NUMERICAL ANALYSIS AND MATHEMATICAL MODELLING
BANACH CENTER PUBLICATIONS, VOLUME 29
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1994

ON AN UNCONVENTIONAL VARIATIONAL METHOD FOR SOLVING THE PROBLEM OF LINEAR ELASTICITY WITH NEUMANN OR PERIODIC BOUNDARY CONDITIONS

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Abstract. A new variational formulation of the linear elasticity problem with Neumann or periodic boundary conditions is presented. This formulation does not require any quotient spaces and is advisable for finite element approximations.

1. Introduction. In this paper we give an unconventional variational approach for solving the Neumann problem of linear elasticity on $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, where Ω is a bounded domain with Lipschitz boundary $\partial\Omega$. The solution will be sought in the standard product Sobolev space

$$(H^1(\Omega))^d = \underbrace{H^1(\Omega) \times \ldots \times H^1(\Omega)}_{d \text{ times}},$$

with no other restrictions upon the solution. Thus the formulation will be suitable for finite element approximations.

Recall first the classical formulation of the Neumann problem in linear elasticity (the so-called first basic problem of linear elasticity — see [8, p. 95]) for a nonhomogeneous and anisotropic material, in general: Find a displacement $u \in (C^2(\overline{\Omega}))^d$ such that, for $i = 1, \ldots, d$, we have

$$(1.1) \qquad -\sum_{j,k,l=1}^{d} \frac{\partial}{\partial x_j}(c_{ijkl}\varepsilon_{kl}(u)) = f_i \quad \text{in } \Omega\,,$$

1991 *Mathematics Subject Classification*: 65N30, 73C60.
The paper is in final form and no version of it will be published elsewhere.

$$\sum_{j,k,l=1}^{d} n_j c_{ijkl} \varepsilon_{kl}(u) = g_i \quad \text{on } \partial\Omega\,, \tag{1.2}$$

where n_j are components of the outward unit normal to $\partial\Omega$,

$$\varepsilon_{kl}(v) = \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right), \qquad v = (v_1, \ldots, v_d)^T \in (H^1(\Omega))^d\,, \tag{1.3}$$

$\varepsilon = (\varepsilon_{kl})_{k,l=1}^d$ is the strain tensor and $c_{ijkl} \in C^1(\overline{\Omega})$ are elastic coefficients satisfying

$$c_{jikl} = c_{ijkl} = c_{klij}\,, \qquad i, j, k, l = 1, \ldots, d\,. \tag{1.4}$$

We moreover assume that there exists a constant $C > 0$ such that

$$\sum_{i,j,k,l=1}^{d} c_{ijkl}(x) e_{ij} e_{kl} > C \sum_{i,j=1}^{d} e_{ij}^2 \tag{1.5}$$

for any $x \in \overline{\Omega}$ and any symmetric matrix $(e_{ij})_{i,j=1}^d$, $e_{ij} \in \mathbb{R}^1$. The body forces $f = (f_1, \ldots, f_d)^T \in (L^2(\Omega))^d$ and surface forces $g = (g_1, \ldots, g_d)^T \in (L^2(\partial\Omega))^d$ are supposed to satisfy the following equilibrium condition (for forces and moments):

$$\int_\Omega f^T p \, dx + \int_{\partial\Omega} g^T p \, ds = 0 \qquad \forall p \in P\,, \tag{1.6}$$

where

$$P = \{p \in (H^1(\Omega))^d \mid \varepsilon(p) = 0\}\,. \tag{1.7}$$

It is known that the space P is finite-dimensional (see e.g. [8, p. 78]),

$$D \equiv \dim P = \frac{d(d+1)}{2} \tag{1.8}$$

($D = D(d)$). Basis functions $p^i = p^i(x_1, \ldots, x_d)$, $i = 1, \ldots, D$, in P can be chosen for instance as follows:

$$\begin{aligned} &p^1 = 1 \text{ for } d = 1\,, \\ &p^1 = (1,0)^T,\ p^2 = (0,1)^T,\ p^3 = (x_2, -x_1)^T \text{ for } d = 2\,, \\ &p^1 = (1,0,0)^T,\ p^2 = (0,1,0)^T,\ p^3 = (0,0,1)^T,\ p^4 = (0, x_3, -x_2)^T\,, \\ &p^5 = (-x_3, 0, x_1)^T,\ p^6 = (x_2, -x_1, 0)^T \text{ for } d = 3\,. \end{aligned} \tag{1.9}$$

We see that if u is a solution of the problem (1.1)–(1.2) then so is $u + p$ for any $p \in P$. Therefore, the standard primal variational formulation of (1.1)–(1.2) is usually given in the quotient space $(H^1(\Omega))^d/P$. However, quotient spaces are not suitable for finite element approximations.

