

CONVERGENCE OF APPROXIMATE SOLUTIONS OF A DIFFERENTIAL EQUATION

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1. Introduction and notation. The paper presents a finite-difference method for solving the differential problem

$$(1.1) \quad Lu(t, x) = \frac{\partial u}{\partial t}(t, x) + \frac{\partial(a u)}{\partial x}(t, x) = 0, \quad (t, x) \in \Omega = (0, T) \times \mathbb{R},$$

$$(1.2) \quad u(0, x) = g(x), \quad x \in \mathbb{R},$$

where a is continuous and g is a bounded and measurable function.

The solution of problem (1.1)–(1.2) is investigated in Section 2 (Theorem 1). Section 3 contains the definition of the finite-difference problem, in Section 4 the problem approximating (1.1)–(1.2) is formulated, and the theorems concerning the convergence of the numerical solution are stated (Theorem 2 for $g \in W_1^2(\mathbb{R})$, Theorem 3 for $g \in L_1(\mathbb{R})$). Section 5 contains some results of numerical computation. In the next sections all the results are proved.

Let us now define some function spaces which will be used in the paper. First, the spaces L_p are defined in the usual way, and we use the following norms and moduli of continuity: if $f \in L_p(A)$, $A \subset \mathbb{R}$, then

$$\|f\|_p = \left(\int_A |f(x)|^p dx \right)^{1/p}, \quad \omega_p^k(h, f) = \sup\{\|\Delta_z^k f\|_p : 0 < z \leq h\}, \quad \text{where}$$

$$\Delta_z^k f = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} f(\cdot + jz) \in L_p(A_{kz}), \quad A_\varepsilon = \{x \in A : (x, x + \varepsilon) \subset A\}.$$

Next, let us consider the two-dimensional case. Let $I = [0, T]$, and let $\varrho_1 :$

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$I \rightarrow \mathbb{R} \cup \{-\infty\}$, $\varrho_2 : I \rightarrow \mathbb{R} \cup \{+\infty\}$ be continuous functions, $Q = \{(t, x) : t \in I, \varrho_1(t) < x < \varrho_2(t)\}$, $Q^t = \{x : (t, x) \in Q\}$.

The space $\mathcal{L}_C(Q)$ is defined as the set of all functions u which are measurable on Q and such that for each $t \in I$, $u(t, \cdot) \in L_1(Q^t)$ and

$$(1.3) \quad \|u\|_* = \sup\{\|u^t\|_1 : t \in I\} < \infty, \quad \|\mathcal{P}u^s - \mathcal{P}u^t\|_1 \rightarrow 0 \quad \text{as } s \rightarrow t \\ (u^t = u(t, \cdot))$$

(the function $\mathcal{P}u^s$ is defined on \mathbb{R} by $\mathcal{P}u^s(x) = u^s(x)$ if $x \in Q^s$, $\mathcal{P}u^s(x) = 0$ if $x \in \mathbb{R} \setminus Q^s$); $\|\cdot\|_*$ is the norm in $\mathcal{L}_C(Q)$. The following moduli of continuity will be used in $\mathcal{L}_C(Q)$:

$$\omega_*^k(h, u) = \sup\{\omega_1^k(h, u^t) : t \in I\}, \\ \omega_0(h, u) = \sup\{\|\mathcal{P}u^t - \mathcal{P}u^s\|_1 : s, t \in I, |s - t| \leq h\}.$$

It can be proved (see Section 7) that

$$(1.4) \quad \text{if } u \in \mathcal{L}_C(Q) \text{ then } \omega_*^k(h, u) \rightarrow 0 \text{ as } h \rightarrow 0;$$

the fact that $\omega_0(h, u) \rightarrow 0$ as $h \rightarrow 0$ directly follows from the definition of $\mathcal{L}_C(Q)$.

It will be convenient to introduce C_{mon} , the set of all nondecreasing functions σ such that $\lim_{h \rightarrow 0} \sigma(h) = 0$. Formula (1.4) can thus be written as $\omega_*^k(\cdot, u) \in C_{\text{mon}}$.

Finally, since we are interested mostly in the derivatives with respect to x , we use the notation

$$(1.5) \quad D^k u = D_x^k u = \frac{\partial^k}{\partial x^k} u.$$

2. Solution of the differential problem. To consider the properties of the solution of problem (1.1)–(1.2) we use the characteristics of the operator L , that is, continuous functions $\varphi : I \rightarrow \mathbb{R}$ satisfying

$$\frac{d}{dt} \varphi(t) = a(t, \varphi(t)), \quad t \in I.$$

Throughout this paper we assume that

$$(2.1) \quad a \in C(\Omega), \quad Da \in \mathcal{L}_C(\Omega) \cap L_\infty(\Omega),$$

and we use the notation

$$(2.2) \quad A = \|a\|_\infty, \quad A' = \|Da\|_\infty, \quad A'_* = \|Da\|_*.$$

With this assumption it can be proved that if φ, ψ are two characteristics then

$$e^{A'(s-t)} |\varphi(t) - \psi(t)| \leq |\varphi(s) - \psi(s)| \leq e^{A'(t-s)} |\varphi(t) - \psi(t)| \quad \text{if } 0 \leq s < t \leq T.$$

Hence, no two characteristics have common points and for each $(t, x) \in \Omega$ we can define the function $\lambda(\cdot, t, x)$ as the characteristic passing through (t, x) , that is,

$$\frac{\partial}{\partial s} \lambda(s, t, x) = a(s, \lambda(s, t, x)), \quad \lambda(t, t, x) = x, \quad \text{if } 0 \leq s < t \leq T;$$

we also use the function κ defined by

$$\kappa(s, t, x) = (x - \lambda(s, t, x))/(t - s) \quad (0 \leq s < t \leq T, x \in \mathbb{R}).$$

We thus have

$$(2.3) \quad \begin{aligned} \lambda(s, t, x) &= x - \int_s^t a(\theta, \lambda(\theta, t, x)) d\theta, \\ \kappa(s, t, x) &= \frac{1}{t-s} \int_s^t a(\theta, \lambda(\theta, t, x)) d\theta; \end{aligned}$$

differentiating (2.3) we get

$$(2.4) \quad \begin{aligned} D\lambda(s, t, x) &= \exp\left(-\int_s^t Da(\theta, \lambda(\theta, t, x)) d\theta\right), \\ D\kappa(s, t, x) &= \frac{1}{t-s} \int_s^t Da(\theta, \lambda(\theta, t, x)) D\lambda(\theta, t, x) d\theta. \end{aligned}$$

Therefore,

$$(2.5) \quad \Lambda_{st}^{-1} \leq D\lambda(s, t, x) \leq \Lambda_{st} = e^{A'(t-s)} \quad \text{if } 0 \leq s < t \leq T, x \in \mathbb{R}.$$

It also follows from (2.3) and (2.4) that

$$(2.6) \quad \|\kappa\|_\infty \leq A, \quad \|D\kappa(s, t, \cdot)\|_1 \leq A'_*.$$

If u is a solution of (1.1)–(1.2), then

$$(2.7) \quad u(t, x) = D\lambda(0, t, x)g(\lambda(0, t, x)),$$

where, as in (1.5), $D^k\lambda = \partial^k\lambda/\partial x^k$. This formula allows us to investigate the properties of the solution of (1.1)–(1.2). First, we see that

$$(2.8) \quad \|u^t\|_1 = \|u^0\|_1 \quad \text{for each } t \in I.$$

Next, the following theorem is true.

