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FINITE ELEMENT DISCRETIZATION OF THE KURAMOTO–SIVASHINSKY EQUATION

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Abstract. We analyze semidiscrete and second-order in time fully discrete finite element methods for the Kuramoto–Sivashinsky equation.

1. Introduction. In this paper we study finite element approximations for the solution of the following periodic initial-value problem for the Kuramoto–Sivashinsky (KS) equation: For $T, \nu > 0$, we seek a real-valued function u defined on $\mathbb{R} \times [0, T]$, 1-periodic in the first variable and satisfying

$$u_t + u\, u_x + u_{xx} + \nu\, u_{xxxx} = 0 \qquad \text{in } \mathbb{R} \times [0, T] \tag{1.1}$$

and

$$u(\cdot, 0) = u^0 \qquad \text{in } \mathbb{R}\,, \tag{1.2}$$

where u^0 is a given 1-periodic function. We assume that (1.1)–(1.2) has a unique, sufficiently smooth solution (cf. [8], [17]).

The KS equation was derived independently by Kuramoto and Sivashinsky in the late 70's and is related to turbulence phenomena in chemistry and combustion. It also arises in a variety of other physical problems such as plasma physics and two-phase flows in cylindrical geometries. For the mathematical theory and the physical significance of the KS equation as well as for related computational work we refer the reader to [7], [16], [3], [4], [17], [5], [6], [8], [9], [13], [14], [1] and the references therein; see also Temam [18] for an overview. In [1] the discretization of (1.1)–(1.2) by a Crank–Nicolson finite difference method and a linearization

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thereof by Newton's method is studied. In the present paper we analyze a semidiscrete method and a second-order in time fully discrete finite element method. The discretization in space is based on the standard Galerkin method; for the time discretization the Crank–Nicolson scheme is used.

For $m \in \mathbb{N}$ let $H^m_{\rm per}$ be the periodic Sobolev space of order m, consisting of the 1-periodic elements of $H^m_{\rm loc}(\mathbb{R})$. We denote by $\|\cdot\|_m$ the norm over a period in $H^m_{\rm per}$, by $\|\cdot\|$ the norm in $L^2(0,1)$, and by $(\cdot,\cdot)$ the inner product in $L^2(0,1)$. A variational form of (1.1) is

$$(u_t, v) + (uu_x, v) - (u_x, v') + \nu(u_{xx}, v'') = 0 \quad \forall v \in H^2_{\rm per}\,. \tag{1.3}$$

Taking $v := u(\cdot,t)$ in (1.3) we obtain by periodicity

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 = \|u_x(\cdot,t)\|^2 - \nu\|u_{xx}(\cdot,t)\|^2\,. \tag{1.4}$$

Now, for $v \in H^2_{\rm per}$, $\|v'\|^2 = -(v, v'')$, i.e.,

$$\|v'\|^2 \le \|v\|\|v''\|, \quad v \in H^2_{\rm per}\,. \tag{1.5}$$

Therefore,

$$\|v'\|^2 \le \nu\|v''\|^2 + \frac{1}{4\nu}\|v\|^2, \quad v \in H^2_{\rm per}\,, \tag{1.6}$$

and (1.4) leads to

$$\frac{d}{dt}\|u(\cdot,t)\|^2 \le \frac{1}{2\nu}\|u(\cdot,t)\|^2\,,$$

i.e.,

$$\|u(\cdot,t)\| \le \|u^0\|e^{t/(4\nu)}, \quad 0 \le t \le T\,. \tag{1.7}$$

Moreover, using the well-known Wirtinger inequality

$$\|v'\| \le \frac{1}{2\pi}\|v''\|, \quad v \in H^2_{\rm per}\,, \tag{1.8}$$

(cf. [12]), (1.4) yields

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|^2 \le \left(\frac{1}{4\pi^2} - \nu\right)\|u_{xx}(\cdot,t)\|^2\,,$$

and, consequently,

$$\|u(\cdot,t)\| \le \|u(\cdot,s)\|, \quad 0 \le s \le t \le T, \quad \text{for } \nu \ge \frac{1}{4\pi^2}\,. \tag{1.9}$$

