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ON SINGULAR PERTURBATION OF THE STOKES PROBLEM

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In papers [1, 2] the perturbation of the Stokes problem was studied. We mean the case when the original Stokes problem for incompressible media is replaced by elasticity theory equations with Poisson ratio ν approximately equal to 1/2. In this case it was proved that $u_\varepsilon \to u_0$, $p_\varepsilon \to p_0$ where $\varepsilon \sim 1/2 - \nu$, $(u_\varepsilon, p_\varepsilon)$ is a solution of the boundary value problem for elasticity theory equations and (u_0, p_0) is the solution of the Stokes problem. But in the papers mentioned above only the case $\varepsilon = \text{const}$ was considered. We will consider the case $\varepsilon = \varepsilon(x)$. Our technique is different but the results are almost the same. So let us consider the boundary value problem for elasticity theory equations when the Lamé coefficient μ is constant.

For the sake of simplicity of presentation we will consider the Dirichlet boundary conditions

$$\begin{aligned} \mu\Delta u + \nabla(\lambda+\mu)\operatorname{div} u &= F\,, \\ u|_{\partial\Omega} &= 0\,. \end{aligned} \tag{1}$$

Unless otherwise stated, we will assume that $\mu = \text{const}$. In this case, similarly to [3], the boundary value problem (1) can be rewritten in the following form:

$$\begin{gathered} -\Delta u + \nabla p = f\,, \\ \alpha p + \operatorname{div} u = 0\,, \qquad u|_{\partial\Omega} = 0\,; \end{gathered} \tag{2}$$

here $f = -F/\mu$ and $\alpha = \mu/(\lambda+\mu)$. It can be easily seen that the boundary value problem (2) has a more general form than (1) since it covers the case $\alpha = 0$ on part of or on the whole domain Ω.

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Along with (2) let us consider the perturbed boundary value problem

$$(3)\qquad \begin{gathered} -\Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon\,, \\ (\alpha+\varepsilon)p_\varepsilon + \operatorname{div} u_\varepsilon = 0\,, \qquad u_\varepsilon|_{\partial\Omega} = 0\,. \end{gathered}$$

For $\alpha = \text{const}$ it was proved that

$$(4)\qquad \|u_\varepsilon - u\|_{W_2^1} + \|p - p_\varepsilon\| \le c\varepsilon\,.$$

Below we will establish the validity of this estimate for the case of variable α.

Note that $0 \le \alpha \le 1$. If $p \in L_2$ then by p' we will denote the orthogonal projection of p onto the subspace of functions from the space L_2 which are orthogonal to unity (we will denote this subspace by L_2/R_1). Therefore, we have

$$p = s + p'\,, \qquad s = \text{const}\,, \qquad (p', 1) \equiv \int_\Omega p'\,dx = 0\,.$$

Let us prove the uniform boundedness of the functions u_ε and p_ε. To this end, we take the scalar product of the first equation (3) by u_ε and the scalar product of the second equation (3) by p_ε and sum up the results; thus we obtain

$$(5)\qquad \|u_\varepsilon\|_1^2 + ((\alpha+\varepsilon)p_\varepsilon, p_\varepsilon) = (f_\varepsilon, u_\varepsilon)\,.$$

From the first equation (3) for an arbitrary nonzero vector-function ϕ we obtain

$$(6)\qquad \frac{|(\nabla p_\varepsilon, \varphi)|}{\|\phi\|_1} \le \frac{|(\nabla u_\varepsilon, \nabla\varphi)|}{\|\phi\|_1} + \frac{|(f_\varepsilon, \varphi)|}{\|\phi\|_1} \le \|u_\varepsilon\|_1 + \|f_\varepsilon\|_{-1}\,.$$

We use the Babuška–Brezzi inequality from [4]:

$$\|q\|_{L_2} \le c_0 \sup_{\varphi \in \overset{\circ}{W}{}_2^1} \frac{|(q, \operatorname{div}\varphi)|}{\|\varphi\|_1}\,.$$

This inequality and the estimate (6) yield

$$(7)\qquad \|p'_\varepsilon\| \le c_0(\|u_\varepsilon\|_1 + \|f_\varepsilon\|_{-1})\,.$$

Here and above we use the standard notations

$$\|q\| = \|q\|_{L_2}\,, \qquad \|f\|_{-1} = \sup_{\varphi \in W_2^1} \frac{(\varphi, f)}{\|\varphi\|_1}\,, \qquad \|\varphi\|_1 = \|\nabla\varphi\|\,.$$

Since $\alpha + \varepsilon \ge \varepsilon > 0$, the equality (5) gives the estimate

$$\|u_\varepsilon\|_1 \le \|f_\varepsilon\|_{-1}\,.$$

From this estimate and the inequality (7) we obtain

$$(8)\qquad \|u_\varepsilon\|_1 + \|p'_\varepsilon\| \le (2c_0 + 1)\|f_\varepsilon\|_{-1}\,.$$

Finally, since $f_\varepsilon \to f$ the inequality (8) implies the existence of a constant c_1 independent of ε such that the following inequality is satisfied:

$$(9)\qquad \|u_\varepsilon\|_1 + \|p'_\varepsilon\| \le c_1\|f\|_{-1}\,.$$

This means that the norms $\|u_\varepsilon\|_1$ and $\|p'_\varepsilon\|$ are uniformly bounded with respect to ε.