When $d > 1$, it is not advisable to look for a finite element solution u_h of (1.1)–(1.2) which would be fixed at D points. Note that the true solution $u \in (H^1(\Omega))^d$ may have singularities just at some of these points, as $H^1(\Omega) \not\subset C(\Omega)$ if $d > 1$. Such finite element approximations can then be incorrect (see e.g. [1, p. 13]).

When we employ a dual variational formulation to (1.1)–(1.2), then there is no trouble with the uniqueness of the solution, but a stress tensor satisfying the equilibrium equations must be known a priori (see [3, p. 50]). This fact often makes the dual approach inapplicable.

Another variational formulation of (1.1)–(1.2) is presented in [8, p. 99]. Here the authors define linear functionals q_i, $i = 1, \ldots, D$, by

$$q_i(v) = \int_S v^T p^i \, ds, \qquad v \in (H^1(\Omega))^d, \tag{1.10}$$

where the p^i are given by (1.9) and $S \neq \emptyset$ is an arbitrary open part of the boundary $\partial\Omega_0$ of a domain $\Omega_0 \subseteq \Omega$ with Lipschitz boundary. They further prove that the bilinear form

$$a(v, w) = \int_\Omega \sum_{i,j,k,l=1}^{d} c_{ijkl}\varepsilon_{ij}(v)\varepsilon_{kl}(w) \, dx, \qquad v, w \in (H^1(\Omega))^d\,, \tag{1.11}$$

associated with the problem (1.1)–(1.2) is V_p-elliptic for

$$V_p = \{v \in (H^1(\Omega))^d \mid q_i(v) = 0,\ i = 1, \ldots, D\}\,. \tag{1.12}$$

However, this space is again unsuitable for finite element approximations due to the constraints $q_i(v) = 0$.

In Section 2 we will introduce a new variational formulation of (1.1)–(1.2), where the linear functionals q_i will appear in a modified bilinear form $\widetilde{a}(\cdot,\cdot)$, but they will not appear in the space of test functions. Section 3 is devoted to finite element approximations. A weak formulation of the Neumann problem with periodic boundary conditions is presented in Section 4.

2. New variational approach to the Neumann problem in linear elasticity. Throughout the paper the symbols C, C_1, C_2 stand for the so-called generic positive constants which need not be the same at each occurrence. The standard norm in $(H^k(\Omega))^m$ (k, m integers) is denoted by $\|\cdot\|_{k,\Omega}$. Set

$$V = (H^1(\Omega))^d\,,$$

and for $c_{ijkl} \in L^\infty(\Omega)$ define a symmetric bilinear form

$$\widetilde{a}(v, w) = a(v, w) + \sum_{i=1}^{D} \gamma_i q_i(v) q_i(w), \qquad v, w \in V\,, \tag{2.1}$$

where $\gamma_i > 0$ are fixed constants, and D, q_i and $a(\cdot,\cdot)$ are defined by (1.8), (1.10) and (1.11), respectively. Further, we put

$$b(v) = \int_\Omega f^T v \, dx + \int_{\partial\Omega} g^T v \, ds, \qquad v \in V\,. \tag{2.2}$$

We will show that there is a unique solution of the following problem: Find

$u \in V$ such that

$$\widetilde{a}(u,v) = b(v) \quad \forall v \in V\,. \tag{2.3}$$

If, moreover, $u \in (C^2(\overline{\Omega}))^d$, we shall prove that u solves (1.1)–(1.2). First, we introduce an important lemma.

LEMMA 2.1. *There exist positive constants C_1, C_2 such that*

$$C_1 \|v\|_{1,\Omega}^2 \le \|\varepsilon(v)\|_{0,\Omega}^2 + \sum_{i=1}^{D} q_i^2(v) \le C_2 \|v\|_{1,\Omega}^2 \quad \forall v \in V\,, \tag{2.4}$$

where the q_i are defined by (1.10).

The proof is based upon Korn's inequality and can be found in [2, p. 309]. For the case $d = 3$ see also [8, p. 97]. ∎

THEOREM 2.2. *There exists a unique solution $u \in V$ of problem* (2.3). *This solution satisfies the conditions*

$$q_i(u) = 0, \quad i = 1, \ldots, D\,, \tag{2.5}$$

where the q_i are defined by (1.10).