THEOREM 1. *Let $\varrho \in C^1(I)$, $\Omega_< = \{(t, x) : t \in I, x < \varrho(t)\}$, $\Omega_> = \{(t, x) : t \in I, x > \varrho(t)\}$, $a_< = a|_{\Omega_<}$, $a_> = a|_{\Omega_>}$. We assume that (2.1) is satisfied and*

$$(2.9) \quad D^2a_< \in \mathcal{L}_C(\Omega_<) \cap L_\infty(\Omega_<), \quad D^2a_> \in \mathcal{L}_C(\Omega_>) \cap L_\infty(\Omega_>),$$

$$(2.10) \quad \exists \beta_0 > 0 \forall s \in I |a(s, \varrho(s)) - \varrho'(s)| \geq \beta_0,$$

and we use notation (2.2) and

$$(2.11) \quad \|\varrho'\|_\infty = M, \quad \omega_\infty^1(\varepsilon, \varrho') = \sigma_0(\varepsilon), \quad \omega_0(h, Da) = \sigma_{10}(h),$$

$$(2.12) \quad \|a'_{\varrho\Delta}\|_\infty = A'_\Delta, \quad \text{where } a'_{\varrho\Delta} = a'_{\varrho^+} - a'_{\varrho^-}, \quad a'_{\varrho^\pm} = Da(\cdot, \varrho(\cdot) \pm 0),$$

$$(2.13) \quad \max(\|D^2a_<\|_\infty, \|D^2a_>\|_\infty) = A'', \quad \|D^2a_<\|_* + \|D^2a_>\|_* = A''_*,$$

$$(2.14) \quad \begin{aligned} \omega_0(h, D^2a_<) + \omega_0(h, D^2a_>) &= \sigma_{20}(h), \\ \max(\omega_*^1(h, D^2a_<), \omega_*^1(h, D^2a_>)) &= \sigma_2(h). \end{aligned}$$

Then for every s, t , $D^2\lambda(s, t, \cdot) \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ and

$$(2.15) \quad \begin{aligned} \|D^2\lambda(s, t, \cdot)\|_\infty &\leq e^{2A'(t-s)}((t-s)A'' + A'_\Delta/\beta_0), \\ \|D^2\lambda(s, t, \cdot)\|_1 &\leq e^{A'(t-s)}(t-s)(A''_* + A'_\Delta), \end{aligned}$$

(2.16) there exists a function $\sigma \in C_{\text{mon}}$ such that

$$\omega_1^1(h, D^2\lambda(s, t, \cdot)) \leq \sigma(h) \quad \text{if } 0 \leq s < t \leq T.$$

In the further considerations we use the operators

$$E(s, t) : L_\infty(\mathbb{R}) \rightarrow L_\infty(\mathbb{R}) \quad (0 \leq s < t \leq T)$$

defined by the formula

$$(2.17) \quad [E(s, t)f](x) = D\lambda(s, t, x)f(\lambda(s, t, x)) \quad \forall x \in \mathbb{R}.$$

It follows from (2.7) that the solution of (1.1)–(1.2) satisfies

$$(2.18) \quad u^t = E(s, t)u^s \quad \text{if } 0 \leq s < t \leq T.$$

3. Finite-difference problem. In order to define an approximate solution, we introduce the mesh

$$\begin{aligned} \Omega_h &= \{(t, x) \in \Omega : t = n\tau, x = mh, m, n \in \mathbb{Z}, 0 \leq n \leq N_h\}, \quad N_h = [T/\tau], \\ \Omega'_h &= \{(n\tau, mh) \in \Omega_h : n \leq N_h - 1\}, \quad \mathbb{R}_h = \{x \in \mathbb{R} : x = mh, m \in \mathbb{Z}\}, \end{aligned}$$

where h (the step size) is a parameter from the interval $(0, 1)$, $\tau = \mu h$, and μ is a fixed number (independent of h).

Let $m(A)$ be the set of all functions defined on A . We introduce the following notation for any $w_h \in m(\mathbb{R}_h)$ and $v_h \in m(\Omega_h)$:

$$(3.1) \quad \begin{aligned} v_m^n &= v_h(n\tau, mh), \quad v^n = v_h(n\tau, \cdot), \quad w_m = w_h(mh), \\ \|w_h\|_\infty &= \sup\{|w_m| : m \in \mathbb{Z}\}, \quad \|w_h\|_1 = h \sum_{m \in \mathbb{Z}} |w_m|, \\ \|v_h\|_\infty &= \max\{\|v^n\|_\infty : 0 \leq n \leq N_h\}, \\ \|v_h\|_* &= \max\{\|v^n\|_1 : 0 \leq n \leq N_h\}. \end{aligned}$$

Next, we introduce the difference operator, $L_h : m(\Omega_h) \rightarrow m(\Omega'_h)$, by

$$(3.2) \quad (L_h v_h)_m^n = \frac{1}{\tau} \left[v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n) \right] + \frac{1}{2h} [\alpha_{m+1}^n v_{m+1}^n - \alpha_{m-1}^n v_{m-1}^n],$$

for $v_h \in m(\Omega_h)$, $m \in \mathbb{Z}$, $n = 0, 1, \dots, N_h - 1$,

where $\alpha \in m(\Omega'_h)$ is given, and we formulate the following difference problem: find $v_h \in m(\Omega_h)$ such that

$$(3.3) \quad (L_h v_h)_m^n = 0 \quad \text{for } (n\tau, mh) \in \Omega'_h, \quad v^0 \in m(\mathbb{R}_h) \text{ given.}$$

It can be easily seen that problem (3.3) has a unique solution v_h , and

$$(3.4) \quad \text{if } \mu\|\alpha\|_\infty \leq 1 \text{ then } \|v_h\|_* \leq \|v^0\|_1.$$

4. Approximation of the differential equation. Let us define the operators of restriction (see [1]), $r_h^0 : L_1^{\text{loc}}(\mathbb{R}) \rightarrow m(\mathbb{R}_h)$, $r_h : \mathcal{L}_C(\Omega) \rightarrow m(\Omega_h)$, by the formulas

$$(4.1) \quad \begin{aligned} (r_h^0 f)_m &= \frac{1}{2h} \int_{(m-1)h}^{(m+1)h} f(x) dx \quad \text{if } f \in L_\infty(\mathbb{R}), \\ (r_h u)_m^n &= (r_h^0 u^{n\tau})_m \quad \text{if } u \in \mathcal{L}_C(\Omega) \end{aligned}$$

(u^t is defined in (1.3)).

We consider problem (3.3) where

$$(4.2) \quad \alpha_m^n = a(n\tau, mh), \quad v^0 = r_h^0 g,$$

and assume that

$$(4.3) \quad \mu A = \mu \|a\|_\infty \leq 1.$$

If the operator $F_h : L_\infty(\mathbb{R}) \rightarrow m(\Omega_h)$ is defined by the formula

$$(4.4) \quad \begin{aligned} (F_h g)^0 &= r_h^0 g, \\ (F_h g)_m^{n+1} &= \frac{1}{2}(1 - \mu\alpha_{m+1}^n)(F_h g)_{m+1}^n + \frac{1}{2}(1 + \mu\alpha_{m-1}^n)(F_h g)_{m-1}^n, \\ &\quad (n\tau, mh) \in \Omega'_h, \end{aligned}$$

then $F_h g$ is the unique solution of problem (3.3), (4.2).

Let u be the solution of problem (1.1)–(1.2) and v_h the solution of (3.3), (4.2). Our purpose is to estimate the error of approximation, that is, the function

$$(4.5) \quad z_h = v_h - r_h u,$$

in the norm $\|\cdot\|_*$.

Using definitions (2.17) and (4.4), we can write the error z_h in the form

$$z_h = F_h g - r_h(E(0, \cdot)g),$$

that is, $z^n = (F_h g)^n - r_h^0(E(0, n\tau)g)$ for $0 \leq n \leq N_h$.

The estimate for $\|z_h\|_*$ depends on the regularity of the solution u , and hence of g and a . First, we have the following result.

THEOREM 2. *Assume that conditions (2.1), (4.3) are satisfied and that $D^2\lambda$ satisfies (2.16), and use notation (2.2), (2.11) and*

$$(4.6) \quad \|D\lambda\|_\infty = \Lambda, \quad \|D^2\lambda(0, \cdot, \cdot)\|_* = \Lambda', \quad \omega_*^1(h, Da) = \sigma_1(h).$$

Then for each $g \in W_1^2(\mathbb{R})$,

$$(4.7) \quad \|z_h\|_* \leq M_1(h)\|Dg\|_1 + M_2 h \|D^2g\|_1,$$

where $M_1(h) = T\{(\Lambda + \Lambda'/4)\sigma_{10}(\mu h) + \frac{5}{2}\Lambda\sigma_1(h) + (\mu_1/2)\sigma(h) + h\Lambda'(e^{A'\mu h}(\Lambda(1 + (\mu/2)A'_*) + \Lambda'/2) + \Lambda/2 + \Lambda'/4)\}$, $M_2 = T\Lambda\mu_1(\Lambda + \frac{3}{2}\Lambda')$, $\mu_1 = 81/(4\mu)$.

In the next theorem the initial function g has a lower regularity.