We shall discretize (1.1)–(1.2) in space by the standard Galerkin method. To this end, let $0 = x_0 < x_1 < \ldots < x_J = 1$ be a partition of $[0,1]$, $h := \max_j(x_{j+1} - x_j)$, and $\underline{h} := \min_j(x_{j+1} - x_j)$. Setting $x_{jJ+s} := x_s$, $j \in \mathbb{Z}$, $s = 0, \ldots, J-1$, this partition is extended periodicaly to a partition of $\mathbb{R}$. For integer $r \ge 4$, let S_h^r denote a space of continuously differentiable, 1-periodic splines of degree $r-1$ in which approximations to the solution $u(\cdot,t)$ of (1.1)–(1.2) will be sought for

$0 \le t \le T$. The following approximation property for the family $(S_h^r)_{0<h<1}$ is well known:

$$\inf_{\chi \in S_h^r} \sum_{j=0}^{2} h^j \|v-\chi\|_j \le ch^s \|v\|_s, \quad v \in H_{\text{per}}^s,\ 2 \le s \le r\,, \tag{1.10}$$

(cf., e.g., Schumaker [15], §8.1). Motivated by (1.3) we define the semidiscrete approximation $u_h(\cdot,t) \in S_h^r$, $0 \le t \le T$, to u by

$$(u_{ht},\chi) + (u_h u_{hx},\chi) - (u_{hx},\chi') + \nu(u_{hxx},\chi'') = 0 \quad \forall \chi \in S_h^r\,, \tag{1.11}$$

where $u_h(\cdot,0) := u_h^0 \in S_h^r$, and u_h^0 is such that

$$\|u^0 - u_h^0\| \le ch^r\,. \tag{1.12}$$

In Section 2 we show existence and uniqueness of the semidiscrete approximation, and derive the optimal-order error estimate

$$\max_{0\le t\le T} \|u(\cdot,t) - u_h(\cdot,t)\| \le ch^r\,. \tag{1.13}$$

In analogy to the exact solution, for the semidiscrete approximation the following inequalities hold:

$$\|u_h(\cdot,t)\| \le \|u_h^0\| e^{t/(4\nu)}, \quad 0 \le t \le T\,, \tag{1.14}$$

and

$$\|u_h(\cdot,t)\| \le \|u_h(\cdot,s)\|, \quad 0 \le s \le t \le T, \quad \text{for } \nu \ge \frac{1}{4\pi^2}\,. \tag{1.15}$$

Section 3 is devoted to a second-order in time fully discrete finite element method for (1.1)–(1.2). Let $N \in \mathbb{N}$, $k := T/N$, and $t^n := nk$, $n = 0,\dots,N$. For $v(\cdot,t) \in L^2(0,1)$, $0 \le t \le T$, let

$$v^n := v(\cdot,t^n), \quad \partial v^n := \frac{1}{k}(v^{n+1} - v^n), \quad \text{and} \quad v^{n+1/2} := \frac{1}{2}(v^n + v^{n+1})\,.$$

The Crank–Nicolson approximations $U^n \in S_h^r$ to u^n are then given by $U^0 := u_h^0$, and for $n = 0,\dots,N-1$

$$(\partial U^n,\chi) + (U^{n+1/2}U_x^{n+1/2},\chi) - (U_x^{n+1/2},\chi') + \nu(U_{xx}^{n+1/2},\chi'') = 0 \quad \forall \chi \in S_h^r\,. \tag{1.16}$$

The following discrete analogs to (1.7) and (1.8), respectively, can be easily proved:

$$\|U^n\| \le \|U^0\| e^{\alpha/(4\nu)t^n}, \quad \alpha > 1,\ k \le 8\nu\frac{\alpha-1}{\alpha},\ n = 1,\dots,N\,, \tag{1.17}$$

and

$$\|U^{n+1}\| \le \|U^n\|, \quad n = 0,\dots,N-1, \quad \text{for } \nu \ge \frac{1}{4\pi^2}\,. \tag{1.18}$$

Further, we show existence of the Crank–Nicolson approximations for $k < 8\nu$, and derive the optimal-order error estimate

$$\max_{0\le n\le N} \|u^n - U^n\| \le c(k^2 + h^r)\,. \tag{1.19}$$

We also prove uniqueness of the fully discrete approximations under a mild mesh condition.