Proof. From (1.11) and (1.5) we see that there exists a constant $C > 0$ such that

$$a(v,v) \ge C \|\varepsilon(v)\|_{0,\Omega}^2 \quad \forall v \in V\,. \tag{2.6}$$

Moreover, by (1.7),

$$a(v,p) = 0 \quad \forall v \in V \; \forall p \in P\,. \tag{2.7}$$

(So, in particular, $a(p,p) = 0$ for any $p \in P$, i.e., the bilinear form $a(\cdot,\cdot)$ is not V-elliptic.) By (2.1), (2.6) and (2.4), we find that $\widetilde{a}(\cdot,\cdot)$ is V-elliptic,

$$\begin{aligned}\widetilde{a}(v,v) &= a(v,v) + \sum_{i=1}^{D} \gamma_i q_i^2(v) \\ &\ge C \|\varepsilon(v)\|_{0,\Omega}^2 + \sum_{i=1}^{D} \gamma_i q_i^2(v) \ge C_1 \|v\|_{1,\Omega}^2 \quad \forall v \in V\,.\end{aligned}$$

From (1.10), the Cauchy–Schwarz inequality and the trace theorem we see that the q_i are continuous,

$$|q_i(v)| \le \int_S |v^T p^i| \, ds \le \|v\|_{0,S} \|p^i\|_{0,S} \le C \|v\|_{1,\Omega} \qquad \forall v \in V,\ i = 1, \ldots, D\,. \tag{2.8}$$

Hence, by (2.1), (1.11), (2.8) and (1.3) we deduce that $\widetilde{a}(\cdot,\cdot)$ is also continuous,

$$\begin{aligned}|\widetilde{a}(v,w)| &\le |a(v,w)| + \sum_{i=1}^{D} \gamma_i |q_i(v)||q_i(w)| \\ &\le C_1 \|\varepsilon(v)\|_{0,\Omega} \|\varepsilon(w)\|_{0,\Omega} + C_2 \|v\|_{1,\Omega} \|w\|_{1,\Omega} \\ &\le C \|v\|_{1,\Omega} \|w\|_{1,\Omega} \quad \forall v, w \in V\,.\end{aligned}$$

Since the linear form (2.2) is continuous as well, the existence of a unique $u \in V$ satisfying (2.3) follows from the well-known Lax–Milgram lemma.

Further, we prove (2.5). So, let $u \in V$ be the solution of problem (2.3). Then according to (2.7), (2.1), (2.3), (2.2) and (1.6), we obtain

$$\sum_{i=1}^{D} \gamma_i q_i(u) q_i(p) = a(u,p) + \sum_{i=1}^{D} \gamma_i q_i(u) q_i(p) = \widetilde{a}(u,p) = b(p) = 0 \tag{2.9}$$

for all $p \in P$. Letting

$$\alpha_i = \gamma_i q_i(u)\,, \tag{2.10}$$

we shall prove that

$$\sum_{i=1}^{D} \alpha_i q_i(p) = 0 \ \forall p \in P \quad \Rightarrow \quad \alpha_i = 0,\ i = 1, \ldots, D. \tag{2.11}$$

Let us take, in particular,

$$p = \sum_{i=1}^{D} \alpha_i p^i\,, \tag{2.12}$$

where the p_i are defined by (1.9). Then by (2.9), (2.10), (1.10) and (2.12) we get

$$0 = \sum_{i=1}^{D} \alpha_i q_i(p) = \sum_{i=1}^{D} \alpha_i \int_S p^T p^i \, ds = \int_S p^T p \, ds\,,$$

which yields

$$p = 0 \quad \text{on } S\,. \tag{2.13}$$

If $d = 1$ then clearly $\alpha_1 = 0$. Consider the case $d = 2$. Since $S \neq \emptyset$ is an open part of the boundary $\partial\Omega_0$, there exist at least two different points $(x_1, x_2)^T$ and $(y_1, y_2)^T$ lying on S. So by (2.12), (1.9) and (2.13) we have

$$\begin{pmatrix} \alpha_1 + \alpha_3 x_2 \\ \alpha_2 - \alpha_3 x_1 \end{pmatrix} = 0, \qquad \begin{pmatrix} \alpha_1 + \alpha_3 y_2 \\ \alpha_2 - \alpha_3 y_1 \end{pmatrix} = 0\,,$$

which yields $\alpha_3 = 0$ and thus also $\alpha_1 = \alpha_2 = 0$. For $d = 3$ we also have $\alpha_1 = \ldots = \alpha_6 = 0$, which is proved in [7, p. 91]. Hence (2.11) holds, and by (2.10) we find that $q_i(u) = 0$ for every $i = 1, \ldots, D$. ■

