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NATURAL DUALITIES FOR VARIETIES OF DISTRIBUTIVE LATTICES WITH A QUANTIFIER

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1. Introduction. This paper exploits and illustrates techniques in duality theory developed in the past ten years. It concerns certain varieties whose study is motivated by predicate logic. A duality based on hom functors into a schizophrenic object is presented for each of these varieties. As a byproduct, we are able to describe free algebras.

The varieties in question appear in a recent paper of R. Cignoli [3] concerning Q-distributive lattices. An algebra $(A; \vee, \wedge, 0, 1, \nabla)$ of type $(2,2,0,0,1)$ is a *Q-distributive lattice* if $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operator ∇ is assumed to satisfy, for any $a, b \in A$,

(Q_0) $\quad \nabla(0) = 0,$

(Q_1) $\quad a \wedge \nabla a = a,$

(Q_2) $\quad \nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b,$

(Q_3) $\quad \nabla(a \vee b) = \nabla a \vee \nabla b.$

The laws (Q_0)–(Q_3) were introduced by P. R. Halmos (see [19]) in a quest for an algebraic counterpart of the logical notion of an existential quantifier. Halmos' investigations concerned quantifiers on Boolean algebras; Cignoli's paper reveals the richer structure that emerges if the Boolean negation is dropped.

It is established in [3] that the lattice $\Lambda(\mathbf{Q})$ of subvarieties of the variety $\mathbf{Q}$ of Q-distributive lattices forms an $(\omega+1)$-chain

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$$\begin{aligned}&\mathbf{D}_{00} \subset\\&\mathbf{D}_{10} \subset \mathbf{D}_{01} \subset\\&\mathbf{D}_{20} \subset \mathbf{D}_{02} \subset \mathbf{D}_{11} \subset\\&\mathbf{D}_{30} \subset \mathbf{D}_{03} \subset \mathbf{D}_{12} \subset \mathbf{D}_{21} \subset\\&\mathbf{D}_{40} \subset \mathbf{D}_{04} \subset \mathbf{D}_{13} \subset \mathbf{D}_{22} \subset \mathbf{D}_{31} \subset \ldots\end{aligned}$$

The subvariety $\mathbf{D}_{pq}$ is $\mathbb{HSP}(\underline{D}_{pq})$, where, up to isomorphism, the finite subdirectly irreducible algebras in $\mathbf{Q}$ are $\underline{D}_{pq}$, $(p,q) \in \omega \times \omega$. These are given by

$$\underline{D}_{pq} = (B_p \times C_q; \vee, \wedge, 0, 1, \nabla)\,,$$

defined as follows. The algebra B_p is the p-atom Boolean lattice ($p \geqslant 0$), $C_p := B_p \oplus \mathbf{1}$ ($p \geqslant 1$), and ∇ is the *simple quantifier* given by $\nabla(a) = 1$ if $a \neq 0$, $\nabla(0) = 0$.

To identify $\Lambda(\mathbf{Q})$ Cignoli derived a dual category equivalence between $\mathbf{Q}$ and a category $\mathcal{Y}^{\mathbf{Q}}$ of structures (X, E), where X is a compact totally order-disconnected space and E is an equivalence relation satisfying suitable conditions. This is an extension of the duality due to Halmos in the Boolean case; we elaborate below. For related work see [5], [17], [20]. Cignoli's duality is obtained by restricting Priestley duality for distributive lattices, and is similar in spirit to dualities for many well-known varieties of distributive-lattice-ordered algebras. While such hand-me-down dualities have the benefit of representing algebras concretely by lattices of sets, they frequently have the major drawback that products in the dual category fail to be cartesian. This happens for $\mathcal{Y}^{\mathbf{Q}}$, as is pointed out by Cignoli in a note [4] which extends the arguments employed in [19] to give a description of free algebras in $\mathbf{Q}$.

We present a completely different approach to free algebras. This involves setting up a natural duality, in the sense of Davey and Werner [14] (see also [6]) for each subvariety $\mathbf{D}_{pq}$. Our procedures, and to some extent our results, parallel those in [11], [12] for the ω-chain of proper subvarieties of distributive p-algebras. The major difference is that here we require, except when $q \leqslant 1$, the generalised duality theory introduced in [9], employing a multi-faceted schizophrenic object. As in [11] we make crucial use of Davey and Werner's piggyback method ([15], [16]) in its generalised form from [9] to identify a suitable dual category. We use Cignoli's duality to find the relations required for piggybacking, thereby, as in [11], exploiting the "logarithmic" character of distributive lattice duality. This strategy enables us to perform, with computer assistance, calculations which would be quite prohibitive otherwise. The duality machinery on which we rely is considerable, but, as Section 3 shows, once this machinery is in place we can manufacture dualities with extreme ease—in stark contrast to the treatment of the first natural duality examples presented in [14], by algebraic methods adequate only to handle, laboriously, varieties with very small generating algebras.

In the next section we summarise, very briefly indeed, the theory we need. Our recent survey article [23] (concentrating on distributive-lattice-ordered algebras) and B. A. Davey's introductory survey [7] provide fuller accounts. We should stress that a complete understanding of this background material is not essential for an appreciation of the applications we give. We shall, however, assume familiarity with Priestley duality for distributive lattices, for which reference may be made to [10] or [21].

2. Natural dualities. Let $\underline{P}$ be a finite algebra, and consider the variety

$$\mathcal{A} := \mathbb{HSP}(\underline{P})\,.$$

We seek to define a category $\mathcal{X}$ of topological relational structures in such a way that each algebra in $\mathcal{A}$ is concretely represented as an algebra of continuous relation-preserving maps. Following the historical development of the theory we consider first the special case in which the quasi-variety generated by $\underline{P}$ is in fact a variety, so that

$$\mathcal{A} = \mathbb{ISP}(\underline{P})\,.$$

This happens whenever $\mathcal{A}$ is an equational class in which every subdirectly irreducible algebra is isomorphic to a subalgebra of some subdirectly irreducible algebra $\underline{P}$. (When $\mathcal{A} = \mathbf{D}_{pq}$ we shall see later that this is so precisely when $q \leqslant 1$.)

Let τ be the discrete topology on the underlying set P of $\underline{P}$. Let R be a family of relations on P such that each $r \in R$ is *algebraic* in the sense that r is a subalgebra of some finite power of $\underline{P}$. We wish to form a structure $\underset{\sim}{P} = (P; \tau, R)$ so that

$$(\forall A \in \mathcal{A})\ A \cong ED(A)\,,$$

where D and E are well-defined hom functors given on objects by

$$D : A \mapsto \mathcal{A}(A, \underline{P}) \leq \underset{\sim}{P}^{A} \in \mathcal{X} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\underset{\sim}{P})\,,$$

$$E : X \mapsto \mathcal{X}(X, \underset{\sim}{P}) \leq \underline{P}^{X} \in \mathcal{A} = \mathbb{ISP}(\underline{P})\,,$$

and on morphisms by composition. Here $\leq$ means "is a substructure of", and the subscript c serves as a reminder that substructures in $\mathcal{X}$ are required to be topologically closed. Operations (in $\underline{P}$) and relations (in $\underset{\sim}{P}$) are extended pointwise to subsets of powers, and the topology on subsets of powers of $\underset{\sim}{P}$ is that induced by the product topology derived from τ. We say that R *yields a duality on* $A \in \mathcal{A}$ if $A \cong ED(A)$ and R *yields a duality on* $\mathcal{A}$ if R yields a duality on each $A \in \mathcal{A}$. We call $D(A)$ the *dual* of $A \in \mathcal{A}$.

If we can choose R to yield a duality on $\mathcal{A}$ then we have immediate access to free algebras: for any κ,

$$F\mathcal{A}(\kappa) \cong \mathcal{X}(\underset{\sim}{P}^{\kappa}, \underset{\sim}{P})\,.$$

In particular, the *alter ego* $\underset{\sim}{P}$ of $\underline{P}$ serves as the dual of $F\mathcal{A}(1)$; see [14] (or [23], Lemma 2.3).

Priestley duality fits into this framework as follows. We have $\mathcal{A} = \mathbf{D}$ (bounded distributive lattices) and $\mathcal{X} = \mathbf{P}$ (compact totally order-disconnected spaces (= Priestley spaces), with continuous order-preserving maps). Then

$$\begin{aligned} \underline{P} = \underline{\mathbf{2}} &:= (\mathbf{2}; 0, 1, \vee, \wedge), && \text{the 2-element lattice in } \mathbf{D}\,, \\ \underset{\sim}{P} = \underset{\sim}{\mathbf{2}} &:= (\mathbf{2}; \tau, \leqslant), && \text{the 2-element chain in } \mathbf{P}\,. \end{aligned}$$

We may identify each $A \in \mathbf{D}$ with the lattice of continuous order-preserving maps from its dual $D(A) := \mathbf{P}(A, \mathbf{2})$ into $\underset{\sim}{\mathbf{2}}$, or, equivalently, with the lattice of clopen up-sets of $D(A)$. In particular, $F\mathbf{D}(\kappa)$ is the lattice of clopen up-sets of $\mathbf{2}^\kappa$.