Let $p_\varepsilon = s_\varepsilon + p'_\varepsilon$. Let us prove the uniform boundedness of $|s_\varepsilon|$. Using the expansion of p_ε and estimating the right-hand side of the equality (5) by the ε-inequality we obtain

$$\tag{10} \tfrac{3}{4}\|u_\varepsilon\|_1^2 + s_\varepsilon^2(\alpha+\varepsilon,1) + 2s_\varepsilon((\alpha+\varepsilon),p'_\varepsilon) + ((\alpha+\varepsilon)p'_\varepsilon,p'_\varepsilon) \le c_1^2\|f_\varepsilon\|_{-1}^2\,.$$

Therefore, from the estimates (9) and (10) we have

$$\tag{11} \tfrac{3}{4}\|u_\varepsilon\|_1^2 + s_\varepsilon^2(\alpha+\varepsilon,1) + 2s_\varepsilon(\alpha+\varepsilon,p'_\varepsilon) + ((\alpha+\varepsilon)p'_\varepsilon,p'_\varepsilon) + \|p'_\varepsilon\|^2 \le 2c_1^2\|f\|_{-1}^2\,.$$

Using the ε-inequality we can estimate the term $2s_\varepsilon(\alpha+\varepsilon,p'_\varepsilon)$ on the left-hand side of (11) as follows:

$$\begin{aligned} 2|s_\varepsilon(\alpha+\varepsilon,p'_\varepsilon)| &= 2|(s_\varepsilon\sqrt{\alpha+\varepsilon},\sqrt{\alpha+\varepsilon}p'_\varepsilon)| \\ &\le \varepsilon_1 s_\varepsilon^2(\alpha+\varepsilon,1) + \frac{1}{\varepsilon_1}((\alpha+\varepsilon)p'_\varepsilon,p'_\varepsilon)\,. \end{aligned}$$

Let $\varepsilon_1 \le 1$; substituting this inequality into the estimate (11) we obtain

$$\tag{12} \frac{3}{4}\|u_\varepsilon\|_1^2 + (1-\varepsilon_1)s_\varepsilon^2(\alpha,1) + \left(1-\frac{1}{\varepsilon_1}\right)((\alpha+\varepsilon)p'_\varepsilon,p'_\varepsilon) + \|p'_\varepsilon\|^2 \le 2c_1^2\|f\|_{-1}^2\,.$$

Since we are mainly interested in the asymptotic behaviour as $\varepsilon\to 0$, we assume that $\varepsilon \le 1$. Taking into account that $\alpha \le 1$ and choosing $\varepsilon_1 = 4/5$ we obtain from the inequality (12) the estimate

$$\tfrac{1}{5}s_\varepsilon^2(\alpha,1) + \tfrac{1}{2}\|p'_\varepsilon\|^2 \le c\|f\|_{-1}^2\,.$$

But the inequality (9) yields the boundedness of the norm p'_ε and thus

$$|s_\varepsilon| \le \frac{c}{\sqrt{(\alpha,1)}}\|f\|_{-1}\,.$$

This gives the estimate of the form

$$\tag{13} \|p_\varepsilon\|^2 = s_\varepsilon^2 + \|p'_\varepsilon\|^2 \le c_2\|f\|_{-1}^2\,.$$

Now we pass to the proof of the estimate (4). Let $q_\varepsilon = p - p_\varepsilon$, $v_\varepsilon = u - u_\varepsilon$. The errors v_ε and q_ε are found by solving the problem

$$\tag{14} \begin{gathered} -\Delta v_\varepsilon + \nabla q_\varepsilon = f - f_\varepsilon\,, \\ \alpha q_\varepsilon + \operatorname{div} v_\varepsilon = \varepsilon p_\varepsilon\,, \qquad v_\varepsilon|_{\partial\Omega} = 0\,. \end{gathered}$$

Let us form the scalar product of the first equation (14) and v_ε and the scalar product of the second equation (14) and q_ε. Summing up the results we obtain the equality

$$\tag{15} \|v_\varepsilon\|_1^2 + (\alpha q_\varepsilon,q_\varepsilon) = (f-f_\varepsilon,v_\varepsilon) + \varepsilon(p_\varepsilon,q_\varepsilon)\,.$$