It is well known and easily seen that $u(\cdot,t)$ is odd for $0 \le t \le T$ if the initial value u^0 is an odd function. This property carries over to the semidiscrete and the fully discrete approximations provided $\chi \in S_h^r$ implies $\chi(-\cdot) \in S_h^r$.

2. Semidiscretization. In this section we briefly study the semidiscrete approximation u_h. The inequality (1.14) can be proved in the same way as (1.7). Now, it is evident from (1.14) and the fact that S_h^r is finite-dimensional that an estimate of the form

$$\max_{0\le t\le T} \|u_h(\cdot,t)\|_{L^\infty} \le c(h)$$

is valid. Combining this with the fact that the "right-hand side" of the system of O.D.E.'s (1.11) is locally Lipschitz continuous we deduce existence and uniqueness of the semidiscrete approximation u_h.

In the error estimation that follows we will compare the semidiscrete approximation with the elliptic projection of the exact solution. This projection $P_E : H^2_{\text{per}} \to S_h^r$ is defined by

$$\nu(v'' - (P_E v)'', \chi'') - (v' - (P_E v)', \chi') + \lambda(v - (P_E v), \chi) = 0 \quad \forall \chi \in S_h^r\,, \tag{2.1}$$

where $\lambda > 1/(2\nu)$. For the elliptic projection we have the following estimate:

$$\sum_{j=0}^{2} h^j \|v - P_E v\|_j \le c h^s \|v\|_s, \qquad v \in H^s_{\text{per}},\ 2 \le s \le r \tag{2.2}$$

(cf. [11]). This estimate can be proved in the usual manner. First, using the fact that the bilinear form a,

$$a(v,w) := \nu(v'', w'') - (v', w') + \lambda(v,w)\,,$$

is continuous and coercive in H^2_{per} (cf. (1.5)), the Lax–Milgram lemma yields, in view of the approximation property (1.10),

$$\|v - P_E v\|_2 \le c h^{s-2} \|v\|_s, \qquad v \in H^s_{\text{per}},\ 2 \le s \le r\,. \tag{2.3}$$

Next, to estimate $\|v - P_E v\|$ consider the auxiliary problem

$$a(\psi, w) = (v - P_E v, w) \quad \forall w \in H^2_{\text{per}}\,.$$

Then, for $\chi \in S_h^r$ we have

$$\|v - P_E v\|^2 = a(\psi - \chi, v - P_E v) \le c\|\psi - \chi\|_2 \|v - P_E v\|_2\,.$$

Therefore, the well-known regularity estimate $\|\psi\|_4 \le c\|v - P_E v\|$, easily established in our one-dimensional case, and (1.10), (2.3) yield

$$\|v - P_E v\| \le ch^s \|v\|_s, \qquad v \in H^s_{\mathrm{per}},\ 2 \le s \le r\,. \tag{2.4}$$

The estimate (2.2) now follows from (2.3), (2.4) and (1.5).

THEOREM 2.1. *Let the solution u of* (1.1)–(1.2) *be sufficiently smooth*, *and let* (1.12) *hold. Then*

$$\max_{0\le t\le T} \|u(\cdot,t) - u_h(\cdot,t)\| \le ch^r\,. \tag{2.5}$$

Proof. Let $W(\cdot,t) := P_E u(\cdot,t)$, $\varrho(\cdot,t) := u(\cdot,t) - W(\cdot,t)$, and $\vartheta(\cdot,t) := W(\cdot,t) - u_h(\cdot,t)$. Then $u - u_h = \varrho + \vartheta$ and by (2.2)

$$\max_{0\le t\le T} \|\varrho(\cdot,t)\| \le ch^r\,. \tag{2.6}$$

Thus, it remains to estimate $\|\vartheta(\cdot,t)\|$. Using (1.11), (2.1) and (1.3) we have, for $\chi \in S^r_h$,

$$\begin{aligned}(\vartheta_t,\chi) + a(\vartheta,\chi) &= (W_t,\chi) + a(W,\chi) - (u_{ht},\chi) - a(u_h,\chi)\\ &= (W_t,\chi) + a(u,\chi) + (u_h u_{hx},\chi) - \lambda(u_h,\chi)\\ &= (\lambda\varrho - \varrho_t,\chi) - (uu_x - u_h u_{hx},\chi) + \lambda(\vartheta,\chi)\,,\end{aligned}$$

i.e.,

$$\begin{aligned}&(\vartheta_t,\chi) + \nu(\vartheta_{xx},\chi'') - (\vartheta_x,\chi')\\ &\qquad = (\lambda\varrho - \varrho_t + \varrho\varrho_x + \vartheta\vartheta_x,\chi) + (u\varrho + W\vartheta,\chi') \qquad \forall \chi \in S^r_h\,.\end{aligned} \tag{2.7}$$