THEOREM 3. *Let the assumptions of Theorem 2 be satisfied. If $g \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ then*

$$(4.8) \quad \|z_h\|_* \leq M_3(h)\omega_1^1(\psi(h), g) + M_4\omega_1^2(\psi(h), g),$$

where $\psi(h) = \max(\sqrt{h}, M_1(h))$, $M_3(h) = M_1(h)/\psi(h)$, $M_4 = \frac{13}{3} + 3M_2$, and $M_1(h)$, M_2 are taken from Theorem 2.

These theorems are proved in Section 8.

5. Numerical examples. We present here some numerical results. We consider problem (1.1)–(1.2) where a is constant, and g has two values:

$$g(x) = u_- \quad \text{if } x < 0, \quad g(x) = u_+ \quad \text{if } x > 0.$$

In this case

$$u(t, x) = u_- \quad \text{if } x < at, \quad u(t, x) = u_+ \quad \text{if } x > at.$$

We also consider problem (3.3) with the coefficients given by (4.2), and the error z_h defined by (4.5). The norm of z_h can be estimated with the use of Theorem 3, where $\psi(h) = \sqrt{h}$, $M_3(h) = 0$, $M_4 = 13/3 + 243/(4\mu)$. We also see that $\omega_1^1(\varepsilon, g) = \varepsilon|u_+ - u_-|$, $\omega_1^2(\varepsilon, g) = 2\varepsilon|u_+ - u_-|$. Theorem 3 says that

$$\|z_h\|_* \leq 2M_4|u_+ - u_-|\sqrt{h}.$$

Below, we present some results of computation for $T = 1$ and different values of a , u_- , u_+ , h .

$u_- = 1.00, u_+ = 2.00, a = 0.00, \mu = 1.00$		
h	$\ z^N\ _1$	$\ z^N\ _1/\sqrt{h}$
0.010000000	0.051630	0.516302
0.005000000	0.037469	0.529891
0.002500000	0.026986	0.539718
0.001250000	0.019331	0.546776
0.000625000	0.013796	0.551822
0.000312500	0.009818	0.555417

$u_- = 0.00, u_+ = 1.00, a = 1.60, \mu = 0.50$		
h	$\ z^N\ _1$	$\ z^N\ _1/\sqrt{h}$
.025000000	0.064448	0.407604
.012500000	0.047719	0.426813
.006250000	0.034880	0.441199
.003125000	0.025255	0.451779
.001562500	0.018162	0.459467
.000781250	0.012997	0.465005

$u_- = 2.00, u_+ = 3.00, a = 0.80, \mu = 1.00$		
h	$\ z^N\ _1$	$\ z^N\ _1/\sqrt{h}$
0.020000000	0.039135	0.276724
0.010000000	0.029301	0.293008
0.005000000	0.021596	0.305414
0.002500000	0.015732	0.314644
0.001250000	0.011363	0.321402
0.000625000	0.008157	0.326297

$u_- = 2.00, u_+ = 3.00, a = 0.80, \mu = 0.50$		
h	$\ z^N\ _1$	$\ z^N\ _1/\sqrt{h}$
0.020000000	0.093952	0.664340
0.010000000	0.068316	0.683164
0.005000000	0.049276	0.696864
0.002500000	0.035337	0.706746
0.001250000	0.025238	0.713832
0.000625000	0.017972	0.718892

$u_- = 0.80, u_+ = 2.10, a = -1.00, \mu = 0.60$			$u_- = -1.00, u_+ = 1.00, a = 0.17, \mu = 0.80$		
h	$\ z^N\ _1$	$\ z^N\ _1/\sqrt{h}$	h	$\ z^N\ _1$	$\ z^N\ _1/\sqrt{h}$
0.020833333	0.096772	0.670454	0.016666667	0.145127	1.124150
0.010416667	0.070888	0.694555	0.008333333	0.105862	1.159659
0.005208333	0.051405	0.712290	0.004166667	0.076573	1.186262
0.002604167	0.037007	0.725180	0.002083333	0.054977	1.204496
0.001302083	0.026503	0.734471	0.001041667	0.039304	1.217783
0.000651042	0.018910	0.741128	0.000520833	0.028004	1.227092

Thus, we observe that the convergence of $\|z_h\|_*$ is of order \sqrt{h} , as stated in Theorem 3.

6. Auxiliary formulas and lemmas. All the results presented in this section are proved in Section 7.

Let us start from a lemma which allows us to obtain estimates for functions of low regularity.

LEMMA 4. *Let X be a Banach space and consider the operator $\Phi : L_p(\mathbb{R}^n) \rightarrow X$. Assume that there exist nonnegative numbers $M, \eta, C_0, C_1, \dots, C_k$ such that*

$$(6.1) \quad \forall g, g' \in L_p(\mathbb{R}^n) \quad \|\Phi(g) - \Phi(g')\|_X \leq M\|g - g'\|_p,$$

$$(6.2) \quad \forall f \in W_p^k(\mathbb{R}^n) \quad \|\Phi(f)\|_X \leq \eta \sum_{l=0}^k C_l |f|_p^{(l)}.$$

Then there exist constants N, N_0, \dots, N_k (depending only on k, n, p) such that for every $g \in L_p(\mathbb{R}^n)$,

$$(6.3) \quad \|\Phi(g)\|_X \leq (MN + N_k C_k) \omega_p^k(\eta^{1/k}, g) + \sum_{l=0}^{k-1} \eta^{1-l/k} N_l C_l \omega_p^l(\eta^{1/k}, g).$$

It can be checked that

$$(6.4) \quad \begin{aligned} &\text{if } n = k = p = 1, \text{ then } N = \frac{3}{2}, N_0 = N_1 = 1; \\ &\text{if } n = p = 1, k = 2, \text{ then } N = \frac{13}{3}, N_0 = N_1 = N_2 = 3. \end{aligned}$$

The next two lemmas will be used in the proof of Theorem 2.

LEMMA 5. *Let the operators $\pi_h : L_1^{\text{loc}}(\mathbb{R}) \rightarrow L_\infty^{\text{loc}}(\mathbb{R})$ ($h \in H \subset \mathbb{R}_+$) be defined by*

$$(6.5) \quad (\pi_h f)(x) = \int_{\mathbb{R}} W_h(x, z) f(x + zh) dz,$$

where W_h are bounded measurable functions on \mathbb{R}^2 satisfying

$$(6.6) \quad \forall h \in H \exists \beta_h > 0 \forall x \in \mathbb{R} \quad \text{supp } W_h(x, \cdot) \subset [-\beta_h, \beta_h].$$

For $x \in \mathbb{R}$ let $\psi_{jh}(x) = \int_{\mathbb{R}} W_h(x, z) z^j dz$. If $\psi_{jh} \in L_{p(j)}(\mathbb{R})$ ($j = 0, 1, \dots, k - 1$,

$1 \leq p(j) \leq \infty$), then

$$(6.7) \quad \forall f \in W_1^k(\mathbb{R}) \quad \|\pi_h f\|_1 \leq \sum_{j=0}^k \vartheta_{jh} h^j \|D^j f\|_{p'(j)} + \vartheta'_{kh} h^k \omega_1^1(h\beta_h, D^k f),$$

where $\vartheta_{jh} = (1/j!) \|\psi_{jh}\|_{p(j)}$, $\vartheta'_{kh} = (2/(k+1)!) \beta_h^{k+1} \|W_h\|_\infty$, $1/p(j) + 1/p'(j) = 1$.

LEMMA 6. Let $f \in L_1(\mathbb{R})$,

$$\varphi_{m,h}(f) = \left| \int_{(m-1)h}^{(m+1)h} f(x) dx - \int_{\phi_{m-1}}^{\phi_{m+1}} f(y) dy \right| \quad \text{for } m \in \mathbb{Z},$$

and $|\phi_m - mh| \leq h$, $he^{-Bh} \leq \Delta\phi_m \leq he^{Bh}$. Then

$$(6.8) \quad \sum_{m \in \mathbb{Z}} \varphi_{m,h}(f) \leq 10\omega_1^1(h, f) + 2Bhe^{Bh} \|f\|_1.$$

The following lemma is needed for proving Theorem 1.