R e m a r k 2.3. From (2.9) we see that the equilibrium condition (1.6) in fact implies (2.5). The converse is also true. Namely, by (2.1) and (2.5) we find that

$\widetilde{a}(u,v) = a(u,v)$ for any $v \in V$, and thus the variational solution $u \in V$ satisfies, by (2.3), the relation

$$a(u,v) = b(v) \qquad \forall v \in V\,. \tag{2.14}$$

From here, (2.7) and (2.2) we get the equilibrium condition (1.6):

$$0 = a(u,p) = b(p) = \int\limits_{\Omega} f^T p\,dx + \int\limits_{\partial\Omega} g^T p\,ds \qquad \forall p \in P\,.$$

Next we prove two theorems characterizing a connection between the classical and variational Neumann problem.

THEOREM 2.4. *Let the coefficients c_{ijkl} occurring in* (1.11) *belong to $C^1(\overline{\Omega})$ and let the variational solution u of problem* (2.3) *belong to $(C^2(\overline{\Omega}))^d$. Then u is also the classical solution of problem* (1.1)–(1.2).

P r o o f. Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with a compact support in Ω. Then by (2.14), the symmetry of $a(\cdot,\cdot)$, (1.4) and by Green's formula

$$\begin{aligned}
0 &= a(u,v) - b(v) = a(v,u) - b(v) \\
&= \int\limits_{\Omega} \Big(\frac{1}{2}\sum_{i,j,k,l} c_{ijkl}\varepsilon_{kl}(u)\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) - f^T v\Big)\,dx \\
&= -\int\limits_{\Omega} \Big(\frac{1}{2}\sum_{i,j,k,l} \Big(\frac{\partial}{\partial x_j}(c_{ijkl}\varepsilon_{kl}(u))v_i + \frac{\partial}{\partial x_i}(c_{jikl}\varepsilon_{kl}(u))v_j\Big) + f^T v\Big)\,dx \\
&= -\sum_i \int\limits_{\Omega} \Big(\Big(\sum_{j,k,l} \frac{\partial}{\partial x_j}(c_{ijkl}\varepsilon_{kl}(u)) + f_i\Big)v_i\Big)\,dx \\
&\qquad\qquad\qquad\qquad \forall v = (v_1,\ldots,v_d)^T \in (C_0^\infty(\Omega))^d\,,
\end{aligned} \tag{2.15}$$

which yields (1.1) due to the density $\overline{C_0^\infty(\Omega)} = H^1(\Omega)$.

We proceed analogously to derive the boundary condition (1.2). We use the just derived (1.1), again the symmetry of $a(\cdot,\cdot)$ and c_{ijkl}, Green's formula and (2.14) to get

$$\begin{aligned}
0 &= a(v,u) - b(v) \\
&= \int\limits_{\partial\Omega} \Big(\frac{1}{2}\sum_{i,j,k,l} (n_j c_{ijkl}\varepsilon_{kl}(u)v_i + n_i c_{jikl}\varepsilon_{kl}(u)v_j) - g^T v\Big)\,ds \\
&= \sum_i \int\limits_{\partial\Omega} \Big(\sum_{j,k,l} n_j c_{ijkl}\varepsilon_{kl}(u) - g_i\Big)v_i\,ds \qquad \forall v = (v_1,\ldots,v_d)^T \in V\,.
\end{aligned} \tag{2.16}$$

Denote by $H^{1/2}(\partial\Omega)$ the space of traces of all functions from $H^1(\Omega)$. Then due to [7, p. 87] we have the following density:

$$L^2(\partial\Omega) = \overline{H^{1/2}(\partial\Omega)}\,,$$

where the closure is taken under the $\|\cdot\|_{0,\partial\Omega}$-norm. From here and (2.16) we find that (1.2) holds. ∎

THEOREM 2.5. *Let $u \in (C^2(\overline{\Omega}))^d$ be the classical solution of problem* (1.1)–(1.2)*. Then u is the variational solution of problem* (2.3) *provided* (2.5) *holds.*