A straightforward consequence of the commutativity of P_E with time differentiation is

$$\max_{0\le t\le T} \|\varrho_t(\cdot,t)\| \le ch^r\,. \tag{2.8}$$

Further, (2.2) yields in our one-dimensional case

$$\max_{0\le t\le T} \|W(\cdot,t)\|_{L^\infty} \le c\,. \tag{2.9}$$

Taking $\chi := \vartheta(\cdot,t)$ in (2.7) and using (2.6), (2.8) and (2.9) we obtain by periodicity

$$\frac{1}{2}\frac{d}{dt}\|\vartheta(\cdot,t)\|^2 + \nu\|\vartheta_{xx}\|^2 - \|\vartheta_x\|^2 \le ch^{2r} + c\|\vartheta\|^2 + \|\vartheta_x\|^2\,.$$

Therefore, using (1.5) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\vartheta(\cdot,t)\|^2 \le ch^{2r} + c\|\vartheta\|^2\,,$$

and Gronwall's lemma yields, in view of (1.12),

$$\max_{0\le t\le T} \|\vartheta(\cdot,t)\| \le ch^r\,, \tag{2.10}$$

which concludes the proof. ∎

3. Crank–Nicolson discretization. In this section we show existence of the Crank–Nicolson approximations $U^1, \dots, U^N$ for $k < 8\nu$, derive the optimal-order error estimate (1.19), and under a mild mesh condition prove uniqueness of the Crank–Nicolson approximations. We also briefly discuss the case of an odd initial value.

Taking $\chi := U^{n+1/2}$ in (1.16) we obtain by periodicity

$$\|U^{n+1}\|^2 - \|U^n\|^2 = 2k\{\|U_x^{n+1/2}\|^2 - \nu\|U_{xx}^{n+1/2}\|^2\}, \tag{3.1}$$

and (1.18) follows using (1.8). Further, using (1.6) we obtain from (3.1),

$$\|U^{n+1}\|^2 - \|U^n\|^2 \le \frac{k}{2\nu}\|U^{n+1/2}\|^2,$$

i.e.,

$$\left(1 - \frac{k}{8\nu}\right)\|U^{n+1}\| \le \left(1 + \frac{k}{8\nu}\right)\|U^n\|, \qquad n = 0, \dots, N-1. \tag{3.2}$$

For $\alpha > 1$ obviously

$$\frac{8\nu + k}{8\nu - k} \le 1 + \frac{\alpha}{4\nu}k \quad \text{ for } k \le 8\nu\frac{\alpha - 1}{\alpha},$$

and (1.17) follows easily from (3.2).

Existence. We shall use the following well-known variant of the Brouwer fixed-point theorem (see, e.g., Browder [2]).

LEMMA 3.1. *Let $(H, (\cdot,\cdot)_H)$ be a finite-dimensional inner product space and denote by $\|\cdot\|_H$ the induced norm. Suppose that $g : H \to H$ is continuous and there exists an $\alpha > 0$ such that $(g(x), x)_H > 0$ for all $x \in H$ with $\|x\|_H = \alpha$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \le \alpha$.* ∎

The proof of existence of $U^0, \dots, U^N$ for $k < 8\nu$ is by induction. Assume that $U^0, \dots, U^n$, $n < N$, exist and let $g : S_h^r \to S_h^r$ be defined by

$$(g(V), \chi) = 2(V - U^n, \chi) + k(VV', \chi) - k(V', \chi') + \nu k(V'', \chi'') \quad \forall V, \chi \in S_h^r.$$

This mapping is obviously continuous. Furthermore, by periodicity we have

$$(g(V), V) = 2(V - U^n, V) - k\{\|V'\|^2 - \nu\|V''\|^2\},$$

and via (1.6),

$$(g(V), V) \ge 2\|V\|\left\{\left(1 - \frac{k}{8\nu}\right)\|V\| - \|U^n\|\right\} \quad \forall V \in S_h^r.$$

Therefore, assuming $k < 8\nu$, for $\|V\| = \frac{8\nu}{8\nu - k}\|U^n\| + 1$ obviously $(g(V), V) > 0$ and the existence of a $V^* \in S_h^r$ such that $g(V^*) = 0$ follows from Lemma 3.1. Then $U^{n+1} := 2V^* - U^n$ satisfies (1.16).