In general, it can happen that no choice of R yields a duality on $\mathcal{A}$, although it is the case that the assumption that the relations in R are algebraic at least ensures that

(i) the functors D and E are well-defined and yield an adjunction, and
(ii) for each $A \in \mathcal{A}$, the evaluation map $e_A : A \to ED(A)$ is an embedding

(see [14], Section 1). Thus the point at issue is whether each e_A is surjective. It turns out that if $\underline{P}$ has a lattice reduct then R can be chosen to achieve this: the NU Duality Theorem of Davey and Werner ([14], Theorem 1.18) implies that if we choose R to consist of all subalgebras of $\underline{P}^2$ then R yields a duality on $\mathcal{A}$. Even in the simplest cases this choice will be very uneconomical, in the sense that a smaller set of relations will suffice (for example, taking $R = \mathbb{S}(\mathbf{2}^2)$ for $\mathbf{D}$ would give us two trivial relations (serving no purpose) and both $\leqslant$ and $\geqslant$, either of which alone suffices). We say that R *generates* a relation $r' \notin R$ if for each $A \in \mathcal{A}$ any continuous map $\varphi : D(A) \to \underset{\sim}{P}$ preserving each $r \in R$ also preserves r'. Also, for $R_1 \subset R$, we say that R_1 *generates* R if R_1 generates each $r \in R \setminus R_1$. It is obvious that if R yields a duality on $\mathcal{A}$ then so does any generating subset of R.

Before explaining how a good choice of R can be made we remove the restriction $\mathbb{HSP}(\underline{P}) = \mathbb{ISP}(\underline{P})$. We shall then consider redundancy of relations in the more general setting. We shall continue to assume that $\underline{P}$ is finite, and shall also assume that $\underline{P}$ has a lattice reduct. Birkhoff's Subdirect Product Theorem implies that

$$\mathcal{A} = \mathbb{ISP}(\underline{\mathit{\Pi}})\,,$$

where $\underline{\mathit{\Pi}}$ is a finite set of subdirectly irreducible algebras. The single object $\underset{\sim}{P}$ of the earlier theory is replaced by

$$\underset{\sim}{\mathit{\Pi}} = (\mathit{\Pi}; \tau, R)\,,$$

where $\mathit{\Pi}$ means $\dot{\bigcup}\{P \mid \underline{P} \in \underline{\mathit{\Pi}}\}$, τ is the discrete topology and R is a set of relations each of which is a subalgebra of $\underline{Q}_1 \times \ldots \times \underline{Q}_n$ for some $\underline{Q}_1, \ldots, \underline{Q}_n \in \underline{\mathit{\Pi}}$.

For any set S,

$$\mathit{\Pi}^S := \dot{\bigcup}\{P^S \mid \underline{P} \in \underline{\mathit{\Pi}}\}$$

is given the obvious topology, and relations obtained by pointwise extension of those in R; the resulting structure is referred to as $\underset{\sim}{\Pi}^S$. More generally, we may consider structures $\mathbf{X} = (X; \tau, R)$ such that

$$X := \dot{\bigcup}\{X_{\underline{P}} \mid \underline{P} \in \underline{\Pi}\},$$

where τ is the union topology on X and for each relation

$$r \subseteq \underline{Q}_1 \times \ldots \times \underline{Q}_n$$

in R there is an associated relation

$$r \subseteq X_{\underline{Q}_1} \times \ldots \times X_{\underline{Q}_n}.$$

Given two such $\underline{\Pi}$-*indexed structures*, $\mathbf{X}$ and $\mathbf{Y}$, a *morphism from* $\mathbf{X}$ *to* $\mathbf{Y}$ is a map which takes $X_{\underline{P}}$ into $Y_{\underline{P}}$ for each $\underline{P} \in \underline{\Pi}$ and which is structure-preserving in the obvious sense. We may now define $\mathcal{X}$ to be the category of all $\underline{\Pi}$-indexed structures which take the form of an isomorphic copy of a substructure of some power $\underset{\sim}{\Pi}^S$ of $\underset{\sim}{\Pi}$ (in symbols, $\mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\Pi})$).

To attempt to set up a duality between $\mathcal{A}$ and $\mathcal{X}$ we proceed as follows. For each $A \in \mathcal{A}$, let $X_{\underline{P}} := \mathcal{A}(A, \underline{P})$. Then

$$D(A) := \dot{\bigcup}\{\mathcal{A}(A, \underline{P}) \mid \underline{P} \in \underline{\Pi}\}$$

is an $\mathcal{X}$-substructure of $\underset{\sim}{\Pi}^A$. For each $\mathbf{X} \in \mathcal{X}$,

$$E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\Pi})$$

is an $\mathcal{A}$-subalgebra of $\prod\{\underline{Q_P} \mid \underline{P} \in \underline{\Pi}\}$ where $\underline{Q_P}$ is $\underline{P}$ raised to the power $X_{\underline{P}}$. Just as in the case $|\underline{\Pi}| = 1$ we then have contravariant functors

$$D : \mathcal{A} \to \mathcal{X} \quad \text{and} \quad E : \mathcal{X} \to \mathcal{A}$$

for which all the maps

$$e_A : A \to ED(A), \qquad e_A(a) : x \mapsto x(a)$$

are embeddings.

The definitions given earlier concerning the set R (yielding a duality on an individual algebra or on the whole variety, and being generated by a subset) are extended in the obvious manner to the present setting. The NU Duality Theorem extends in the anticipated way and we deduce, given $\underline{P}$ has a lattice reduct, that the set R yields a duality on $\mathcal{A}$ so long as R generates every subalgebra of $\underline{P} \times \underline{Q}$ for any choice of $\underline{P}, \underline{Q} \in \underline{\Pi}$ (see [9], Theorem 1.9). Further, one can establish that $\underset{\sim}{\Pi}^S$ is the S-fold power of $\underset{\sim}{\Pi}^S$ in $\mathcal{X}$ and is isomorphic to $D(F_S)$ ([9], Lemma 1.5).

We now turn to the problem of finding a good generating set for

$$\mathcal{S} := \bigcup\{\mathbb{S}(\underline{P} \times \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi}\}.$$

We assume that each algebra in $\mathcal{A}$ has a $\mathbf{D}$-reduct, to which the duality for $\mathbf{D}$ applies. This allows us to invoke Davey and Werner's piggyback technique. We shall not attempt to explain the method here. Accounts can be found in [9], [15], [16] and in [23], where a self-contained proof is given of the following theorem, derived from Theorem 2.5 of [9] and tailormade for our needs. We use the following notation: given $\mathcal{E} \subseteq \bigcup\{\mathcal{A}(\underline{P}, \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi}\}$ we let $\langle\mathcal{E}\rangle$ be the subset of $\bigcup\{\mathcal{A}(\underline{P}, \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi}\}$ generated from $\mathcal{E}$ by forming all possible compositions which are defined.

Theorem 2.1 (The Generalised Piggyback Duality Theorem, for distributive-lattice-ordered algebras). *Suppose that $\mathcal{A} = \mathbb{ISP}(\underline{\Pi})$, where $\underline{\Pi}$ is a finite set of finite algebras each having a $\mathbf{D}$-reduct. For each $\underline{P}$ in $\underline{\Pi}$ let $\Omega_{\underline{P}}$ be a* (*possibly empty*) *subset of $\mathbf{D}(\underline{P}, \underline{\mathbf{2}})$.*

Let $\underset{\sim}{\Pi} = (\Pi; \tau, R)$ be the topological relational structure on $\dot{\bigcup}\{P \mid \underline{P} \in \underline{\Pi}\}$ in which

(i) *τ is the discrete topology*,
(ii) *R is a generating set for $S \cup G$, where*
  (a) *S is the collection of maximal $\mathcal{A}$-subalgebras of sublattices of the form*

$$(\alpha, \beta)^{-1}(\leqslant) := \{(a, b) \in \underline{P} \times \underline{Q} \mid \alpha(a) \leqslant \beta(b)\},$$

  where $\alpha \in \Omega_{\underline{P}}$, $\beta \in \Omega_{\underline{Q}}$ ($\underline{P}, \underline{Q} \in \underline{\Pi}$), and
  (b) *G is the set of graphs of a set $\mathcal{E} \subseteq \bigcup\{\, \mathcal{A}(\underline{P}, \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi}\,\}$ satisfying the following separation condition*:
    (S) *for all $\underline{P} \in \underline{\Pi}$, given $a, b \in \underline{P}$ with $a \neq b$, there exists $\underline{Q} \in \underline{\Pi}$, $u \in \mathcal{A}(\underline{P}, \underline{Q}) \cap \langle\mathcal{E}\rangle$ and $\alpha \in \Omega_{\underline{Q}}$ such that $\alpha(u(a)) \neq \alpha(u(b))$.*

Then R yields a duality on $\mathcal{A}$.