Proof. Let $v = (v_1, \ldots, v_d)^T \in V$ be arbitrary and let us multiply (1.1) by v_i. Then the integration over Ω, summation over $i = 1, \ldots, d$, and the use of Green's formula yield (as in (2.15) and (2.16))

$$\begin{aligned} 0 &= -\sum_{i,j,k,l} \int_{\Omega} \left(\left(\frac{\partial}{\partial x_j}(c_{ijkl}\varepsilon_{kl}(u)) + f_i \right) v_i \right) dx \\ &= \sum_{i,j,k,l} \left(\int_{\Omega} \left(\frac{1}{2} c_{ijkl}\varepsilon_{kl}(u) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - f_i v_i \right) dx - \int_{\partial\Omega} n_j c_{ijkl}\varepsilon_{kl}(u) v_i \, ds \right). \end{aligned}$$

Hence, by (1.2), (1.3), (1.11) and (2.2) we have

$$0 = a(v, u) - b(v) \qquad \forall v \in V\,.$$

Finally, from (2.1), (2.5) and the symmetry of $a(\cdot,\cdot)$ we arrive at

$$\widetilde{a}(u, v) = a(u, v) = b(v) \quad \forall v \in V\,. \ ∎$$

Remark 2.6. Since $\widetilde{a}(\cdot,\cdot)$ is symmetric, the weak formulation (2.3) of the Neumann problem is obviously equivalent to minimizing the functional $J(v) = \frac{1}{2}\widetilde{a}(v,v) - b(v)$ over V for arbitrary positive $\gamma_1, \ldots, \gamma_D$. This is the so-called *variational formulation.* Such an approach leads, by (2.1), to exact penalties methods (see [5, 11]).

3. Finite element approximation. Throughout this section we assume that $V_h \subset V$ is an arbitrary finite element space such that $P \subset V_h$. Let the set S from (1.10) be chosen so that $\overline{S}$ is a union of some faces of several elements. A discrete analogue of problem (2.3) consists in finding $u_h \in V_h$ so that

$$\widetilde{a}(u_h, v_h) = b(v_h) \quad \forall v_h \in V_h\,. \tag{3.1}$$

THEOREM 3.1. *There exists a unique solution $u_h \in V_h$ of problem* (3.1)*. This solution satisfies the conditions*

$$q_i(u_h) = 0, \qquad i = 1, \ldots, D, \tag{3.2}$$

where the q_i are defined by (1.10).

The proof is similar to that of Theorem 2.2. ∎

THEOREM 3.2. *The discrete solution u_h of problem* (3.1) *is independent of the parameters γ_i from* (2.1).

Proof. Let u_h^1 and u_h^2 be two solutions of (3.1) corresponding to two different sequences $\{\gamma_i^1\}$ and $\{\gamma_i^2\}$, respectively. Then, by (3.2),

$$q_i(u_h^1) = q_i(u_h^2) = 0, \quad i = 1, \ldots, D\,. \tag{3.3}$$

From this and (2.1) we see that

$$\widetilde{a}(u_h^k, v_h) = a(u_h^k, v_h) = b(v_h) \qquad \forall v_h \in V_h,\ k = 1, 2\,,$$

which yields

$$a(u_h^1 - u_h^2, v_h) = 0 \qquad \forall v_h \in V_h\,.$$

Setting here $v_h = u_h^1 - u_h^2$, we find by (2.6) that

$$C\|\varepsilon(u_h^1 - u_h^2)\|_{0,\Omega}^2 \le a(u_h^1 - u_h^2, u_h^1 - u_h^2) = 0\,,$$

which implies that $\varepsilon(u_h^1 - u_h^2) = 0$ almost everywhere in Ω. Hence, by (3.3) and (2.4), we have $u_h^1 - u_h^2 = 0$. ∎

Let $m = \dim V_h$ and let $\{v^i\}_{i=1}^m$ be a finite element basis of V_h. Set

$$\widetilde{A} = (\widetilde{a}(v^i, v^j))_{i,j=1}^m, \qquad A = (a(v^i, v^j))_{i,j=1}^m\,. \tag{3.4}$$

Notice that the width of the band of the stiffness matrix $\widetilde{A}$ depends upon the choice of the set S occurring in (1.10). We will illustrate this fact in the following example.

EXAMPLE 3.3. Let $\Omega = (0,1)\times(0,1)$ and let $N > 1$ be a given integer. Divide each side of $\overline{\Omega}$ into $N-1$ equal parts and consider a uniform triangulation of $\overline{\Omega}$ as sketched in Figure 1.