Convergence. The main result in this paper is given in the following theorem.

THEOREM 3.1. *Let the solution* u *of* (1.1)–(1.2) *be sufficiently smooth*, $U^0, \ldots$ $\ldots, U^N$ *satisfy* (1.16), *and* (1.12) *hold. Then*, *for* k *sufficiently small*,

$$\max_{0\le n\le N} \|u^n - U^n\| \le c(u)(k^2 + h^r)\,. \tag{3.3}$$

Proof. Let $W^n := W(\cdot, t^n)$, $\varrho^n := u^n - W^n$, and $\zeta^n := W^n - U^n$. Then $u^n - U^n = \varrho^n + \zeta^n$ and by (2.6),

$$\max_{0\le n\le N} \|\varrho^n\| \le ch^r\,. \tag{3.4}$$

Thus it remains to estimate $\|\zeta^n\|$. Using (1.16), (2.1) and (1.3) we have, for $\chi \in S_h^r$,

$$\begin{aligned}(\partial\zeta^n, \chi) + a(\zeta^{n+1/2}, \chi) &= (\partial W^n, \chi) + a(W^{n+1/2}, \chi) - (\partial U^n, \chi) - a(U^{n+1/2}, \chi)\\ &= (\partial W^n, \chi) + a(u^{n+1/2}, \chi) + (U^{n+1/2}U_x^{n+1/2}, \chi) - \lambda(U^{n+1/2}, \chi)\\ &= (\partial W^n - u_t^{n+1/2} - \tfrac{1}{2}(u^n u_x^n + u^{n+1}u_x^{n+1})\\ &\quad + \lambda\varrho^{n+1/2} + \lambda\zeta^{n+1/2} + U^{n+1/2}U_x^{n+1/2}, \chi)\,,\end{aligned}$$

i.e.,

$$\begin{aligned}(\partial\zeta^n, \chi) + \nu(\zeta_{xx}^{n+1/2}, \chi'') - (\zeta_x^{n+1/2}, \chi')\\ = (\omega^n + \varrho^{n+1/2}\varrho_x^{n+1/2} + \zeta^{n+1/2}\zeta_x^{n+1/2}, \chi)\\ + (u^{n+1/2}\varrho^{n+1/2} + W^{n+1/2}\zeta^{n+1/2}, \chi')\,,\end{aligned} \tag{3.5}$$

where $\omega^n = \omega_1^n + \omega_2^n + \omega_3^n + \lambda\varrho^{n+1/2}$, and

$$\begin{aligned}\omega_1^n &:= \partial W^n - \partial u^n,\\ \omega_2^n &:= \partial u^n - u_t^{n+1/2},\\ \omega_3^n &:= u^{n+1/2}u_x^{n+1/2} - \tfrac{1}{2}(u^n u_x^n + u^{n+1}u_x^{n+1})\,.\end{aligned}$$

It is easily seen that

$$\max_{0\le n\le N} \|\omega^n\| \le c(k^2 + h^r)\,. \tag{3.6}$$

Taking $\chi := \zeta^{n+1/2}$ in (3.5) and using (3.4), (3.6) and (2.9) we obtain by periodicity

$$\begin{aligned}\frac{1}{2k}(\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2) + \nu\|\zeta_{xx}^{n+1/2}\|^2 - \|\zeta_x^{n+1/2}\|^2\\ \le c(k^2 + h^r)^2 + c\|\zeta^{n+1/2}\|^2 + \|\zeta_x^{n+1/2}\|^2\,.\end{aligned}$$