As [11] and [12] strikingly show, even choosing R as prescribed in Theorem 2.1 may lead to a duality which is far from optimal. The test algebra technique introduced in [12] allows us unequivocally to decide whether a given member of R is redundant. Suppose we know that R yields a duality on $\mathcal{A}$ and wish to find out whether $R \setminus \{r\}$ generates R. This is certainly so if $A \cong ED(A)$ for all $A \in \mathcal{A}$ (with the functors D and E taken relative to $R \setminus \{r\}$). It turns out that, instead of having to try out $R \setminus \{r\}$ on the *whole* class $\mathcal{A}$, we only need to test it on a *single* "test algebra", namely $\underline{r}$, by which we mean r *qua* member of $\mathcal{A}$ (remember our relations are algebraic!). This observation is made precise in the following easy theorem (proved, as stated here, in [23] (Theorem 2.7)).

Theorem 2.2. *Let $\underline{\Pi}$ be a finite set of finite algebras, let $\mathcal{A} := \mathbb{ISP}(\underline{\Pi})$ and let R be a finite set of algebraic relations on $\underline{\Pi}$. If R yields a duality on a subalgebra $\underline{r}$ of $\underline{Q}_1 \times \ldots \times \underline{Q}_n$ (where each $\underline{Q}_i \in \underline{\Pi}$), then R generates r.*

3. Natural dualities for the proper subvarieties of Q. For consistency with our other papers using the piggyback method we shall use H and K to denote the functors setting up Priestley duality, and for their restrictions to subcategories. Thus the dual of $A \in \mathbf{D}$ is $H(A) := \mathbf{D}(A, \mathbf{2})$, with relative product topology from $\mathbf{2}^A$, and pointwise order. When A is finite, the topology is discrete (and so ignorable) and $H(A)$ is just the set of join-irreducible elements of A, with the reverse of the order induced from A (see Chapter 8 of [10]). The algebra A will be identified with $KH(A)$, the continuous order-preserving maps from $H(A)$ into $\underset{\sim}{\mathbf{2}}$ or, where more convenient, with the clopen up-sets of $H(A)$. When A is finite it is identified with the order-preserving maps from $H(A)$ into a 2-element chain, or, equivalently, the up-sets of $H(A)$.

We now recall Cignoli's duality for $\mathbf{Q}$ [3]. As Cignoli shows, quantifiers on an algebra $A \in \mathbf{D}$ correspond bijectively with certain equivalence relations on $H(A)$. To explain how this works we need the following notation and definitions. Given a set X and an equivalence relation E on X we denote, for each $x \in X$, the equivalence class of x by $[x]$. We then let, for each $U \subseteq X$,

$$\mathbf{E}U := \bigcup\{[x] \mid U \cap [x] \neq \emptyset\}.$$

Definitions 3.1 ([3], 2.4). A $\mathbf{Q}$-*space* is a structure (X, E) (or, when we need to emphasise the order, $(X, E, \leqslant)$) such that $X \in \mathbf{P}$ and E is an equivalence relation on X satisfying the following conditions:

(E_1) $\mathbf{E}U$ is a clopen up-set in X whenever U is a clopen up-set in X;

(E_2) $[x]$ is closed in X for each $x \in X$.

Given $\mathbf{Q}$-spaces (X, E) and (Y, F), a $\mathbf{Q}$-*map from* (X, E) *to* (Y, F) is a $\mathbf{P}$-morphism $\varphi : X \to Y$ such that, for every clopen up-set $V \subseteq Y$,

$$\mathbf{E}(\varphi^{-1}(V)) = \varphi^{-1}(\mathbf{F}V).$$

A $\mathbf{Q}$-map $\varphi : X \to Y$ is an *isomorphism* if it is a homeomorphism and order-isomorphism for which $(x, y) \in E$ if and only if $(\varphi(x), \varphi(y)) \in F$.

The category of $\mathbf{Q}$-spaces and $\mathbf{Q}$-maps will be denoted by $\mathcal{Y}^{\mathbf{Q}}$. There is then a dual category equivalence between $\mathbf{Q}$ and $\mathcal{Y}^{\mathbf{Q}}$, as indicated by the following theorem. There $\nabla(A)$ denotes the range of the quantifier ∇, viz. $\{\nabla(a) \mid a \in A\}$.

Theorem 3.2 ([3], §2). (1) *Given* $(A; \vee, \wedge, 0, 1, \nabla) \in \mathbf{Q}$, *the structure* $(H(A), E(\nabla))$ *belongs to* $\mathcal{Y}^{\mathbf{Q}}$, *where*

$$E(\nabla) := \{(x, y) \in H(A) \times H(A) \mid x \restriction \nabla(A) = y \restriction \nabla(A)\}.$$

(2) *For each* $f \in \mathbf{Q}(A, B)$, *the* $\mathbf{P}$-*morphism* $H(f) := - \circ f : H(A) \to H(B)$ *is a* $\mathcal{Y}^{\mathbf{Q}}$-*morphism.*

(3) *Given* $(X, E) \in \mathcal{Y}^{\mathbf{Q}}$, *the algebra* $(K(X), \mathbf{E})$ *belongs to* $\mathbf{Q}$.

(4) *For each* $\varphi \in \mathcal{Y}^{\mathbf{Q}}(X, Y)$, *the* $\mathbf{D}$-*morphism* $K(\varphi) : - \circ \varphi : K(Y) \to K(X)$ *is a* $\mathbf{Q}$-*morphism.*

(5) *For each* $A \in \mathbf{Q}$ *and each* $X \in \mathcal{Y}^{\mathbf{Q}}$,

$$A \cong KH(A) \in \mathbf{Q} \quad \textit{and} \quad X \cong HK(X) \in \mathcal{Y}^{\mathbf{Q}},$$

where $\cong$ *in* $\mathcal{Y}^{\mathbf{Q}}$ *means isomorphism in the sense defined in* 3.1.

Note that in order to set up natural dualities for the varieties $\mathbf{D}_{pq}$, and to determine finitely generated free algebras, we only work with finite structures, and the topology disappears. From here on we shall assume that all algebras and spaces are finite.

To identify the subalgebras and homomorphisms to which the Piggyback Duality Theorem refers we adopt the duality method which was first employed in [11] and which is explained and illustrated in Section 4 of [23]. We need the following facts from distributive lattice duality.

PROPOSITION 3.3. *Let* $A, B \in \mathbf{D}$ *and* $f \in \mathbf{D}(A, B)$. *Then*

(i) $H(A \times B) = H(A) \dot{\cup} H(B)$ *and* $H(A \oplus \mathbf{1}) = \mathbf{1} \oplus H(A)$;
(ii) f *is injective if and only if* $H(f) := - \circ f$ *is surjective.*

Let $(X, E), (Y, F) \in \mathcal{Y}^{\mathbf{Q}}$. The following facts are too elementary to need proof: the quantifier $\mathbf{E}$ on $K(X)$ is the simple quantifier if and only if $E = X \times X$, and, in $\mathbf{Q}$, $K(X) \times K(Y)$ is isomorphic to $(K(X \dot{\cup} Y), \mathbf{G})$, where $G = E \cup F$.

We next examine how these observations apply to the subvarieties $\mathbf{D}_{pq} = \mathbb{HSP}(\underline{D}_{pq})$ whose definition we recalled in Section 1. Henceforth we assume $p+q > 0$ (thereby excluding the trivial variety $\mathbf{D}_{00}$). We denote

(i) a p-element antichain with elements $1, \dots, p$ by A_p, and
(ii) the ordered set $\{0, 1, \dots, p, p+1, \dots, p+q\}$ ($p \geqslant 0,\ q \geqslant 1$) in which $\{1, \dots, p+q\}$ is an antichain and $0 < i$ ($i = p+1, \dots, p+q$) by T_{pq} (Figure 1).

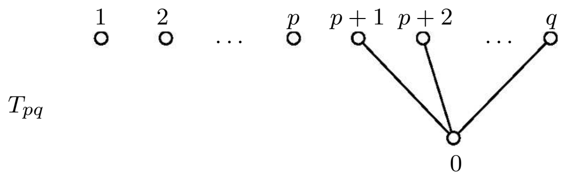


Fig. 1

We have

$$\underline{D}_{p0} = (K(A_p), A_p \times A_p) \qquad (p \geqslant 0),$$
$$\underline{D}_{pq} = (K(T_{pq}), T_{pq} \times T_{pq}) \qquad (p \geqslant 0,\ q \geqslant 1),$$

Lemma 4.1 of [3] gives the following lemma.