From the first equation (14) we obtain as above the following estimate:

$$\|q'_\varepsilon\| \le c_0(\|v_\varepsilon\|_1 + \|f-f_\varepsilon\|_{-1})\,.$$

Let us multiply both sides of the last inequality by an arbitrary constant λ and take squares. Summing up this result and the equality (15) and estimating the right-hand side, we have

$$\left(\frac{1}{2}-c_0^2\lambda\right)\|v_\varepsilon\|_1^2+(\alpha q_\varepsilon,q_\varepsilon)+\lambda\|q'_\varepsilon\|^2 \le \varepsilon\delta\|q_\varepsilon\|^2+\frac{\varepsilon}{4\delta}\|p_\varepsilon\|^2+\left(\frac{1}{2}-c_0^2\lambda_0\right)\|f-f_\varepsilon\|_{-1}^2\,. \tag{16}$$

Let $q_\varepsilon = l_\varepsilon + q'_\varepsilon$ and $l_\varepsilon = \mathrm{const}$. Then the following sequence of relations holds:

$$\begin{aligned}(\alpha q_\varepsilon,q_\varepsilon)+\lambda\|q'_\varepsilon\|^2 &= l_\varepsilon^2(\alpha,1)+2l_\varepsilon(\alpha,q'_\varepsilon)+(\alpha q'_\varepsilon,q'_\varepsilon)+\lambda\|q'_\varepsilon\|^2\\ &\ge l_\varepsilon^2(\alpha,1)+\alpha(q'_\varepsilon,q'_\varepsilon)+(\lambda q'_\varepsilon,q'_\varepsilon)-\varepsilon_1 l_{\varepsilon_1}^2(\alpha,1)+\frac{1}{\varepsilon_1}(\alpha q'_\varepsilon,q'_\varepsilon)\\ &\ge (1-\varepsilon_1)l_\varepsilon^2(\alpha,1)+\left(\lambda+1-\frac{1}{\varepsilon_1}\right)\|q'_\varepsilon\|^2\,.\end{aligned}$$

Let $\varepsilon_1 = 2/(2+\lambda)$; then the last inequality yields the estimate

$$(\alpha q_\varepsilon,q_\varepsilon)+\lambda\|q'_\varepsilon\|^2 \ge \frac{\lambda(\alpha,1)}{2+\lambda}l_\varepsilon^2+\frac{\lambda}{2}\|q'_\varepsilon\|^2 \ge c_3\|q_\varepsilon\|^2$$

where

$$c_3 = c_3(\lambda) = \min\left\{\frac{\lambda(\alpha,1)}{2+\lambda},\frac{\lambda}{2}\right\}.$$

Note that $c > 0$ since $(\alpha,1) > 0$ by assumption. Choose $\lambda = 1/(8c_0^2)$ and $\delta = c_3/(2\varepsilon)$. Then from the inequality (16) we obtain the estimate

$$\frac{1}{4}\|v_\varepsilon\|_1^2+\frac{c_3}{2}\|q_\varepsilon\|^2 \le \frac{\varepsilon^2}{2c_3}\|p_\varepsilon\|^2+c_4\|f-f_\varepsilon\|_{-1}^2\,. \tag{17}$$

This estimate implies the convergences $v_\varepsilon \to 0$ and $q_\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, if the f_ε converge to f so that the estimate $\|f_\varepsilon - f\|_{-1} \le c\varepsilon$ is satisfied then the convergence will be of first order in ε. Thus the estimate (4) is true.

So we have proved the following theorem:

THEOREM 1. *Let the domain Ω, the functions f and f_ε and the coefficient α be such that the generalised solutions of the problems* (2) *and* (3) *exist and are unique and $f_\varepsilon - f \to 0$. Then the solution $(u_\varepsilon, p_\varepsilon)$ of the problem* (3) *converges to (u,p) with $\varepsilon \to 0$ where (u,p) is the solution of the problem* (2). *Moreover, if $\|f_\varepsilon - f\|_{-1} \le c\varepsilon$ then we can estimate the rate of convergence of $(u_\varepsilon,p_\varepsilon)$ to (u,p). Namely, in this case*

$$\|u_\varepsilon - u\|_1 + \|p_\varepsilon - p\| \le c\varepsilon\,.$$

Remark 1. The convergence of $(u_\varepsilon,p_\varepsilon)$ to (u,p) holds true not only in the continuous case but also in the discrete case when the boundary value problems (2) and (3) are approximated by a finite element method or by a finite difference one. Here it is necessary that the discrete analogue of the Babuška–Brezzi inequality is valid while the operators ∇ and div are approximated so that $(\nabla^h)^* = -\operatorname{div}^h$.

References

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