LEMMA 7. Let $f \in L_1(a, b)$, $\varphi : [c, d] \rightarrow [a, b]$, $0 < P_2^{-1} \leq D\varphi(x) \leq P_1$ for almost every $x \in [c, d]$. Then

$$(6.9) \quad \omega_1^1(h, f(\varphi(\cdot))D\varphi) \leq (3 + P_1)\omega_1^1(h, \mathcal{P}f) + P_2\omega_\infty^1(h, D\varphi) \|f\|_1.$$

Finally, we formulate some properties of measurable functions:

$$(6.10) \quad \text{if } g \in W_1^1(a, b) \text{ then } g \in L_\infty(a, b) \text{ and } \|g\|_\infty \leq (2/(b-a))\|g\|_1 + \|Dg\|_1,$$

$$(6.11) \quad \text{if } g \in W_1^1(\mathbb{R}) \text{ then } g \in L_\infty(\mathbb{R}) \text{ and } \|g\|_\infty \leq \frac{1}{2}\|Dg\|_1,$$

and a formula which can be proved with the use of the mean value theorem:

$$(6.12) \quad \forall a, b \in \mathbb{R} \exists \xi \in (0, 1) \quad e^a - e^b = (\xi e^a + (1-\xi)e^b)(a-b).$$

7. Proofs of auxiliary formulas. In this section all the results from the previous section and formula (1.4) are proved.

PROOF OF LEMMA 4. We first give a definition of multivariate box splines (cf. [1] or [2]), which will be used in the proof. We introduce the class \mathcal{S}_k of all systems of vectors from \mathbb{Z}^n of the form

$$Y = [x_1, \dots, x_r], \quad \text{where } r > nk, \quad x_{lk+j} = e_{l+1} \text{ if } 0 \leq l \leq n-1, \quad 1 \leq j \leq k$$

(e_i is the unit vector of the i th axis) which satisfy the condition: each subsystem of Y consisting of $r-k$ vectors spans the space \mathbb{R}^n .

The multivariate box spline B_Y is the function satisfying the identity

$$\int_{\mathbb{R}^n} B_Y(x) f(x) dx = \int_0^1 \dots \int_0^1 f\left(\sum_{j=1}^r \xi_j x_j\right) d\xi_1 \dots d\xi_r$$

for every $f \in C(\mathbb{R}^n)$.

Let $g \in L_p(\mathbb{R}^n)$ be fixed. If $f \in W_p^k(\mathbb{R}^n)$ is an arbitrary function, it follows from the triangle inequality and assumptions (6.1), (6.2) that

$$(7.1) \quad \|\Phi(g)\|_X \leq M\|g - f\|_p + \eta \sum_{l=0}^k C_l |f|_p^{(l)}.$$

Now, as in the proof of Lemma 2 in [2], we construct f for which the right-hand side of (7.1) can be estimated by the right-hand side of (6.3).

Let $Y \in \mathcal{S}_k$ and let B_Y be the corresponding box spline. Let $t > 0$ and

$$f_t(x) = - \sum_{j=1}^k (-1)^j \binom{k}{j} \int_{\mathbb{R}^n} B_Y(y) g(x + jty) dy;$$

the number t will be chosen later. It is shown in [2] that

$$\|g - f_t\|_p \leq N\omega_p^k(t, g), \quad |f_t|_p^{(l)} \leq t^{-l} N_l \omega_p^l(t, g), \quad l = 0, 1, \dots, k,$$

where N and N_l depend only on k, p and Y . Taking $t = \eta^{1/k}$ we thus obtain the estimate

$$M\|g - f_t\|_p + \eta \sum_{l=0}^k C_l |f|_p^{(l)} \leq MN\omega_p^k(\eta^{1/k}, g) + \sum_{l=0}^k \eta^{1-l/k} N_l C_l \omega_p^l(\eta^{1/k}, g).$$

Inequality (6.3) follows from this formula and (7.1). ■

Proof of (6.4). We use here the notation from the proof of Lemma 4. Let B_l ($l \in \mathbb{Z}_+$) be the Schoenberg splines satisfying the recurrence relation $B_0 = \chi_{[0,1]}$, $B_{l+1} = \int_{-1}^0 B_l(x + \xi) d\xi$. Then B_l is the spline B_Y where $Y = [1, \dots, 1] \in \mathbb{Z}^{l+1}$. Defining the operator $\mathcal{M}_\varepsilon^k$ by $\mathcal{M}_\varepsilon^k g(x) = \int_{\mathbb{R}} B_k(y) g(x + \varepsilon y) dy$, we see that $f_t = - \sum_{j=1}^k (-1)^j \binom{k}{j} \mathcal{M}_{jt}^k g$. Hence $g - f_t = (-1)^k \int_{\mathbb{R}} B_k(y) \Delta_{ty}^k(\cdot) dy$, and consequently

$$\|g - f_t\|_1 \leq \int_{\mathbb{R}} B_k(y) \int_{\mathbb{R}} |\Delta_{ty}^k g(x)| dx dy = \int_{\mathbb{R}} B_k(y) \|\Delta_{ty}^k g\|_1 dy.$$

Since for $j \in \mathbb{Z}$, $\|\Delta_{tj}^k g\|_1 \leq j^k \|\Delta_t^k g\|_1$, and $\|\Delta_{ty}^k g\|_1 \leq \omega_1^k(tj, g)$ if $0 < y \leq j$, we obtain

$$\|g - f_t\|_1 \leq N\omega_1^k(t, g), \quad \text{where } N = \sum_{j=1}^{k+1} j^k \int_{j-1}^j B_k(y) dy.$$

Hence $N = \frac{3}{2}$ if $k = 1$, $N = \frac{13}{3}$ if $k = 2$.

It is shown in the proof of Lemma 2 in [2] that (in the one-dimensional case) $D^l \mathcal{M}_\varepsilon^k g = \varepsilon^{-l} \mathcal{M}_\varepsilon^{k-l}(\Delta_\varepsilon^l g)$, $\|\mathcal{M}_\varepsilon^k g\|_1 \leq \|g\|_1$. Thus

$$\|D^l f_t\|_1 \leq \sum_{j=1}^k \binom{k}{j} (jt)^{-l} \|\mathcal{M}_{jt}^{k-l}(\Delta_{jt}^l g)\|_1 \leq t^{-l} \sum_{j=1}^k \binom{k}{j} \|\Delta_t^l g\|_1 (2^k - 1) t^{-l} \omega_1^l(t, g).$$

Hence $N_i = 2^k - 1$ for $i = 0, 1, \dots, k$, which was to be proved. ■

Proof of Lemma 5. Let $f \in W_1^k(\mathbb{R})$. Applying Taylor's formula we deduce from (6.5) that at each $x \in \mathbb{R}$,

$$\begin{aligned} \pi_h f(x) = \int_{\mathbb{R}} W_h(x, z) \left\{ \sum_{j=0}^k \frac{z^j h^j}{j!} D^j f(x) \right. \\ \left. + \frac{z^k h^k}{(k-1)!} \int_0^1 (1-\xi)^{k-1} \Delta_{\xi z h} D^k f(x) d\xi \right\} dz. \end{aligned}$$

This formula can be transformed to

$$\begin{aligned} \pi_h f(x) = \sum_{j=0}^k \frac{h^j}{j!} \psi_{jh}(x) D^j f(x) \\ + \frac{h^k}{(k-1)!} \int_{\mathbb{R}} W_h(x, z) z^k \int_0^1 (1-\xi)^{k-1} \Delta_{\xi z h} D^k f(x) d\xi dz. \end{aligned}$$

Applying assumption (6.6) and Hölder's inequality we obtain estimate (6.7). ■

Proof of Lemma 6. First, let $f \in W_1^1(\mathbb{R})$. Introducing a new variable into the second integral, $y = \Delta_0 \phi_m(x - (m-1)h)/2h$ (where $\Delta_0 \phi_m = \phi_{m+1} - \phi_{m-1}$), we deduce that

$$\begin{aligned} \varphi_{m,h}(f) \leq \int_{(m-1)h}^{(m+1)h} \left| f(x) - f\left(\phi_{m-1} + \frac{\Delta_0 \phi_m}{2h}(x - (m-1)h)\right) \right| dx \\ + \left| 1 - \frac{2h}{\Delta_0 \phi_m} \right| \int_{\phi_{m-1}}^{\phi_{m+1}} |f(y)| dy. \end{aligned}$$