LEMMA 3.4. *A surjective order-preserving map* $\varphi : (X, X \times X) \to (Y, Y \times Y)$ *is a* $\mathbf{Q}$*-map if and only if* $\operatorname{Max} Y \subseteq \operatorname{Im} \varphi$, *where* $\operatorname{Max} Y$ *denotes the maximal points of* Y.

It follows from this, and is implicit in Section 4 of [3], that $\underline{D}_{rs}$ is a subalgebra of $\underline{D}_{pq}$ if and only if $r+s \leqslant p+q$ and at least one of the following holds:

(a) $q \leqslant s$,
(b) $r+s \leqslant p+1$ and $q \geqslant 2$,
(c) $p = 0$.

Up to isomorphism, the subdirectly irreducible algebras in $\mathbf{D}_{pq}$ are those $\underline{D}_{rs}$ such that $\mathbf{D}_{rs} \subseteq \mathbf{D}_{pq}$ according to the chain ordering indicated in Section 1 and specified precisely in [3], Remark 4.6. We deduce the proposition below. It implies in particular that every subvariety $\mathbf{D}_{pq}$ of $\mathbf{D}_{30}$ is such that $\mathbf{D}_{pq} = \mathbb{ISP}(\underline{D}_{pq})$ except for $\mathbf{D}_{02}$.

PROPOSITION 3.5. (1) $\mathbf{D}_{pq} = \mathbb{ISP}(\underline{D}_{pq})$ *if and only if* $q \leqslant 1$.
(2) $\mathbf{D}_{pq} = \mathbb{ISP}(\underline{D}_{pq}, \underline{D}_{(p+q)0})$ *if* $q > 1$.

Define

$$\underline{\Pi} = \begin{cases} \{\underline{D}_{pq}\} & \text{if } q \leqslant 1, \\ \{\underline{D}_{pq}, \underline{D}_{(p+q)0}\} & \text{if } q > 1. \end{cases}$$

Our next task is to consider how to choose the sets $\Omega_{\underline{Q}}$ ($\underline{Q} \in \underline{\Pi}$) and the subset $\mathcal{E}$ of $\bigcup\{\mathcal{A}(\underline{P}, \underline{Q}) \mid \underline{P}, \underline{Q} \in \underline{\Pi}\}$ so as to satisfy the separation condition (S) in Theorem 2.1. Note that each $\Omega_{\underline{Q}}$ is a subset of some A_i or T_{kl} and so, by our choice of labelling, may be designated by a subset of $\mathbb{N} \cup \{0\}$. By Theorem 3.2 we may identify $\mathbf{Q}(\underline{P}, \underline{Q})$ with $\mathcal{Y}^{\mathbf{Q}}(H(\underline{Q}), H(\underline{P}))$. According to Lemma 3.4 this set certainly contains all order-preserving maps φ from $H(\underline{Q})$ to $H(\underline{P})$ such that $\operatorname{Im}\varphi$ contains $\operatorname{Max} H(\underline{P})$.

What follows is complicated only because we have to distinguish many different cases. We let α_1 be the point 0 in T_{pq} and let α_2 be the point 1 in A_r for $r > 0$. We denote the identity map on a set S by id_S. We define homomorphisms as follows.

(i) When $r \geqslant 2$ let f_σ and f_τ be the automorphisms of B_r for which $H(f_\sigma)$ is the permutation $\sigma := (1\ 2 \ldots r)$ and $H(f_\tau)$ is the permutation $\tau := (1\ 2)$, and, similarly, let f'_σ and f'_τ be generators of the automorphism group of C_r.

(ii) Let g be the endomorphism of $B_p \times C_1$ for which $H(g)$ is given by $H(g)(0) = H(g)(p) = 1$ and $H(g)(i) = i$ $(1 < i < p)$. When $p \geqslant 2$ let g_σ and g_τ be the automorphisms of $B_p \times C_1$ for which

$$H(g_\sigma){\restriction}\{1,\ldots,p\} = (1\ 2 \ldots p), \quad H(g_\sigma)(0) = 0, \quad H(g_\sigma)(p+1) = p+1, \quad \text{and}$$
$$H(g_\tau){\restriction}\{1,\ldots,p\} = (1\ 2), \quad H(g_\tau)(0) = 0, \quad H(g_\tau)(p+1) = p+1\,.$$

(iii) Let h be the homomorphism from $B_p \times C_q$ onto B_{p+q} for which $H(h)(i) = i$ for $i = 1, \ldots, p+q$.

PROPOSITION 3.6. *Condition* (S) *in Theorem* 2.1 *is satisfied in each of the cases below.*

(1) *Let* $\mathcal{A} := \mathbf{D}_{p0} = \mathbb{ISP}(\underline{\Pi})$ *where* $\underline{\Pi} = \{D_{p0}\}$ $(p > 0)$, *let* $\Omega_{\underline{D}_{p0}} := \{\alpha_2\}$ *and let*

$$\mathcal{E} := \begin{cases} \emptyset & \text{if } p = 1,\\ \{f_\sigma\} & \text{if } p = 2,\\ \{f_\sigma, f_\tau\} & \text{if } p \geqslant 3.\end{cases}$$

(2) *Let* $\mathcal{A} := \mathbf{D}_{p1} = \mathbb{ISP}(\underline{\Pi})$ *where* $\underline{\Pi} = \{\underline{D}_{p1}\}$ $(p \geqslant 0)$, *let* $\Omega_{\underline{D}_{p1}} := \{\alpha_1\}$ *and let*

$$\mathcal{E} := \begin{cases} \{\mathrm{id}_{T_{p1}}, g\} & \text{if } p = 0,\\ \{\mathrm{id}_{T_{p1}}, g_\sigma\} & \text{if } p = 1,\\ \{g, g_\sigma, g_\tau\} & \text{if } p \geqslant 2.\end{cases}$$

(3) *Let* $\mathcal{A} := \mathbf{D}_{pq} = \mathbb{ISP}(\underline{\Pi})$ *where* $\underline{\Pi} = \{\underline{D}_{pq}, \underline{D}_{(p+q)0}\}$ $(p \geqslant 0,\ q \geqslant 2)$, *let* $\Omega_{\underline{D}_{pq}} := \{\alpha_1\}$, $\Omega_{\underline{D}_{(p+q)0}} := \{\alpha_2\}$, *and let*

$$\mathcal{E} \subseteq \mathbf{Q}(\underline{D}_{(p+q)0}, \underline{D}_{(p+q)0}) \cup \mathbf{Q}(\underline{D}_{pq}, \underline{D}_{(p+q)0}) \cup \mathbf{Q}(\underline{D}_{pq}, \underline{D}_{pq})$$

be given by

$$\mathcal{E} := \begin{cases} \{\mathrm{id}_{T_{pq}}, h, f_\sigma\} & \text{if } p = 0, q = 2,\\ \{\mathrm{id}_{T_{pq}}, h, f_\sigma, f_\tau\} & \text{if } p + q > 2.\end{cases}$$

Proof. We prove (2), and indicate the modifications needed for (3); we omit the (simpler) proof of (1). Note that σ, τ acting on $\{1, \ldots, r\}$ generate the permutation group S_r if $r \geqslant 3$, and that σ $(= \tau)$ generates S_r if $r = 2$.

Consider (2), with $p \geqslant 2$ (the cases $p = 0, 1$ are similar, but simpler). Let $a \neq b$ in $\underline{D}_{p1}$ (which we elect to regard as the up-sets of its dual space T_{p1}). We must show that there exists $e \in \mathcal{E}$ such that $\alpha_1(e(a)) \neq \alpha_1(e(b))$, in other words such that 0 belongs to just one of $e(a)$, $e(b)$. Remember that $e(c) = (H(e))^{-1}(c)$ for any $c \in D_{p1}$. If 0 belongs to just one of a, b, we take $e = \mathrm{id}_{T_{p1}}$. Otherwise we may assume without loss of generality that there exists $i \geqslant 1$ such that $i \in a \setminus b$. We can find $e \in \langle\{g_\sigma, g_\tau\}\rangle$ such that $H(e)(i) = 1$. Then $0 \in (H(e))^{-1}(a) \setminus (H(e))^{-1}(b)$, so $0 \in e(a) \setminus e(b)$.

Now consider (3), with $p + q > 2$ (the case $p + q = 2$ is similar). We have two cases to treat.