Therefore by (1.5) we see that

$$\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2 \le ck\{(k^2 + h^r)^2 + \|\zeta^{n+1}\|^2 + \|\zeta^n\|^2\}$$

and the discrete Gronwall lemma yields in view of (1.12) for k sufficiently small

$$\max_{0\le n\le N} \|\zeta^n\| \le c(k^2 + h^r)\,, \tag{3.7}$$

which concludes the proof. ■

Uniqueness. In addition to our assumptions on S_h^r we suppose here for the corresponding partition that for a positive constant μ,

$$\underline{h} \geq ch^{2\mu}\,. \tag{3.8}$$

It is well known that this inequality implies

$$\|\chi\|_{L^\infty} \leq ch^{-\mu}\|\chi\| \quad \forall \chi \in S_h^r\,, \tag{3.9}$$

(cf. Nitsche [10]). Let now $V^0 = U^0$ and $V^0, \dots, V^N \in S_h^r$ satisfy

$$(\partial V^n, \chi) + (V^{n+1/2}V_x^{n+1/2}, \chi) - (V_x^{n+1/2}, \chi') + \nu(V_{xx}^{n+1/2}, \chi'') = 0 \quad \forall \chi \in S_h^r\,, \tag{3.10}$$

for $n = 0, \dots, N-1$. Letting $E^n := U^n - V^n$, $n = 0, \dots, N$, from (1.16), (3.10) we obtain

$$\begin{aligned}(\partial E^n, \chi) &+ \nu(E_{xx}^{n+1/2}, \chi'') - (E_x^{n+1/2}, \chi')\\ &= (E^{n+1/2}E_x^{n+1/2}, \chi) + (U^{n+1/2}E^{n+1/2}, \chi') \quad \forall \chi \in S_h^r\,.\end{aligned}$$

Taking $\chi := E^{n+1/2}$ we obtain by periodicity

$$\begin{aligned}\frac{1}{2k}(\|E^{n+1}\|^2 - \|E^n\|^2) &+ \nu\|E_{xx}^{n+1/2}\|^2 - \|E_x^{n+1/2}\|^2\\ &= (U^{n+1/2}E^{n+1/2}, E_x^{n+1/2})\\ &\leq \tfrac{1}{2}(\|W^{n+1/2}\|_{L^\infty}^2 + \|\zeta^{n+1/2}\|_{L^\infty}^2)\|E^{n+1/2}\|^2 + \|E_x^{n+1/2}\|^2\\ &\leq (c + ch^{-2\mu}(k^4 + h^{2r}))\|E^{n+1/2}\|^2 + \|E_x^{n+1/2}\|^2\end{aligned}$$

where (2.9), (3.9) and (3.7) have been used. Then (1.5) yields

$$\|E^{n+1}\|^2 - \|E^n\|^2 \leq Ck(1 + k^4h^{-2\mu} + h^{2(r-\mu)})(\|E^{n+1}\|^2 + \|E^n\|^2)\,. \tag{3.11}$$

For $k^5h^{-2\mu}$ and $kh^{2(r-\mu)}$ sufficiently small, assuming $E^n = 0$, (3.11) implies $E^{n+1} = 0$. Summarizing, for sufficiently smooth u and $k^5h^{-2\mu}$, $kh^{2(r-\mu)}$ sufficiently small, assuming (3.9) we deduce uniqueness of the Crank–Nicolson approximations.

Odd initial value. We assume here that the initial value u^0 is an odd function. Then $v(x,t) := -u(-x,t)$ is a solution of (1.1)–(1.2). Thus $v = u$, i.e., $u(\cdot,t)$ is odd for $0 \leq t \leq T$.

Assume now that if x_i is a knot of our spline space then $-x_i$ is a knot as well, and moreover that the same differentiability conditions are posed at x_i and $-x_i$, $i \in \mathbb{Z}$. As a consequence, $\chi \in S_h^r$ implies $\chi(-\cdot) \in S_h^r$. Let u_h^0 be an odd function as is natural for odd u^0. Then the semidiscrete approximation $u_h(\cdot,t)$ is odd for $0 \leq t \leq T$, and moreover under our assumptions implying uniqueness of the Crank–Nicolson approximations U^n, they are odd, since $V^n := -U^n(-\cdot)$ are also Crank–Nicolson approximations. This fact is of significant practical importance, since in (1.16) we only have to take the odd χ's thus reducing the number of equations to about 50%.

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