According to our assumptions,

$$\left| x - \left(\phi_{m-1} + \frac{\Delta_0 \phi_m}{2h}(x - (m-1)h) \right) \right| \leq h \quad \text{and} \quad \left| 1 - \frac{2h}{\Delta_0 \phi_m} \right| \leq Bhe^{Bh}.$$

Hence

$$\varphi_{m,h}(f) \leq \int_{(m-1)h}^{(m+1)h} \int_{x-h}^{x+h} |Df(y)| dy dx + Bhe^{Bh} \int_{\phi_{m-1}}^{\phi_{m+1}} |f(y)| dy,$$

and therefore

$$\sum_{m \in \mathbb{Z}} \varphi_{m,h}(f) \leq 2Bhe^{Bh} \|f\|_1 + 4h \|Df\|_1.$$

At the same time, for any two functions $f, f' \in L_1(\mathbb{R})$,

$$\sum_{m \in \mathbb{Z}} |\varphi_{m,h}(f) - \varphi_{m,h}(f')| \leq 4 \|f - f'\|_1.$$

Using Lemma 4 with $X = l_1$, $\Phi(f) = (\varphi_{m,h}(f))_{m \in \mathbb{Z}}$, and remark (6.4), we obtain (6.8). ■

Proof of Lemma 7. We use Lemma 4. First, we have

$$(7.2) \quad \|\Delta_h(f(\varphi(\cdot))D\varphi)\|_1 = \int_c^{d-h} |f(\varphi(x+h))D\varphi(x+h) - f(\varphi(x))D\varphi(x)| dx.$$

Let us take $g \in W_1^1(\mathbb{R})$. Then

$$\begin{aligned} & \int_c^{d-h} |g(\varphi(x+h))D\varphi(x+h) - g(\varphi(x))D\varphi(x)| dx \\ & \leq \int_c^{d-h} \left| D\varphi(x+h) \int_{\varphi(x)}^{\varphi(x+h)} Dg(y) dy + \Delta_h D\varphi(x)g(\varphi(x)) \right| dx \\ & \leq \int_c^{d-h} D\varphi(x+h) \int_{\varphi(x+h)-P_1h}^{\varphi(x+h)} |Dg(y)| dy dx + \int_c^{d-h} |\Delta_h D\varphi(x)| |g(\varphi(x))| dx. \end{aligned}$$

Introducing new variables of integration, $z = \varphi(x+h)$ in the first integral, and $z = \varphi(x)$ in the second one, and using (7.2), we obtain

$$\|\Delta_h(g(\varphi(\cdot))D\varphi)\|_1 \leq P_1h\|Dg\|_1 + P_2\omega_\infty^1(h, D\varphi)\|g\|_1.$$

At the same time, if $f, f' \in L_1(a, b)$ then

$$\begin{aligned} & \|\Delta_h(f(\varphi(\cdot))D\varphi) - \Delta_h(f'(\varphi(\cdot))D\varphi)\|_1 \\ & \leq \int_c^{d-h} D\varphi(x+h)|(f - f')(\varphi(x+h))| + D\varphi(x)|(f - f')(\varphi(x))| dx \\ & \leq 2\|\mathcal{P}f - \mathcal{P}f'\|_1. \end{aligned}$$

Thus, applying Lemma 4 with (6.4) we obtain (6.9). ■

Proof of (6.10). For almost every $x \in ((a+b)/2, b)$ we have

$$g(x) = \frac{1}{\delta} \int_a^x \left(g(y) + \int_y^x Dg(z) dz \right) dy, \quad \text{where } \delta = x - a > \frac{b-a}{2},$$

and therefore

$$|g(x)| \leq \frac{1}{\delta} \int_a^x |g(y)| dy + \frac{1}{\delta} \int_a^x |Dg(z)| \int_a^z dy dz \leq \frac{2}{b-a}\|g\|_1 + \|Dg\|_1.$$

Similarly, if $x \in (a, (a+b)/2)$ then $g(x) = (1/(b-x)) \int_x^b (g(y) - \int_x^y Dg(z) dz) dy$. Hence, formula (6.10) is proved. ■

Proof of (6.11). For almost every $x \in \mathbb{R}$,

$$g(x) = \int_{-\infty}^x Dg(y) dy = - \int_x^\infty Dg(y) dy.$$

Thus,

$$|g(x)| \leq \min\left(\int_{-\infty}^x |Dg(y)| dy, \int_x^{\infty} |Dg(y)| dy\right) \leq \frac{1}{2}\|Dg\|_1,$$

which was to be proved. ■

Proof of (1.4). For every $t \in I$, $\omega_1^k(h, u^t) \rightarrow 0$ as $h \rightarrow 0$, hence

$$\forall \delta > 0 \forall t > 0 \exists \varepsilon = \varepsilon(t) \forall z < \varepsilon \quad \|\Delta_z^k u^t\|_1 \leq \frac{\delta}{2}.$$

At the same time, the function $t \mapsto \mathcal{P}u^t$ is continuous on I , thus

$$\forall \delta > 0 \exists \eta \forall s, t \ |s - t| \leq \eta \Rightarrow \forall z \quad \|\Delta_z^k(\mathcal{P}u^s - \mathcal{P}u^t)\|_1 \leq \frac{\delta}{2}.$$

Let us take the numbers $0 = t_0 < t_1 < \dots < t_n = T$ such that $t_{i+1} - t_i < 2\eta$, and $\bar{\varepsilon} = \min(\varepsilon(t_0), \dots, \varepsilon(t_n))$. Then for every $t \in I$, if t_i is the point nearest to t , and $z < \bar{\varepsilon}$, we have

$$\|\Delta_z^k u^t\|_1 \leq \|\Delta_z^k u^{t_i}\|_1 + \|\Delta_z^k(\mathcal{P}u^t - \mathcal{P}u^{t_i})\|_1 \leq \delta,$$

which was to be proved. ■

8. Proofs of the main results

Proof of Theorem 1. First, we show that the functions $a_\varrho = a(\cdot, \varrho(\cdot))$ and $a'_{\varrho+}, a'_{\varrho-}$ defined in (2.12) are continuous on I . If s, t are fixed, we have

$$\begin{aligned} |a_\varrho(s) - a_\varrho(t)| &\leq |a(s, \varrho(s)) - a(s, \varrho(t))| + |a(s, \varrho(t)) - a(t, \varrho(t))| \\ (8.1) \quad &\leq \sigma_{30}(|s - t|), \\ \sigma_{30}(\varepsilon) &= A'M\varepsilon + \frac{1}{2}\sigma_{10}(\varepsilon); \end{aligned}$$

the last inequality follows from (2.2), (2.11) and (6.11). Similarly, if $\varrho(s) \leq \varrho(t)$ then applying (2.13), (2.14) and (6.10) we obtain

$$\begin{aligned} |a'_{\varrho+}(s) - a'_{\varrho+}(t)| &\leq |Da(s, \varrho(s) + 0) - Da(s, \varrho(t))| \\ (8.2) \quad &+ |Da(s, \varrho(t)) - Da(t, \varrho(t) + 0)| \leq \sigma_{40}(|s - t|), \\ \sigma_{40}(\varepsilon) &= A''M\varepsilon + \sigma_{20}(\varepsilon). \end{aligned}$$

A similar estimate can be obtained for $a'_{\varrho-}$, hence, $a_\varrho, a'_{\varrho+}, a'_{\varrho-}$ are continuous.

Therefore, since (2.10) is assumed, $a_\varrho - \varrho'$ has a constant sign on I . Without loss of generality we may suppose that it is negative, hence

$$(8.3) \quad \forall s \in I \quad \beta(s) = \varrho'(s) - a(s, \varrho(s)) \geq \beta_0.$$

Let now s, t be fixed, let $\eta = t - s > 0$ and

$$[x', x''] = \{x \in \mathbb{R} : \exists \psi(x) \in [s, t] \ \lambda(\psi(x), t, x) = \varrho(\psi(x))\}.$$

Differentiating the definition of ψ we deduce that

$$(8.4) \quad D\psi(x) = D\lambda(\psi(x), t, x)/\beta(\psi(x)).$$

Hence

$$(8.5) \quad 1/C_1 \leq D\psi(x) \leq C_2, \quad C_1 = \Lambda_{st}(A + M), \quad C_2 = \Lambda_{st}/\beta_0,$$

and there exists a function ξ inverse to ψ , that is, $\xi(\psi(x)) = x$ for $x \in [x', x'']$. For later convenience, we extend ψ onto \mathbb{R} , setting $\psi(x) = s$ for $x < x'$, $\psi(x) = t$ for $x > x''$. Differentiating (2.4) we obtain

$$D^2\lambda(s, t, x) = -D\lambda(s, t, x) \int_s^t D^2a(\theta, \lambda(\theta, t, x)) D\lambda(\theta, t, x) d\theta$$

if $x < x'$ or $x > x''$.