(i) If $a \neq b$ in $\underline{D}_{(p+q)0}$ then there exists $e \in \langle f_\sigma, f_\tau\rangle$ such that $\alpha_2(e(a)) \neq \alpha_2(e(a))$.

(ii) Suppose $a \neq b$ in $\underline{D}_{pq}$. If 0 belongs to just one of a, b then $\alpha_1(\mathrm{id}_{T_{pq}}(a)) \neq \alpha_1(\mathrm{id}_{T_{pq}}(b))$. Now assume there exists $i \in \operatorname{Max} T_{pq}$ such that $i \in a \setminus b$. Then there exists $e \in \langle\{f_\sigma, f_\tau\}\rangle$ such that $\alpha_2(e(h(a))) \neq \alpha_2(e(h(b)))$. ∎

To set up a piggyback duality for $\mathbf{D}_{pq}$ we need to identify the maximal $\mathbf{Q}$-subalgebras of $(\alpha_i, \alpha_j)^{-1}(\leqslant)$ for $i, j \in \{1, 2\}$. We use duality, in the manner described in Section 4 of [23]. We can extract all the information we need by considering $\underline{\Pi} = \{\underline{D}_{pq}, \underline{D}_{(p+q)0}\}$, $\Omega_{\underline{D}_{pq}} := \{0\}$ and $\Omega_{\underline{D}_{(p+q)0}} := \{1\}$ $(p \geqslant 1$, $q \geqslant 0)$. For $\underline{P} \in \underline{\Pi}$ denote the single member of $\Omega_{\underline{P}}$ by β_P. Denote $H(\underline{P})$ by W_P, with order $\leqslant_P$.

Fix $\underline{P}$, $\underline{Q}$ (not necessarily distinct) in $\underline{\Pi}$. Label the elements of W_P by members of $\mathbb{N} \cup \{0\}$ as before, and the elements of W_Q by members of $\mathbb{N} \cup \{0\}$ with $\smile$ overset. As the dual of $H(\underline{P}) \times H(\underline{Q})$ *qua* $\mathbf{Q}$-algebra, $W_P \dot{\cup} W_Q$ carries the equivalence relation $E(P,Q) := (W_P \times W_P) \cup (W_Q \times W_Q)$. We denote the associated equivalence relation on subsets by $\mathbf{E}(P,Q)$; see 3.1.

From here on it will be convenient to identify any A in $\mathbf{D}$ or $\mathbf{Q}$ with $KH(A)$. Let L be the $\mathbf{D}$-subalgebra

$$(\beta_P, \beta_Q)^{-1}(\leqslant) := \{\, (a,b) \in \underline{P} \times \underline{Q} \mid a(\beta_P) \leqslant b(\beta_Q) \,\}$$

of $\underline{P} \times \underline{Q}$. Note that (a,b) in $\underline{P} \times \underline{Q}$ belongs to L if and only if $a(\beta_P) = 1$ implies $b(\beta_Q) = 1$.

Proposition 3.3 implies that $H(L) = W_P \dot{\cup} W_Q$ ordered by $\leqslant_{PQ}$, the transitive closure of the relation $\leqslant_P \cup \leqslant_Q \cup (\beta_P, \beta_Q)$. The $\mathbf{Q}$-subalgebras of L correspond bijectively to surjective $\mathcal{Y}^{\mathbf{Q}}$-morphisms $\varphi : W_P \dot{\cup} W_Q \to Z$ such that $\varphi = \iota \circ \eta$ for some $\eta \in \mathbf{P}(H(L), Z)$, where $\iota : H(\underline{P} \times \underline{Q}) \to H(L)$ is the dual of the natural embedding:

$$\begin{tikzcd} H(\underline{P} \times \underline{Q}) \arrow[r, "\varphi"] \arrow[dr, "\iota"'] & Z \\ & H(L) \arrow[u, "\eta"'] \end{tikzcd}$$

Since the subalgebra associated with φ is determined by the image Z of φ we denote it by $S(Z)$. We have

$$S(Z) = \{(a \circ \varphi{\restriction}W_P, a \circ \varphi{\restriction}W_Q) \mid a \in K(Z)\}\,.$$

The Q-algebra $S(Z)$ is maximal in L if the following holds: whenever $Z' \in \mathcal{Y}^{\mathbf{Q}}$, $\varphi' \in \mathcal{Y}^{\mathbf{Q}}(H(\underline{P} \times \underline{Q}), Z')$ is surjective, $\theta \in \mathcal{Y}^{\mathbf{Q}}(Z', Z)$ and $\varphi = \theta \circ \varphi'$, we have $Z' \cong Z$ in $\mathcal{Y}^{\mathbf{Q}}$.

Suppose we construct (Z, F) to satisfy the following:

(i) (Z,F) is a $\mathbf{Q}$-space whose underlying set is $W_P \dot{\cup} W_Q$;

(ii) $\eta : (H(L), E(P,Q)) \to (Z,F)$ is a $\mathbf{Q}$-map.

Then $S(Z)$ will be a $\mathbf{Q}$-subalgebra of L, and it will be a maximal subalgebra if and only if the ordering, $\leqslant^*$, on Z is minimal (with respect to set inclusion of relations) among partial orders on Z such that (i) and (ii) hold. The map η will be order-preserving precisely when $\leqslant_{PQ} \subseteq \leqslant^*$. Because η is a bijection, it will then be a $\mathbf{Q}$-map just when F is chosen to be $E(P,Q)$. We therefore wish to know which orders $\leqslant^*$ make $(Z, E(P,Q))$ into a $\mathbf{Q}$-space. We call such orders *admissible*.

Lemma 3.7. *Let $Z := W_P \dot{\cup} W_Q$ carry an order $\leqslant^*$ containing $\leqslant_{PQ}$. Then $(Z, E(P,Q))$ is a $\mathbf{Q}$-space if and only if*

(a) *W_Q is an up-set, and*

(b) *given any point $x \in W_P$, there exists a point $y \in W_Q$ such that $x <^* y$.*

P r o o f. Suppose the given condition is satisfied. By (b), every non-empty up-set of Z meets W_Q. Then for any up-set V of Z,

$$\mathbf{E}(P,Q)V = \begin{cases} \emptyset & \text{if } V = \emptyset, \\ W_Q & \text{if } \emptyset \neq V \subseteq W_Q, \\ Z & \text{if } V \cap W_P \neq \emptyset. \end{cases}$$

Since (E_2) is automatic in a finite space, $(Z, E(P,Q))$ is a $\mathbf{Q}$-space.

Conversely, suppose $(Z, E(P,Q))$ is a $\mathbf{Q}$-space. Then $\mathbf{E}(P,Q){\uparrow}x$ is an up-set for each $x \in Z$, where ${\uparrow}x := \{y \in Z \mid y \geqslant x\}$. So, if there were x such that ${\uparrow}x \cap W_Q = \emptyset$, then W_P would be an up-set (and likewise with Q and P interchanged). But we know W_P is not an up-set, since $\beta_P <^* \beta_Q$. Hence (b) holds. Let $x \in \operatorname{Max} Z$. If x were in W_P we would have $\mathbf{E}(P,Q)\{x\} = W_P$, which is impossible. Hence $x \in W_Q$ and $\mathbf{E}(P,Q)\{x\} = W_Q$, which must therefore be an up-set. Thus (a) holds. ∎

The next lemma is needed in our proof that every maximal subalgebra of L is associated with an admissible order on $W_P \dot\cup W_Q$.

LEMMA 3.8. *Let* $\varphi : (W_P \dot\cup W_Q, E(P,Q), \leqslant_{PQ}) \to (Z, F, \leqslant)$ *be a surjective, but not injective,* $\mathcal{Y}^{\mathbf{Q}}$*-morphism. Then there exists an admissible order* $\leqslant^*$ *on* $W_P \dot\cup W_Q$ *such that* $\varphi : (W_P \dot\cup W_Q, E(P,Q), \leqslant^*) \to (Z, F, \leqslant)$ *is a* $\mathbf{Q}$*-morphism.*

P r o o f. Assume first that $\varphi(W_P) \cap \varphi(W_Q) = \emptyset$. It is clear that, since φ is a $\mathbf{Q}$-map, the equivalence classes of F are $\varphi(W_P)$ and $\varphi(W_Q)$. Then, by the same argument as in the proof of Lemma 3.7, $\varphi(W_Q)$ is an up-set and each element of $\varphi(W_P)$ is majorised by some point of $\varphi(W_Q)$. For each $x \in \operatorname{Max} W_P$ let $y_x \in \operatorname{Max} W_Q$ be such that $\varphi(x) < \varphi(y_x)$. Let

$$\leqslant^* := \leqslant_{PQ} \cup \bigcup_{x \in \operatorname{Max} W_P} \{(\beta_P, y_x), (x, y_x)\}.$$

Then $\leqslant^*$ is a well-defined admissible order and φ remains a $\mathbf{Q}$-map when $\leqslant_{PQ}$ is strengthened to $\leqslant^*$.