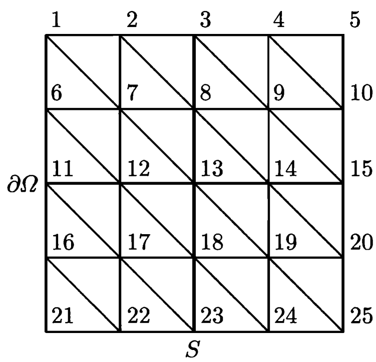

Fig. 1

Assume that the nodes x^j, $j = 1, \dots, N^2$, are numbered e.g. row-wise and denote by φ^i the standard Courant piecewise linear basis functions, i.e., $\varphi^i(x^j) = \delta_{ij}$ for $i, j = 1, \dots, N^2$. Let the space V_h be generated by continuous piecewise linear vector fields over each triangulation. The functions

$$v^{2i-1} = (\varphi^i, 0)^T, \quad v^{2i} = (0, \varphi^i)^T, \qquad i = 1, \dots, m\ (= 2N^2)\,,$$

can obviously be taken as a basis in V_h. Then the symmetric matrix A from (3.4) has a band structure. The half bandwidth equals $2N+2$. We can easily see by (2.1) that the half bandwidth of $\widetilde{A}$ will be the same if we choose S from (1.10) as the lower (or upper) side of $\overline{\Omega}$ — cf. Figure 1. On the other hand, although the choice $S = \partial\Omega$ would yield a sparse matrix $\widetilde{A}$, the band structure would be lost. That is why the set S has to be chosen appropriately. Let us note that the choice of $\gamma_1, \ldots, \gamma_D$ from (2.1) has no influence upon the structure of $\widetilde{A}$, but has a certain influence on the condition number of $\widetilde{A}$ (see [5]).

Let us further note that the classical setting of the discrete Neumann problem consists in minimizing the quadratic functional

$$I(y) = y^T Ay - 2y^T \mathbf{b} \tag{3.5}$$

subject to the linear constraints

$$Qy = 0, \tag{3.6}$$

where A given in (3.4) is singular, $\mathbf{b} = (b(v^i))_{i=1}^m$ and Q is a $D \times m$ matrix corresponding to a discretization of the constraints $q_i(v) = 0$ $(i = 1, \ldots, D)$ from (1.12). The method of Lagrange's multipliers applied to (3.5)–(3.6) then yields the system (cf. [6, p. 1263])

$$\begin{pmatrix} A & Q^T \\ Q & 0 \end{pmatrix} \begin{pmatrix} y \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix},$$

whose matrix is nonsingular but indefinite. On the contrary, formulation (3.1) leads to the system

$$\widetilde{A}y = \mathbf{b}\,,$$

whose matrix is positive definite.

Remark 3.4. The problem of convergence of $\|u - u_h\|_{1,\Omega}$, where u and u_h are defined by (2.3) and (3.1), respectively, can be transformed by Céa's well-known lemma (see e.g. [4, p. 41]) to the study of approximation properties of the system $\{V_h\}$ of finite element subspaces in V.

Remark 3.5. According to [2, p. 309 or p. 317], Lemma 2.1 remains valid if the integration domain S in (1.10) is replaced by Ω_i or S_i, $i = 1, \ldots, D$, where $\Omega_i \neq \emptyset$ are subdomains of Ω with Lipschitz boundaries and $S_i \neq \emptyset$ are arbitrary open parts of $\partial\Omega_i$. Thus all the previous theorems may be formulated also for this case.

4. Periodic boundary conditions. Assume that Ω is rectangular, $\Omega = (0, r_1) \times \ldots \times (0, r_d)$, and consider the problem given by (1.1) with the following periodic boundary conditions:

$$\begin{gathered} u_{|F_m} = u_{|F'_m}, \quad m = 1, \ldots, d, \\ \sum_{j,k,l=1}^{d} n_j c_{ijkl} \varepsilon_{kl}(u)_{|F_m} = \sum_{j,k,l=1}^{d} n_j c_{ijkl} \varepsilon_{kl}(u)_{|F'_m}, \quad i, m = 1, \ldots, d, \end{gathered} \tag{4.1}$$

where F_m and F'_m are opposite faces of Ω which are perpendicular to the axis x_m, $m = 1, \ldots, d$, and c_{ijkl} are from $C^1(\overline{\Omega})$ for the time being. The necessity of solving such a problem arises in studying materials with periodic structure. For single elliptic equations with periodic boundary conditions see e.g. [9, 10]. Set