Now, let $x \in \mathbb{R}$ and $h > 0$. Let $\nu(\theta) = \lambda(\theta, t, x)$, $\nu_h(\theta) = \lambda(\theta, t, x + h)$. It follows from (2.5) that

$$(8.6) \quad \nu_h(\theta) - \nu(\theta) \leq \Lambda_{st}h = e^{A'\eta}h.$$

Applying (2.4) and (6.12) we deduce that there exists a number $\xi_h \in (0, 1)$ such that

$$(8.7) \quad D\lambda(s, t, x + h) - D\lambda(s, t, x) = -(\xi_h D\lambda(s, t, x) + (1 - \xi_h) D\lambda(s, t, x + h)) \times \int_s^t (Da(\theta, \nu_h(\theta)) - Da(\theta, \nu(\theta))) d\theta.$$

We divide the interval of integration into three parts: $(s, \psi(x)) \cup (\psi(x), \psi(x + h)) \cup (\psi(x + h), t)$, and use assumption (2.9) to the first and third parts:

$$(8.8) \quad \int_s^t (Da(\theta, \nu_h(\theta)) - Da(\theta, \nu(\theta))) d\theta = \int_s^{\psi(x)} \int_{\nu(\theta)}^{\nu_h(\theta)} D^2a_{>}(\theta, y) dy d\theta + \int_{\psi(x)}^{\psi(x+h)} (Da(\theta, \nu_h(\theta)) - Da(\theta, \nu(\theta))) d\theta + \int_{\psi(x+h)}^t \int_{\nu(\theta)}^{\nu_h(\theta)} D^2a_{<}(\theta, y) dy d\theta.$$

We deduce from (8.7), (2.5), (8.8), (8.6), (2.13), (8.5) and (2.2) that

$$(8.9) \quad |D\lambda(s, t, x + h) - D\lambda(s, t, x)| \leq C_3h, \quad C_3 = \Lambda_{st}^2(\eta A'' + 2A'/\beta_0).$$

Hence, $D\lambda(s, t, \cdot)$ is Lipschitz-continuous on \mathbb{R} and therefore $D^2\lambda(s, t, \cdot)$ is bounded.

Further, we see that $h^{-1}(\nu_h(\theta) - \nu(\theta)) \rightarrow D\lambda(\theta, t, x)$ and $h^{-1}(\psi(x + h) - \psi(x)) \rightarrow D\psi(x)$ as $h \rightarrow 0$. Next, $\nu(\theta) < \varrho(\theta) < \nu_h(\theta)$ if $\theta \in (\psi(x), \psi(x + h))$. Hence, it follows from (8.7) and (8.8) (majorized convergence of integrals) that

$$D^2\lambda(s, t, x) = -D\lambda(s, t, x)(B_{>}(x) - B(x) + B_{<}(x)), \quad \text{where}$$

$$(8.10) \quad B_{>}(x) = \int_s^{\psi(x)} b_{>}(\theta, x) d\theta, \quad B_{<}(x) = \int_{\psi(x)}^t b_{<}(\theta, x) d\theta,$$

$$B(x) = D\psi(x)a'_{\varrho\Delta}(\psi(x)),$$

$b_\varepsilon(\theta, x) = D\lambda(\theta, t, x)D^2a_\varepsilon(\theta, \lambda(\theta, t, x))$, ε stands for $<$ or $>$, $a'_{\varrho\Delta}$ is defined in (2.12). Therefore, due to (2.5), (2.13), (8.5), and (2.12), the first inequality in (2.15) is true. Next, changing the order of integration we deduce that

$$(8.11) \quad \|B_{>}\|_1 \leq \int_s^t \int_{\xi(\theta)}^\infty |b_{>}(\theta, x)| dx d\theta = \int_s^t \|D^2a_{>}^\theta\|_1 d\theta \leq \eta \|D^2a_{>}\|_*,$$

$$\|B_{<}\|_1 \leq \eta \|D^2a_{<}\|_*.$$

Setting $\theta = \psi(x)$ and using (2.12) we obtain

$$(8.12) \quad \|B\|_1 = \int_s^t |a'_{\varrho\Delta}(\theta)| d\theta \leq \eta A'_\Delta.$$

Thus, applying (2.5) and (2.13) we prove that the second inequality in (2.15) is true. Now, we want to show (2.16). First, we see from (8.10) that

$$(8.13) \quad \|\Delta_h D^2\lambda(s, t, \cdot)\|_1 \leq \omega_\infty^1(h, D\lambda(s, t, \cdot))(\|B_{>}\|_1 + \|B_{<}\|_1 + \|B\|_1) \\ + \|D\lambda(s, t, \cdot)\|_\infty \omega_1^1(h, B_{>} - B + B_{<}).$$

The first component has just been estimated; let us consider the second. First, according to (8.10),

$$|\Delta_h B_{>}(x)| \leq \int_s^{\psi(x)} |\Delta_h b_{>}^\theta(x)| d\theta + \int_{\psi(x)}^{\psi(x+h)} |b_{>}(\theta, x+h)| d\theta.$$

The first term on the right-hand side can be estimated by use of Lemma 7 with $f = D^2a_{>}^\theta$, $\varphi = \lambda(\theta, t, \cdot)$:

$$\int_{\mathbb{R}} \int_s^{\psi(x)} |\Delta_h b_{>}^\theta(x)| d\theta dx = \int_s^t \int_{\xi(\theta)}^\infty |\Delta_h b_{>}^\theta(x)| dx d\theta \\ \leq \int_s^t ((3 + \Lambda_{st})\omega_1^1(h, D^2a_{>}^\theta) + \Lambda_{st}\omega_\infty^1(h, D\lambda(\theta, t, \cdot))A_*'') d\theta,$$

the second — from (2.5) and (2.13):

$$\int_{\mathbb{R}} \int_{\psi(x)}^{\psi(x+h)} |b_{>}(\theta, x+h)| d\theta dx = \int_s^t \int_{\xi(\theta)}^{\xi(\theta)+h} |b_{>}(\theta, x)| dx d\theta \leq h\eta\Lambda_{st}A_*''.$$

Combining these two inequalities and applying (8.9) and (2.14) we obtain

$$(8.14) \quad \omega_1^1(h, B_{>}) \leq \sigma_3(h), \quad \sigma_3(h) = \eta((3 + \Lambda_{st})\sigma_2(h) + h\Lambda_{st}A_*''(1 + C_3)).$$

The same estimate holds for $B_{<}$. Further, it follows from (8.4) that if $x' \leq x \leq x'' - h$ then

$$(8.15) \quad |\Delta_h D\psi(x)| \leq |D\lambda(\psi(x+h), t, x+h) - D\lambda(\psi(x), t, x)|/\beta_0 \\ + \Lambda_{st}|\beta(\psi(x+h))^{-1} - \beta(\psi(x))^{-1}|.$$

Applying (2.4), (6.12), (2.5), and next (8.5) and (2.2) we deduce that

$$\begin{aligned} & |D\lambda(\psi(x+h), t, x+h) - D\lambda(\psi(x), t, x+h)| \\ & \leq \Lambda_{st} \int_{\psi(x)}^{\psi(x+h)} |Da(\theta, \lambda(\theta, t, x+h))| d\theta \leq h\Lambda_{st}^2 A' / \beta_0. \end{aligned}$$

Hence, due to (8.9),

$$|D\lambda(\psi(x+h), t, x+h) - D\lambda(\psi(x), t, x)| \leq C_4 h, \quad C_4 = C_3 + \Lambda_{st}^2 A' / \beta_0.$$

Further, it follows from the definition (8.3) of β , and from (2.11), (8.1) and (8.5) that

$$|\beta(\psi(x+h))^{-1} - \beta(\psi(x))^{-1}| \leq \beta_0^{-2} |\beta(\psi(x+h)) - \beta(\psi(x))| \leq \beta_0^{-2} \sigma_4(h),$$

where $\sigma_4(h) = \sigma_0(C_2 h) + \sigma_{30}(C_2 h)$. Applying these inequalities in (8.15) we conclude that

$$\omega_\infty^1(h, D\psi) \leq \sigma_5(h), \quad \sigma_5(h) = (C_4 h + C_2 \sigma_4(h)) / \beta_0.$$