Assume now that $\varphi(W_P) \cap \varphi(W_Q) \neq \emptyset$. Suppose, for contradiction, that there exists $x \in \operatorname{Max} Z$ such that $x \notin \varphi(W_Q)$. Then

$$W_P = \mathbf{E}(P,Q)(\varphi^{-1}(\{x\})) = \varphi^{-1}(\mathbf{F}\{x\}).$$

This is incompatible with our hypothesis that $\varphi(W_P \cap W_Q) \neq \emptyset$. Thus every maximal point of Z is in $\varphi(W_Q)$, and similarly in $\varphi(W_P)$. Pick any $z \in \operatorname{Max} Z$. Then

$$W_P \dot\cup W_Q = \mathbf{E}(P,Q)(\varphi^{-1}(\{z\})).$$

We deduce that $\mathbf{F}\{z\} = Z$, whence $F = Z \times Z$. It follows from these facts that any order-preserving map onto $(Z, F, \leqslant)$ from a $\mathbf{Q}$-space $(W_P \dot\cup W_Q, E(P,Q), \leqslant^*)$ with some admissible order $\leqslant^*$ is automatically a $\mathbf{Q}$-map. Because every point of Z is majorised by an element of $\varphi(W_Q)$ we may, as previously, find for each

$x \in W_P$ some point $y_x \in \operatorname{Max} W_Q$ such that $\varphi(x) \leqslant \varphi(y_x)$. Now define $\leqslant^*$ in the same way as above. ∎

THEOREM 3.9. *The maximal* $\mathbf{Q}$*-subalgebras of* $H(L)$ *are in one-to-one correspondence with functions* $\theta : \operatorname{Max} W_P \to \operatorname{Max} W_Q$.

Proof. The two preceding lemmas imply that the maximal $\mathbf{Q}$-subalgebras correspond exactly to the admissible orders. As the proof of Lemma 3.7 shows, each admissible order is determined by, and determines, a function $\theta : \operatorname{Max} W_P \to \operatorname{Max} W_Q$. ∎

We can now describe the relations which arise in our piggyback duality. We shall incorporate among our relations the graphs of sufficient endomorphisms to ensure that the graph of every automorphism of $\underline{P}$ and of $\underline{Q}$ is generated. This means that we only need include "essentially different" maximal subalgebras of $(\beta_P, \beta_Q)^{-1}(\leqslant)$. Here "essentially different" will mean, dually, "up to the permutation of labels of maximal elements", with the proviso that in T_{pq} we can only permute $1, \ldots, p$ and $p+1, \ldots, q$ among themselves. As in the case of the varieties $\mathbf{B}_n$ of distributive p-algebras treated in [11], the essentially different maximal subalgebras can be labelled, dually, using partitions of integers. The notation here gets rather complicated, but the ideas are simple, as the illustrative examples after Theorem 3.10 show.

We adopt the notation for partitions used in [2]. We write $(k_1, k_2, \ldots, k_r)$ to denote the partition of n into parts $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_r > 0$ (so $k_1 + k_2 + \ldots + k_r = n$) and we let $k_0 = 0$. Let $(k_1, k_2, \ldots, k_r)$ be a partition of $n > 0$ and let $n' \geqslant 0$. Define

$$\pi(n, n'; (k_1, \ldots, k_r)) = \bigcup_{1 \leqslant j \leqslant n} \{(j, \breve{n'+i}) \mid k_0 + \ldots + k_{i-1} < j \leqslant k_0 + \ldots + k_i\}.$$

To allow a uniform notation encompassing special cases we write (0) in place of $(k_1, \ldots, k_r)$ if no partition is present, and let $\pi(n, n'; (0)) = \emptyset$. In definitions (ii)–(iv) below we permit either $(k_1, \ldots, k_r)$ or $(l_1, \ldots, l_s)$ to be replaced by (0). We define ordered sets as follows.

(i) $O(p; (k_1, \ldots, k_r))$ has underlying set $\{1, \ldots, p, \breve{1}, \ldots, \breve{p}\}$ and order

$$\pi(p, 0; (k_1, \ldots, k_r)).$$

(ii) $N(p, q, m; (k_1, \ldots, k_r); (l_1, \ldots, l_s))$ has underlying set $\{0, 1, \ldots, p+q, \breve{0}, \breve{1}, \ldots, \breve{p+q}\}$ and order

$$\pi(m, 0; (k_1, \ldots, k_r)) \cup \pi(p+q-m, p; (l_1, \ldots, l_s)) \cup \{(0, \breve{0})\} \cup \{(0, \breve{i}) \mid p+1 \leqslant i \leqslant p+q\} \cup \{(\breve{0}, \breve{i}) \mid p+1 \leqslant i \leqslant p+q\}.$$

(iii) $M(p,q;(k_1,\dots,k_r);(l_1,\dots,l_s))$ has underlying set $\{0,1,\dots,p+q,\breve{1},\dots$ $\dots,\breve{p+q}\}$ and order

$$\pi(q,0;(k_1,\dots,k_r))\cup\pi(p,0;(l_1,\dots,l_s))\cup\{(0,\breve{1})\}\,.$$

(iv) $L(p,q,m;(k_1,\dots,k_r);(l_1,\dots,l_s))$ has underlying set $\{1,\dots,p+q,\breve{0},\breve{1},\dots$ $\dots,\breve{p+q}\}$ $(m\geqslant q)$, and order

$$\pi(m,0;(q,\dots,k_r))\cup\pi(p+q-m,q;(l_1,\dots,l_s))\cup\{(1,\breve{0})\}$$
$$\cup\{(\breve{0},i)\mid 1\leqslant i\leqslant p\}\,.$$

As an immediate consequence of previous results we deduce our main theorem.

Theorem 3.10. *Let τ be the discrete topology.*

(1) *For $p\geqslant 1$ let $\mathcal{A}:=\mathbf{D}_{p0}=\mathbb{ISP}(\underline{D}_{p0})$ and let $\underset{\sim}{D}_{p0}:=(D_{p0};\tau,R)$, where R consists of the following relations*:
   (a) *the set of subalgebras $S(O(p;k_1,\dots,k_r))$ where $(k_1,\dots,k_r)$ is a partition of p, and the graphs of the members of*

$$\begin{cases}\emptyset & \text{if } p=1,\\ \{f_\sigma\} & \text{if } p=2,\\ \{f_\sigma,f_\tau\} & \text{if } p\geqslant 3.\end{cases}$$

(2) *For $p\geqslant 0$ let $\mathcal{A}:=\mathbf{D}_{p1}=\mathbb{ISP}(\underline{D}_{p1})$ and let $\underset{\sim}{D}_{p1}:=(D_{p1};\tau,R)$, where R consists of the following relations*:
   (a) *the set of subalgebras $S(N(p,1,m;(k_1,\dots,k_r),(m)))$ where $0\leqslant m\leqslant p+1$ and $(k_1,\dots,k_r)$ is a partition of $p+1$ if $m<p+1$ and (0) if $m=p+1$, and*
   (b) *the graphs of the members of*

$$\begin{cases}\{g\} & \text{if } p=0,\\ \{g_\sigma\} & \text{if } p=1,\\ \{g,g_\sigma,g_\tau\} & \text{if } p\geqslant 2.\end{cases}$$

(3) *For $p\geqslant 0$, $q\geqslant 2$, let $\mathcal{A}:=\mathbf{D}_{pq}=\mathbb{ISP}(\underline{\Pi})$, where $\underline{\Pi}=\{\underline{D}_{pq},\underline{D}_{(p+q)0}\}$, and let*

$$\underset{\sim}{\Pi}=(D_{pq}\,\dot\cup\, D_{(p+q)0};\tau,R)$$

where R consists of the following relations:
   (a) *subalgebras*

$$\begin{aligned}&S(O(p+q;(k_1,\dots,k_r)))\,,\\ &S(N(p,q,m;(k_1,\dots,k_r),(l_1,\dots,l_s)))\,,\\ &S(M(p,q,m;(k_1,\dots,k_r),(l_1,\dots,l_s)))\,,\quad \textit{and}\\ &S(L(p,q,m;(k_1,\dots,k_r)))\,,\end{aligned}$$

where the parameters vary over all possible values, and

(b) *the graphs of the members of*

$$\begin{cases} \{h, f_\sigma, f'_\sigma\} & \text{if } p = 0,\ q = 2, \\ \{h, f_\sigma, f_\tau, f'_\sigma, f'_\tau\} & \text{if } p + q > 2. \end{cases}$$

Then R yields a duality on $\mathcal{A}$.