$$W = \{w \in V \mid w_{|F_m} = w_{|F'_m},\ m = 1, \dots, d\} \tag{4.2}$$

and suppose that some smooth u satisfies (1.1)+(4.1). Multiplying (1.1) by an arbitrary function $w \in W$, integrating this over Ω and using (1.3), (1.4), the Green formula and (4.1), we come to

$$a(u, w) = b(w) \quad \forall w \in W\,, \tag{4.3}$$

where $a(\cdot,\cdot)$ is defined in (1.11), c_{ijkl} may belong to $L^\infty(\Omega)$ now, and

$$b(w) = \int_\Omega f^T w\, dx, \quad w \in W\,. \tag{4.4}$$

Since the functions p^i, $i = 1, \dots, d$, given by (1.9), are constant and belong to W, we find by (1.11) and (4.3) that

$$b(p^i) = 0, \quad i = 1, \dots, d\,. \tag{4.5}$$

Note that $a(\cdot,\cdot)$ is not W-elliptic, since

$$a(p^i, p^i) = 0, \quad i = 1, \dots, d\,. \tag{4.6}$$

It is elliptic only over the space

$$U = \{v \in V \mid q_i(v) = 0,\ v_{|F_i} = v_{|F'_i},\ i = 1, \dots, d\}\,,$$

where the q_i are defined in (1.10). The space U is, however, unsuitable for finite element approximation, as FE-basis functions would have too complicated shape due to the constraints occurring in U. Thus analogously to (2.1), we introduce a modified bilinear form

$$\overset{*}{a}(v, w) = a(v, w) + \sum_{i=1}^d \gamma_i q_i(v) q_i(w), \quad v, w \in W\,, \tag{4.7}$$

where $\gamma_i > 0$ are fixed constants. Consider now a related problem: Find $u \in W$ such that

$$\overset{*}{a}(u, w) = b(w) \quad \forall w \in W\,. \tag{4.8}$$

THEOREM 4.1. *There exists a unique solution $u \in W$ of* (4.8). *This solution satisfies the conditions*

$$q_i(u) = 0, \quad i = 1, \dots, d\,. \tag{4.9}$$

Proof. We first prove by contradiction that

$$\|w\|^2_{1,\Omega} \le C\Big(\|\varepsilon(w)\|^2_{0,\Omega} + \sum_{i=1}^d q_i^2(w)\Big) \quad \forall w \in W\,. \tag{4.10}$$

So, let there exist a sequence $\{w_m\}_{m=1}^{\infty} \subset W$ such that

$$\|w_m\|_{1,\Omega}^2 > m\Big(\|\varepsilon(w_m)\|_{0,\Omega}^2 + \sum_{i=1}^{d} q_i^2(w_m)\Big)\,. \tag{4.11}$$

We may assume, moreover, that the w_m are normalized so that

$$\|w_m\|_{1,\Omega} = 1, \qquad m = 1, 2, \ldots \tag{4.12}$$

From (4.11) and (4.12) we see that

$$\|\varepsilon(w_m)\|_{0,\Omega} \to 0 \qquad \text{as } m \to \infty\,, \tag{4.13}$$

and for any $i \in \{1, \ldots, d\}$,

$$q_i(w_m) \to 0 \qquad \text{as } m \to \infty\,. \tag{4.14}$$

Since the imbedding $H^1(\Omega) \subset L^2(\Omega)$ is compact, there exist $w_0 \in (L^2(\Omega))^d$ and a subsequence of $\{w_m\}$, still denoted by $\{w_m\}$, such that

$$\|w_m - w_0\|_{0,\Omega} \to 0 \qquad \text{as } m \to \infty\,. \tag{4.15}$$

Using now the coercivity of strains, i.e.

$$C\|v\|_{1,\Omega}^2 \le \|\varepsilon(v)\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 \qquad \forall v \in V$$

(see e.g. [8, p. 79]), we get for any $m, r \in \{1, 2, \ldots\}$

$$\begin{aligned} &C\|w_m - w_r\|_{1,\Omega}^2 \\ &\quad \le \|\varepsilon(w_m)\|_{0,\Omega}^2 + 2\|\varepsilon(w_m)\|_{0,\Omega}\|\varepsilon(w_r)\|_{0,\Omega} + \|\varepsilon(w_r)\|_{0,\Omega}^2 + \|w_m - w_r\|_{0,\Omega}^2\,. \end{aligned}$$

This estimate, (4.13) and (4.15) imply that $\{w_m\}$ is a Cauchy sequence in V. Applying (4.15) once again, we find that $w_0 \in V$ and that

$$\|w_m - w_0\|_{1,\Omega} \to 0 \qquad \text{as } m \to \infty\,. \tag{4.16}$$

Since all w_m belong to W, we further deduce by the trace theorem that also $w_0 \in W$. Moreover, (4.13) and (4.16) yield

$$\varepsilon(w_0) = 0\,.$$

From this and (1.9) we observe that w_0 must be constant on Ω as $w_0 \in W$ is periodic. Finally, from definition (1.10) we get

$$w_0 = 0\,, \tag{4.17}$$

since by (2.8), (4.16) and (4.14) we have $q_i(w_0) = 0$ for $i = 1, \ldots, d$. However, (4.17) contradicts (4.12) and (4.16), i.e., (4.10) holds.