Thus, using Lemma 7 with $\varphi = \psi$, $f = a'_{\varrho\Delta}$ and taking into account (8.5) we obtain

$$\omega_1^1(h, B) \leq (3 + C_2)\omega_1^1(h, a'_{\varrho\Delta}) + C_1 \sigma_5(h) \|a'_{\varrho\Delta}\|_1.$$

Finally, due to (8.2) and (2.12), we have

$$(8.16) \quad \omega_1^1(h, B) \leq \sigma_6(h), \quad \sigma_6(h) = \eta(6 + 2C_2)\sigma_{40}(h) + C_1 A'_\Delta \sigma_5(h) T.$$

Applying inequalities (8.9), (8.11), (8.12), (2.5), (8.14) and (8.16) to (8.13) we conclude that (2.16) is true and $\sigma(h) = C_3 h \eta(A'_* + A'_\Delta) + \Lambda_{st}(2\sigma_3(h) + \sigma_6(h))$. ■

Proof of Theorem 2. Step 1. Formula (3.2) yields for each $(n\tau, mh) \in \mathcal{O}'_h$ the equality

$$(8.17) \quad z_m^{n+1} = \frac{1}{2}(1 - \mu\alpha_{m+1}^n)z_{m+1}^n + \frac{1}{2}(1 + \mu\alpha_{m-1}^n)z_{m-1}^n + \tau(L_h z_h)^n_m.$$

Since (4.3) is satisfied, we have

$$(8.18) \quad \mu \|\alpha\|_\infty \leq 1.$$

Therefore, the coefficients in (8.17) are nonnegative and $\|z^{n+1}\|_1 \leq \|z^n\|_1 + \tau \|(L_h z_h)^n\|_1$, and we deduce by induction that

$$\|z^n\|_1 \leq \|z^0\|_1 + \tau \sum_{j=0}^{n-1} \|(L_h z_h)^j\|_1 \quad (0 \leq n \leq N_h).$$

But, from (4.2), (4.5) and (3.3), $z^0 = 0$ and $L_h z_h = L_h v_h - L_h r_h u = -L_h r_h u$. Hence

$$(8.19) \quad \|z^n\|_1 \leq \tau \sum_{j=0}^{n-1} \|(L_h r_h u)^j\|_1.$$

Thus, we must estimate the norm of $(L_h r_h u)^n$ ($0 \leq n \leq N_h - 1$).

Step 2. Let $h \in H$ and n ($0 \leq n \leq N_h - 1$) be fixed and let us introduce the following notation:

$$(8.20) \quad \begin{aligned} f &= u^{n\tau}, \quad f^+ = u^{(n+1)\tau}, \quad \varphi(x) = \lambda(0, n\tau, x), \\ \lambda_m &= \lambda(n\tau, (n+1)\tau, mh), \quad \phi_m(\theta) = \lambda(\theta, (n+1)\tau, mh), \\ \kappa_m^n &= \kappa(n\tau, (n+1)\tau, mh). \end{aligned}$$

It follows from (2.3), (4.3) and (2.5) that for every $m \in \mathbb{Z}$,

$$(8.21) \quad \begin{aligned} |mh - \phi_m(\theta)| &\leq A((n+1)\tau - \theta) \leq A\tau \leq h; \\ he^{-A'\tau} &\leq \phi_{m+1}(\theta) - \phi_m(\theta) \leq he^{A'\tau}. \end{aligned}$$

According to (3.2) and (4.1) we have

$$(L_h r_h u)_m^n = \frac{1}{2\tau h} \left[\int_{(m-1)h}^{(m+1)h} f^+ - \frac{1}{2} \int_{(m-2)h}^{(m+2)h} f \right] + \frac{1}{4h^2} \left[\alpha_{m+1}^n \int_{mh}^{(m+2)h} f - \alpha_{m-1}^n \int_{(m-2)h}^{mh} f \right].$$

Since u is a solution of problem (1.1)–(1.2), it follows from (2.18) that

$$\int_{(m-1)h}^{(m+1)h} f^+ = \int_{\lambda_{m-1}}^{\lambda_{m+1}} f.$$

Thus, $(L_h r_h u)_m^n$ can be written as $l_h f$ where l_h is an operator acting from $L_1^{\text{loc}}(\mathbb{R})$ to $m(\mathbb{R}_h)$, defined by

$$(8.22) \quad \begin{aligned} (l_h f)_m &= \frac{1}{2\tau h} \left[\int_{\lambda_{m-1}}^{\lambda_{m+1}} f - \frac{1}{2} \int_{(m-2)h}^{(m+2)h} f \right] \\ &\quad + \frac{1}{4h^2} \left[\alpha_{m+1}^n \int_{mh}^{(m+2)h} f - \alpha_{m-1}^n \int_{(m-2)h}^{mh} f \right]. \end{aligned}$$

If we define the prolongation operator (cf. [2]), $p_h^0 : m(\mathbb{R}_h) \rightarrow L_\infty^{\text{loc}}(\mathbb{R})$, by

$$(8.23) \quad (p_h^0 w_h)(x) = \sum_{m \in \mathbb{Z}} w_m \chi_{[0,1)}(x/h - m)$$

(where χ_A is the characteristic function of A), then we can see that $\|p_h^0 w_h\|_1 = \|w_h\|_1$ and thus

$$(8.24) \quad \|(L_h r_h u)^n\|_1 = \|p_h^0 l_h f\|_1.$$

Step 3. We prove here that the operator $\pi_h = p_h^0 l_h$ satisfies the assumptions of Lemma 5. First, we show that (6.5) and (6.6) hold. It follows from (8.22), (8.23) and (4.2) that for $m \in \mathbb{Z}$, $\xi \in [0, 1)$,

$$\begin{aligned} W_h((m + \xi)h, \cdot) &= \frac{1}{2\tau} \left[\chi_{[-1 - \mu\kappa_{m-1}^n - \xi, 1 - \mu\kappa_{m+1}^n - \xi)} - \frac{1}{2} \chi_{[-2 - \xi, 2 - \xi)} \right] \\ &\quad + \frac{1}{4h} [\alpha_{m+1}^n \chi_{[-\xi, 2 - \xi)} - \alpha_{m-1}^n \chi_{[-2 - \xi, -\xi)}]. \end{aligned}$$

Thus, (8.18) and (2.6) imply that $\|W_h\|_\infty \leq 3/(4\tau)$, $\beta_h = 3$. Next, the following formulas are true for $x = (m + \xi)h$, $0 \leq \xi < 1$:

$$(8.25) \quad \begin{aligned} \psi_{0h}(x) &= \frac{1}{2h} \Delta_0(\alpha^n - \kappa^n)_m, \\ \psi_{1h}(x) &= \frac{\mu}{4h} \Delta_0 \kappa_m^n (\kappa_{m+1}^n + \kappa_{m-1}^n) \\ &\quad + \frac{1}{2h} [(1 - \xi)(\alpha - \kappa)_{m+1}^n + (1 + \xi)(\alpha - \kappa)_{m-1}^n], \end{aligned}$$

where $\Delta_0 w_m = w_{m+1} - w_{m-1}$. We want to estimate $\|\psi_{0h}\|_1$ and $\|\psi_{1h}\|_\infty$. First, by (8.25) we deduce that

$$(8.26) \quad \|\psi_{0h}\|_1 = \frac{1}{2} \sum_{m \in \mathbb{Z}} |\Delta_0(\alpha^n - \kappa^n)_m|.$$

$$(8.27) \quad \|\psi_{1h}\|_\infty \leq \frac{\mu}{2h} \|\Delta_0 \kappa^n\|_\infty \|\kappa^n\|_\infty + \frac{1}{h} \|\alpha^n - \kappa^n\|_\infty.$$

It follows from (4.2), (8.20) and (2.3) that

$$(8.28) \quad \begin{aligned} (\alpha - \kappa)_m^n &= \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} (a(n\tau, mh) - a(\theta, \phi_m(\theta))) d\theta \\ &= \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} \left(a(n\tau, mh) - a(\theta, mh) - \int_{mh}^{\phi_m(\theta)} Da(\theta, y) dy \right) d\theta. \end{aligned}$$