We show explicitly how the partition-induced relations work out for the varieties $\mathbf{D}_{30}$ (Figure 2) and $\mathbf{D}_{11}$ (Figure 3).

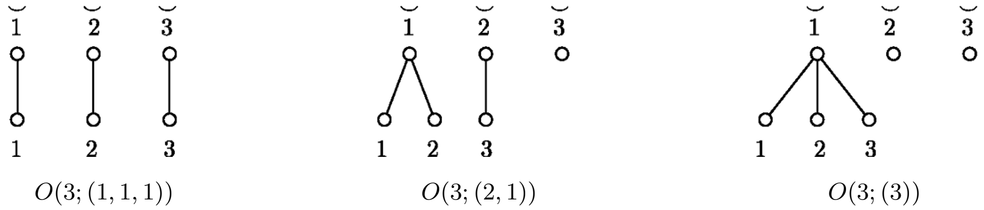


Fig. 2

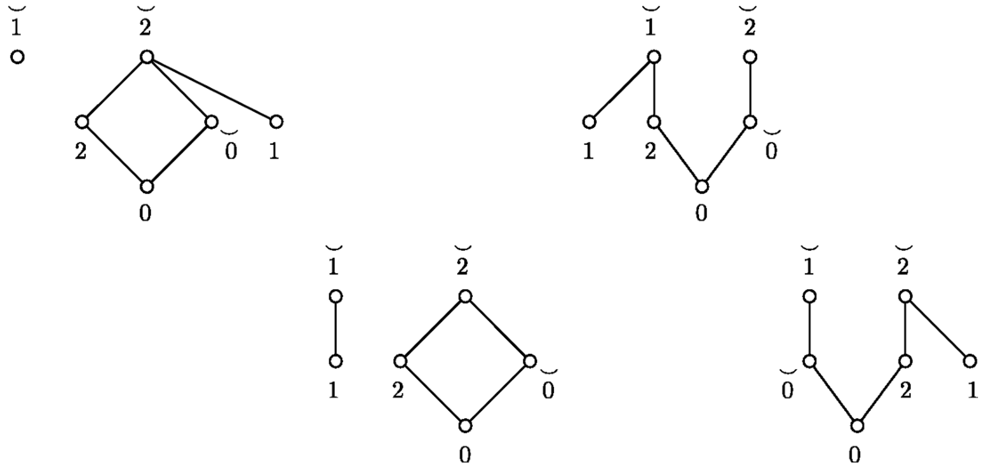


Fig. 3

For the beginning of the chain of subvarieties we have dualities with numbers of relations as indicated in Table 1.

Table 1

	Maximal subalgebras	Endomorphisms
$\mathbf{D}_{10}$	1	0
$\mathbf{D}_{01}$	1	1
$\mathbf{D}_{20}$	2	1
$\mathbf{D}_{02}$	8	3
$\mathbf{D}_{11}$	4	1
$\mathbf{D}_{30}$	3	2

The varieties $\mathbf{D}_{p0}$ are exactly those generated by the finite simple algebras in $\mathbf{Q}$ ([3], Corollary 3.8). Their dualities are less complicated than those for varieties $\mathbf{D}_{pq}$ with $q > 0$. Nevertheless, the number of relations for the duality for $\mathbf{D}_{p0}$ grows exponentially with p. Application of Theorem 2.2 in the cases $p = 3$ and $p = 4$ proves that none of the relations can be discarded. These results, obtained by computer, suggest that our duality for $\mathbf{D}_{p0}$ is best possible, at least relative to piggyback dualities. We have also shown by computer than none of our relations can be discarded in the case of $\mathbf{D}_{11}$. The question of optimality will be pursued further elsewhere.

4. Free algebras. In our introduction we alluded to Cignoli's note [4] in which he gives a construction of the $\mathbf{Q}$-space dual to the $\mathbf{Q}$-algebra freely generated by a member of $\mathbf{D}$. This construction generalises that given by Halmos in [18] (reproduced in Chapter IV of [19]) for the free monadic extension of a Boolean algebra, and is similar in spirit to the dual construction of the free distributive p-algebra generated by a member of $\mathbf{D}$ given by B. A. Davey and M. S. Goldberg in [8]. Cignoli has used his construction to show that $F\mathbf{Q}(1)$ is a 4-element chain $0 < a < \nabla a < 1$ and that the $\mathbf{Q}$-space dual to $F\mathbf{Q}(2)$ is as shown in Figure 5(c).

We approach free algebras in quite a different way, via our natural dualities for the varieties $\mathbf{D}_{pq}$. We should emphasise that we are able to describe free algebras in $\mathbf{D}_{pq}$ in total ignorance either of the identities which determine $\mathbf{D}_{pq}$ (so far unknown beyond $\mathbf{D}_{01}$) or of a subcategory of $\mathcal{Y}^{\mathbf{Q}}$ dual to $\mathbf{D}_{pq}$. Indeed our results may assist in finding the requisite laws (cf. [1], [22]). Theorem 4.2 implies, for example, that $\mathbf{D}_{10}$, $\mathbf{D}_{20}$ and $\mathbf{D}_{11}$ are distinguished by 2-variable identities. We shall pursue this topic further elsewhere.

We recall that, if $\mathcal{A} = \mathbb{ISP}(\underline{\Pi})$ has a natural duality based on a schizophrenic object $\underset{\sim}{\Pi}$, then $\underset{\sim}{\Pi}^{\kappa}$ serves as the natural dual of $F\mathcal{A}(\kappa)$. Theorem 4.1 makes this completely precise in the special cases we require. The theorem extends to free algebras with infinitely many generators by adding the appropriate topological ingredients.

THEOREM 4.1. (1) *Assume $\underline{P}$ is a finite algebra and that R is a set of algebraic relations yielding a duality on $\mathcal{A}$. Then for $1 \leqslant n < \omega$, the algebra $F\mathcal{A}(n)$ is isomorphic to the set of R-preserving maps from P^n (with relations in R extended pointwise) to P, with operations inherited pointwise from $\underline{P}^{P^n}$.*

(2) *Assume $\underline{P}$, $\underline{Q}$ are finite algebras and that*

$$R \subseteq \mathbb{S}(\underline{P}^2) \cup \mathbb{S}(\underline{Q}^2) \cup \mathbb{S}(\underline{P} \times \underline{Q}) \cup \mathbb{S}(\underline{Q} \times \underline{P})$$

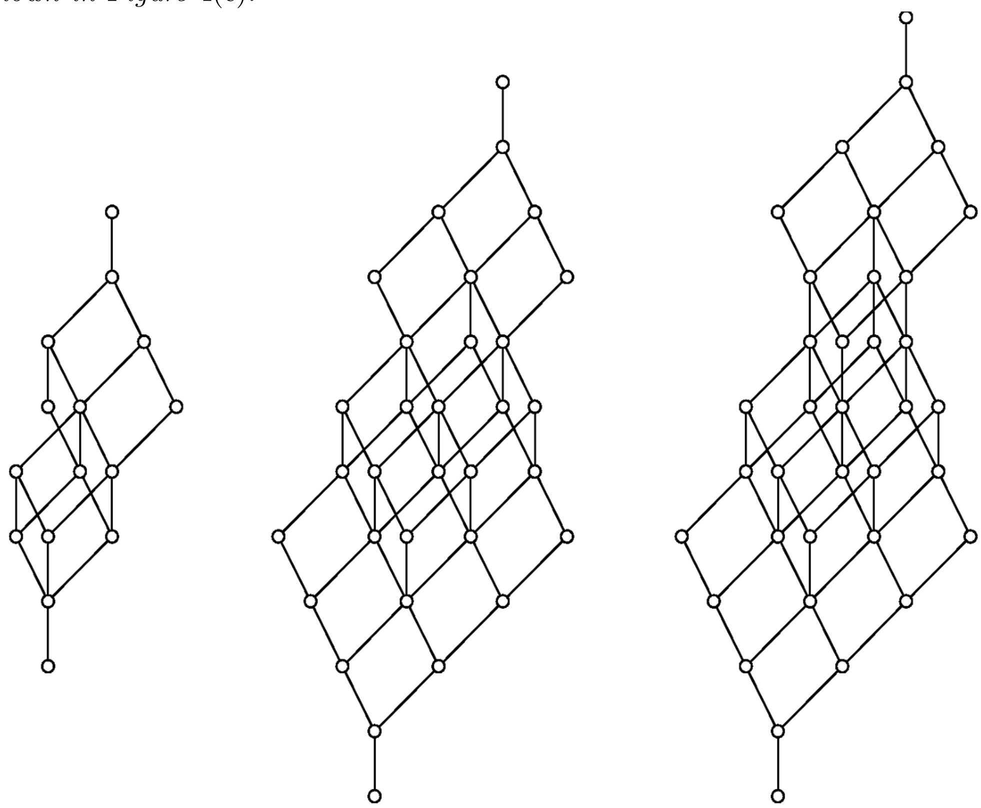
yields a duality on $\mathcal{A}$. Then for $1 \leqslant n < \omega$ the algebra $F\mathcal{A}(n)$ is isomorphic to the set of R-preserving maps from $P^n \dot\cup Q^n$ which map P^n into P and Q^n into Q (with relations extended pointwise from $P \dot\cup Q$), structured pointwise from $\underline{P}^{P^n}$ and $\underline{Q}^{Q^n}$.