Now from (4.7), (1.5) and (4.10), we observe that $\overset{*}{a}(\cdot,\cdot)$ is W-elliptic, i.e.

$$\overset{*}{a}(w, w) = a(w, w) + \sum_{i=1}^{d} \gamma_i q_i^2(w) \ge C_1\|\varepsilon(w)\|_{0,\Omega}^2 + C_2 \sum_{i=1}^{d} q_i^2(w) \ge C_3\|w\|_{1,\Omega}^2$$

for all $w \in W$. Since, moreover, the forms (4.4) and (4.7) are continuous, the existence of a unique solution $u \in W$ follows from the Lax–Milgram lemma.

Further we prove (4.9). By (2.7), (4.7), (4.8) and (4.5),

$$\sum_{i=1}^{d} \gamma_i q_i(u) q_i(p^j) = a(u, p^j) + \sum_{i=1}^{d} \gamma_i q_i(u) q_i(p^j) = \overset{*}{a}(u, p^j) = b(p^j) = 0$$

for any $j = 1, \ldots, d$. Therefore, from (2.10) and (2.11) we come to (4.9). ■

Remark 4.2. Theorems 2.4, 2.5 and Remark 2.6 can be stated analogously also for periodic boundary conditions. If $W_h \subset W$ is an arbitrary finite element space which contains constant functions, then a discrete analogue of (4.8) consists in finding $u_h \in W_h$ such that

$$\overset{*}{a}(u_h, w_h) = b(w_h) \qquad \forall w_h \in W_h\,. \tag{4.18}$$

Theorems 3.1, 3.2 and Remarks 3.4 and 3.5 can be again easily modified to the space W_h. Therefore, the discrete solution u_h is independent of $\gamma_1, \ldots, \gamma_d$, but the stiffness matrix associated with (4.18) and some basis $\{w^i\} \subset W_h$ essentially depends upon $\gamma_1, \ldots, \gamma_d$. This can be seen from the following numerical example.

Example 4.3. Let $d = 3$ and suppose that the elastic body $\Omega = (0,1)\times(0,1)\times(0,1)$ consists of a homogeneous and isotropic material with Lamé's constants $\mu = c_{1212} = 10^{11}[\mathrm{Nm}^{-2}]$ and $\lambda = c_{1122} = 10^{11}[\mathrm{Nm}^{-2}]$. We choose the right-hand side of (1.1) so that

$$u_1(x_1, x_2, x_3) = \frac{1}{100}\sin(2\pi x_1), \qquad u_2 \equiv u_3 \equiv 0$$

is the true solution of the problem (1.1)+(4.1). Let each edge of $\overline{\Omega}$ be divided into N equal parts and let W_h be the corresponding vector finite element space (from (4.18)) generated by the standard trilinear elements, i.e., $\dim W_h = 3N^3$. Let us number the basis functions w^i lexicographically. In order to define the bilinear form (4.7), we further set $S = \{0\}\times(0,1)\times(0,1)$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, where γ is a positive parameter. Due to (4.2), the stiffness matrix $(\overset{*}{a}(w^i, w^j))$ associated with (4.18) will not be a band matrix (as in Example 3.3), but will remain sparse. The system of simultaneous equations has been solved iteratively by the SOR method with $\omega = 1.4$ and zero initial guess in all cases. In the first row of Table 1 we see the values of the expression

$$\frac{\overset{*}{a}(w^1, w^1) - a(w^1, w^1)}{a(w^1, w^1)} \cdot 100\%$$

for various values of the parameter γ. The iteration process has been stopped

Table 1

	3%	7%	15%	30%	300%
$N = 3$	80	29	25	24	23
$N = 4$	110	51	28	26	25
$N = 5$	138	72	36	32	26

when the Euclidean norm of two subsequent iterations was less than 10^{-6}. The corresponding numbers of iterations are given in Table 1.

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