Applying estimate (6.11) and using (2.11) we obtain

$$(8.29) \quad |a(n\tau, mh) - a(\theta, mh)| \leq \|a^{n\tau} - a^\theta\|_\infty \leq \frac{1}{2} \|Da^{n\tau} - Da^\theta\|_1 \leq \frac{1}{2} \sigma_{10}(\tau);$$

formula (8.21) and assumption (2.2) imply that

$$\int_{mh}^{\phi_m(\theta)} |Da(\theta, y)| dy \leq AA'((n+1)\tau - \theta).$$

Combining this inequality with (8.29) and (8.28) we obtain

$$(8.30) \quad |\alpha_m^n - \kappa_m^n| \leq \frac{1}{2}(\sigma_{10}(\tau) + \tau AA').$$

Next, we estimate $\Delta_0 \kappa$. By (2.3), (2.2) and (8.21), for fixed n, m ,

$$(8.31) \quad |\Delta_0 \kappa_m^n| \leq \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} \int_{\phi_{m-1}(\theta)}^{\phi_{m+1}(\theta)} |Da(\theta, y)| dy d\theta \leq 2hA'e^{A'\tau}.$$

Finally, let us consider $\Delta_0(\alpha^n - \kappa^n)_m$. It follows from (4.2) and (2.3) that

$$(8.32) \quad \Delta_0(\alpha^n - \kappa^n)_m = \frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} \left\{ \int_{(m-1)h}^{(m+1)h} (Da(n\tau, y) - Da(\theta, y)) dy \right. \\ \left. + \int_{(m-1)h}^{(m+1)h} Da(\theta, y) dy - \int_{\phi_{m-1}(\theta)}^{\phi_{m+1}(\theta)} Da(\theta, y) dy \right\} d\theta.$$

We first have

$$(8.33) \quad \sum_{m \in \mathbb{Z}} \int_{(m-1)h}^{(m+1)h} |Da(n\tau, y) - Da(\theta, y)| dy = 2 \|Da^{n\tau} - Da^\theta\|_1 \leq 2\sigma_{10}(\tau).$$

To estimate the remaining part of our sum, let us observe that if we take $f = Da^\theta$, $\phi_m = \phi_m(\theta)$, $B = A'\mu$, then the assumptions of Lemma 6 are satisfied due to (8.21). Hence,

$$\sum_{m \in \mathbb{Z}} \left| \int_{(m-1)h}^{(m+1)h} Da(\theta, y) dy - \int_{\phi_{m-1}(\theta)}^{\phi_{m+1}(\theta)} Da(\theta, y) dy \right| \leq 10\omega_1^1(h, Da^\theta) + 2A'e^{A'\tau} \tau \|Da^\theta\|_1.$$

Combining this estimate with (8.33), (4.6), (2.11), (2.2) and (8.32) we obtain

$$(8.34) \quad \sum_{m \in \mathbb{Z}} |\Delta_0(\alpha^n - \kappa^n)_m| \leq 2\sigma_{10}(\tau) + 10\sigma_1(h) + 2A'A'_*e^{A'\tau} \tau.$$

Inequalities (8.30), (8.31), (2.6), (4.3) and (8.34) applied to (8.27) and (8.26) imply the estimates

$$\|\psi_{0h}\|_1 \leq \gamma_0(h), \quad \gamma_0(h) = \sigma_{10}(\mu h) + 5\sigma_1(h) + \mu h A' A'_* e^{A'\mu h}, \\ \|\psi_{1h}\|_\infty \leq h^{-1} \gamma_1(h), \quad \gamma_1(h) = \frac{1}{2} \sigma_{10}(\mu h) + h A' (e^{A'\mu h} + \frac{1}{2}).$$

Hence, Lemma 5 yields the inequality

$$(8.35) \quad \|p_h^0 l_h f\|_1 \leq \gamma_0(h) \|f\|_\infty + \gamma_1(h) \|Df\|_1 + \frac{81}{4\mu} \omega_1^1(h, Df).$$

Step 4. We now estimate the terms occurring on the right-hand side of (8.35) by the given numbers. First, it follows from (2.7) that

$$f(x) = D\varphi(x)g(\varphi(x)), \quad Df(x) = D^2\varphi(x)g(\varphi(x)) + (D\varphi(x))^2 Dg(\varphi(x)), \\ \Delta_h Df(x) = \Delta_h D^2\varphi(x)g(\varphi(x+h)) + D^2\varphi(x)\Delta_h(g \circ \varphi)(x) \\ + \Delta_h((D\varphi)^2)(x)Dg(\varphi(x+h)) + (D\varphi(x))^2 \Delta_h(Dg \circ \varphi)(x).$$

Assumption (4.6) implies the estimates

$$\|D\varphi\|_\infty \leq A, \quad \varphi(x+h) - \varphi(x) \leq Ah, \quad \|D^2\varphi\|_1 \leq A'.$$

Hence

$$|\Delta_h(g \circ \varphi)(x)| \leq \int_{\varphi(x)}^{\varphi(x)+Ah} |Dg(y)| dy \leq Ah \|Dg\|_\infty;$$

$$\begin{aligned}
 |\Delta_h((D\varphi)^2)(x)| &\leq |D\varphi(x+h) + D\varphi(x)| \int_x^{x+h} |D^2\varphi(y)| dy \\
 &\Rightarrow \|\Delta_h((D\varphi)^2)\|_1 \leq 2\Lambda\Lambda'h; \\
 |\Delta_h(Dg \circ \varphi)(x)| &\leq \int_{\varphi(x)}^{\varphi(x)+\Lambda h} |D^2g(y)| dy \\
 &\Rightarrow \int_{\mathbb{R}} (D\varphi(x))^2 |\Delta_h(Dg \circ \varphi)(x)| dx \leq \Lambda^2 h \|D^2g\|_1.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 \|f\|_\infty &\leq \Lambda \|g\|_\infty, \quad \|Df\|_1 \leq \Lambda' \|g\|_\infty + \Lambda \|Dg\|_1, \\
 \omega_1^1(h, Df) &\leq \sigma(h) \|g\|_\infty + 3\Lambda'\Lambda h \|Dg\|_\infty + \Lambda^2 h \|D^2g\|_1.
 \end{aligned}$$

Combining these inequalities with (8.35), (8.24) and (8.19) and using (6.11), we obtain

$$\begin{aligned}
 \|z_h\|_* \leq T \left\{ \left[\frac{\Lambda}{2} \gamma_0(h) + \left(\Lambda + \frac{\Lambda'}{2} \right) \gamma_1(h) + \frac{\mu_1}{2} \sigma(h) \right] \|Dg\|_1 \right. \\
 \left. + h\Lambda\mu_1 \left(\frac{3}{2} \Lambda' + \Lambda \right) \|D^2g\|_1 \right\}.
 \end{aligned}$$

This estimate implies (4.7), hence the proof of Theorem 2 is complete. ■

Proof of Theorem 3. The proof is carried out with the use of Lemma 4. Let us consider the Banach space $L(\Omega_h)$ consisting of all mesh functions y_h which are bounded and such that the norm $\|y_h\|_*$ defined by (3.1) is finite. For $h \in (0, 1)$ let the operator $\Phi_h : L_\infty(\mathbb{R}) \cap L_1(\mathbb{R}) \rightarrow L(\Omega_h)$ be defined by $\Phi_h(g) = F_h g - r_h(E(0, \cdot)g)$. We must show that Φ_h satisfies the assumptions of Lemma 4. First, according to (3.4), $\|F_h g\|_* \leq \|(F_h g)^0\|_1 = \|g\|_1$. Next, it follows from formula (2.8) that for every $n > 0$, $\|r_h^0(E(0, n\tau)g)\|_1 = \|E(0, n\tau)g\|_1 = \|g\|_1$. Since Φ_h is linear, we deduce that (6.1) is satisfied for each $h \in H$ and $M = 2$.

Next, by Theorem 2, if $f \in W_2^1(\mathbb{R})$ then $\|z_h\|_* \leq M_1(h) \|Df\|_1 + M_2 h \|D^2f\|_1$. Applying Lemma 4 with $\eta = \max(h, M_1(h)^2)$, we obtain inequality (4.8) and the proof is complete. ■

References

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