The calculations involved in applying this theorem in practice require a computer, except in the very simplest cases. We use the program for calculating relation-preserving maps which was devised to test optimality of dualities for varieties of distributive p-algebras; see [12] and [24]. The theorem below is obtained by combining our computational results with Cignoli's description of $F\mathbf{Q}(1)$ and $F\mathbf{Q}(2)$. The first part can alternatively be derived from Cignoli's results alone, by noting that $F\mathbf{Q}(1)$ satisfies the identity $\nabla(a \wedge b) = \nabla(a) \wedge \nabla(b)$ for $\mathbf{D}_{01}$ given in [3], Section 4. Before giving the theorem we get a trivial case out of the way: note that $\mathbf{D}_{01}$ is defined by the law $\nabla a = a$, whence it follows that $F\mathbf{D}_{10}(n) = F\mathbf{D}(n)$ as a lattice.

THEOREM 4.2. *Let $\mathcal{A}$ be a subvariety of $\mathbf{Q}$.*

(1) *If $\mathbf{D}_{01} \subseteq \mathcal{A} \subseteq \mathbf{Q}$, then $F\mathcal{A}(1) = F\mathbf{Q}(1)$, a 4-element chain $0 < a < \nabla a = \nabla^2 a < 1$.*

(2) *$F\mathbf{D}_{01}(2)$ and $F\mathbf{D}_{20}(2)$ are shown in Figure 4(a) and (b). The associated Q-spaces are shown in Figure 5. If $\mathbf{D}_{02} \subseteq \mathcal{A} \subseteq \mathbf{Q}$, then $F\mathcal{A}(2) = F\mathbf{Q}(2)$, as shown in Figure 4(c).*



(a) (b) (c)

Fig. 4

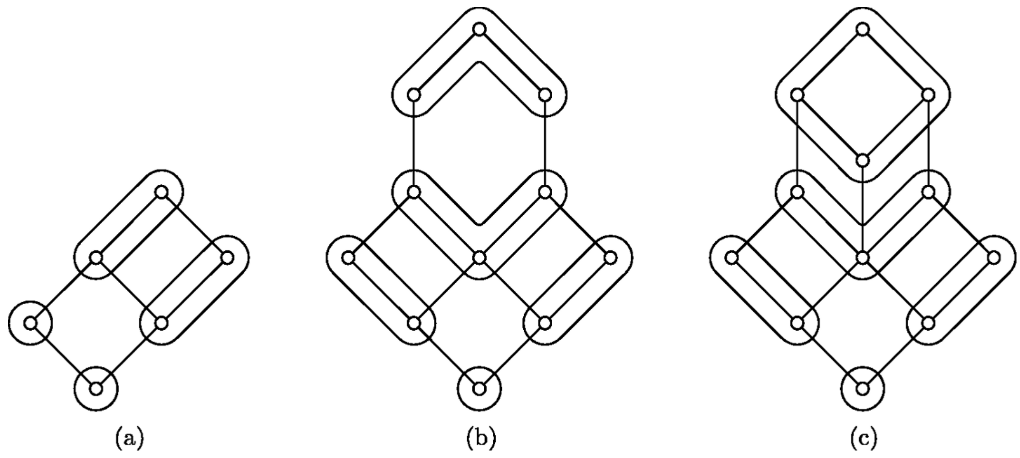


Fig. 5

The free algebra $F\mathbf{D}_{01}(3)$ has 980 elements and $F\mathbf{D}_{20}(3)$ has 22,470 elements.

For the benefit of sceptics, and to give the flavour of our theory, we present input and output data for the calculation of $F\mathbf{D}_{20}(2)$. We have $\underline{P} = \underline{D}_{20}$, a 4-element Boolean lattice whose elements we label as 1, 2, 3, 4 with $1 < 2 < 4$, $1 < 3 < 4$ and the simple quantifier. Our relations are the following subalgebras of $\underline{P}^2$: the graph of the endomorphism g_σ, viz. $\{(1,1),(2,3),(3,2),(4,4)\}$ and the maximal subalgebras associated with the partitions $(1,1)$ and (2) of 2 given in Figure 6.

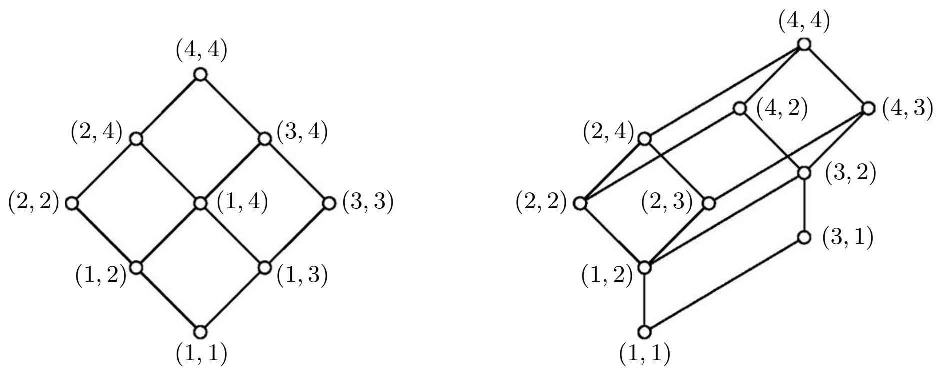


Fig. 6

Table 2 lists the 32 relation-preserving maps from the 16-element set $\underset{\sim}{P}^2$ to the 4-element set $\underset{\sim}{P}$. With lattice operations and quantifier inherited from $\underline{P}$ these maps form $F\mathbf{D}_{20}(2)$.

Table 2

1	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
2	1 1 1 1 1 2 1 2 1 1 3 3 1 2 3 4
3	1 1 1 1 1 2 1 2 1 1 3 3 1 4 4 4
4	1 1 1 1 1 2 1 4 1 1 3 4 1 2 3 4
5	1 1 1 1 1 2 1 4 1 1 3 4 1 4 4 4
6	1 1 1 1 1 2 2 2 1 3 3 3 1 4 4 4
7	1 1 1 1 1 2 2 4 1 3 3 4 1 4 4 4
8	1 1 1 1 1 2 3 4 1 2 3 4 1 2 3 4
9	1 1 1 1 1 2 3 4 1 2 3 4 1 4 4 4
10	1 1 1 1 1 2 4 4 1 4 3 4 1 4 4 4
11	1 1 1 1 1 4 1 4 1 1 4 4 1 4 4 4
12	1 1 1 1 1 4 2 4 1 3 4 4 1 4 4 4
13	1 1 1 1 1 4 3 4 1 2 4 4 1 4 4 4
14	1 1 1 1 1 4 4 4 1 4 4 4 1 4 4 4
15	1 1 1 1 2 2 2 2 3 3 3 3 4 4 4 4
16	1 1 1 1 2 2 2 4 3 3 3 4 4 4 4 4
17	1 1 1 1 2 2 4 4 3 4 3 4 4 4 4 4
18	1 1 1 1 2 4 2 4 3 3 4 4 4 4 4 4
19	1 1 1 1 2 4 4 4 3 4 4 4 4 4 4 4
20	1 1 1 1 4 4 4 4 4 4 4 4 4 4 4 4
21	1 2 3 4 1 2 3 4 1 2 3 4 1 2 3 4
22	1 2 3 4 1 2 3 4 1 2 3 4 1 4 4 4
23	1 2 3 4 1 2 4 4 1 4 3 4 1 4 4 4
24	1 2 3 4 1 4 3 4 1 2 4 4 1 4 4 4
25	1 2 3 4 1 4 4 4 1 4 4 4 1 4 4 4
26	1 2 3 4 2 2 4 4 3 4 3 4 4 4 4 4
27	1 2 3 4 2 4 4 4 3 4 4 4 4 4 4 4
28	1 2 3 4 4 4 4 4 4 4 4 4 4 4 4 4
29	1 4 4 4 1 4 4 4 1 4 4 4 1 4 4 4
30	1 4 4 4 2 4 4 4 3 4 4 4 4 4 4 4
31	1 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4
32	4 4 4 4 4 4 4 4 4 4 4 4 4 4 4 4

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