \) is a known element of \( V \) and \( A, B \) satisfy the conditions (1), (2), (7). A somewhat different projection-iteration method for problems of the type (27) was formulated already in [11] (Kap. V). This method does not need the strong monotonicity of \( A \) and the assumption (7). On the other hand the operator that corresponds to the operator \( U_{p_n} \) used here is contractive on \( X = L^2(S; V) \) only, if one provides this space with the norm
\[
[u]_{V, k} = \left( \int_S e^{-k u(t)} dt \right)^{1/2},
\]
where \( k \) is a sufficiently great positive number. Moreover, the method requires \( A \) and \( B \) to be so-called Volterra operators.

3.2. Evolution equations. We use all notations introduced at the beginning of this section. Moreover, we assume that \( H \) and \( H_n, n = 1, 2, ..., \) are Hilbert spaces such that
\[
V \text{ is continuously and densely imbedded into } H,
\]
\[
H_n \subset H_{n+1} \subset ... \subset H,
\]
\[
V_n \text{ is continuously and densely imbedded into } H_n.
\]
Identifying the space \( H \) and its dual we obtain
\[
V \subset H \subset V^*.
\]
Similarly, we find
\[
V_n \subset H_n \subset V^*_n \quad (n = 1, 2, ...).
\]
We denote now by \( u' \) the derivative of \( u \in X \) or \( u \in X_n \) in the sense of distributions on \( 0, T \) with values in \( V^* \) or \( V^*_n \), respectively. Let
\[
\tilde{\nu} = [u' \in X, u' \in X^*], \quad [u]_{\tilde{\nu}} = [u]_{V^*} + [u']_{V^*}, \quad \forall u \in \tilde{\nu}
\]
and
\[
\tilde{\nu} = [u \in X_n, u' \in X^*_n], \quad [u]_{\tilde{\nu}} = [u]_{V^*_n} + [u']_{V^*_n}, \quad \forall u \in \tilde{\nu}.
\]
We assume that we are given operators \( A \in (X \to X^*) \) and \( C \in (X^* \to X^*) \) such that \( A \) satisfies (1) and \( C \) is strongly monotone and Lipschitzian, which means that \( B := CL \in (X \to X^*) \) satisfies (2). We are now interested in problems of the type
\[
Au + Cu' = 0, \quad u \in \tilde{\nu}, \quad u(0) = u(T).
\]
Defining \( A \in X \times X \) by
\[
A = \{ [u, L^{-1} u'] \mid u \in \tilde{\nu}, u(0) = u(T) \}
\]
we can write (29) as
\[
Au + Bu = 0, \quad [u, v] \in A.
\]
Let
\[
A_n \coloneqq \{ [u, L^{-1} u'] \mid u \in \tilde{\nu}_n, u(0) = u(T) \}.
\]
It is easily proved that (18) is satisfied also in this case. Therefore, it is possible to apply the results of Sections 1 and 2 to problem (29). We do not want to go into details here.

Remark 6. Gajewski and Gröger [5] have considered already two important special cases of problem (29). The first case is
\[
u' + Au = 0, \quad u \in \tilde{\nu}, \quad u(0) = u(T),
\]
and the second
\[
u' + Lu = 0, \quad u \in \tilde{\nu}, \quad u(0) = u(T).
\]
Gajewski and Gröger treated these cases with the aid of two different maximal monotone operators \( A \) whereas we use the same \( A \) in both cases.

Remark 7. We could apply our results also to initial value problems of the form
\[
Au + Cu' = 0, \quad u \in \tilde{\nu}, \quad u(0) = a \in H.
\]

Remark 8. In the case of the problems (29) and (30) we can prove stronger results on the convergence of the methods considered in this paper, provided the operators \( A \) and \( C \) satisfy some further conditions. We shall deal with this question elsewhere (cf. Gajewski-Gröger [6]–[9], where such stronger results were proved in the two special cases mentioned in Remark 6).
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NUMERICAL METHODS FOR SOLVING VARIATIONAL INEQUALITIES

ANDRZEJ WAKULICZ
Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland

1. Variational inequalities

Let \( X \) be a reflexive Banach space and let \( K \) be a convex, closed, non-empty subset of \( X \). We denote by \( X^* \) the dual space of \( X \) and by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( X^* \) and \( X \). For a given map \( A \) which maps \( X \) into \( X^* \) we consider the following problem:

PROBLEM 1. Find \( u \in K \) such that for every \( v \in K \)

\[ \langle A(u), v - u \rangle \geq 0. \]

Problems of this type often arise in practice (see [1], [2]) and there is a natural need of numerical methods for solving them. Since Problem 1 is a generalization of a problem involving variational equations (for \( K = X \) Problem 1 has the form of a variational equation), therefore a study of approximate methods for solving it is important for numerical methods theory.

A very detailed survey of approximate methods for solving variational inequalities is given in [1]. Here we complement the results of Mosco's paper with an estimation of the rate of convergence.

2. An approximation of a Banach space and its dual

Let \( \Theta \) be a subset of the interval \( (0, 1] \) such that \( \inf \Theta = 0 \) and let \( n \) be a function mapping \( \Theta \) into the set of natural numbers \( \{1, 2, 3, \ldots\} \).

A family \( \{X_h, p_h, \tau_h\}_{h \in \Theta} \) will be called an approximation of a Banach space \( X \) iff for every \( h \in \Theta \)

(i) \( X_h = \mathbb{R}^{n(h)} \), the \( n(h) \)-dimensional Euclidean space,

(ii) \( p_h \): \( X_h \to X, p_h \) (prolongation) is an isomorphism from \( X_h \) onto a closed subspace \( P_h \) of \( X \) (the space of approximants),

(iii) \( \tau_h \) is a linear map from \( X \) into \( X_h \) which is a left inverse of \( p_h \), i.e., for every \( u_h \in X_h \) we have \( \tau_h p_h u_h = u_h \).

[119]