ALGEBRAIC METHODS IN LOGIC AND IN COMPUTER SCIENCE
BANACH CENTER PUBLICATIONS, VOLUME 28
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1993

LECTURES ON CYLINDRIC SET ALGEBRAS

J. DONALD MONK

Department of Mathematics, University of Colorado
Campus Box 395, Boulder, Colorado 80309, U.S.A.

These notes are a corrected and revised version of notes which accompanied lectures given at the Banach Center in the fall of 1991. The intent is to give a self-contained introduction to cylindric algebras from the concrete point of view. I hope that after these lectures the reader will be able to digest the basic works on this subject (Henkin, Monk, Tarski [4], [5] and Henkin, Monk, Tarski, Andréka, Németi [6]) more easily, and that even research articles in this area will be readable by one who studies these notes carefully. As the title of the lectures indicates, we are mainly concerned with the topics in [6], which appear in a condensed form in [5]. One of the frightening things about both of these books is that they begin with a mass of definitions and proceed with very detailed discussion of the interrelationships of the defined notions. We are going to introduce just a few of these definitions, little by little, giving important (but not highly technical) results about them as we go along. And we will try to motivate the notions from logic.

Cylindric algebras form the most developed form of algebraic logic. In general, algebraic logic is concerned with algebraic structures which correspond to logics of various sorts. Cylindric algebras correspond to ordinary first-order logics and to certain straightforward modifications of these logics. Other algebraic structures have a similar relationship to first-order logic; the most developed of these are relation algebras (in Tarski's sense) and polyadic algebras. We will not be concerned with these, but the reader should be able to study them more easily after reading these notes.

We will describe only the concrete aspect of cylindric algebras. The axiomatic version, fully developed in [4], will play only a minor role. Also, we will not deal with applications. Such applications exist in several other fields, such as combinatorics and theoretical computer science.

1991 *Mathematics Subject Classification*: 03G15.
The paper is in final form and no version of it will be published elsewhere.

We assume familiarity with the elementary theory of Boolean algebras, elementary first-order logic, and with the basics of universal algebra.

1. Fields of sets. We assume that the reader is familiar with the notion of a field of sets; here we just recall the notion, and establish notation. Any more extended apparatus which we need will be mentioned later on. A *field of sets* over a set X is a collection A of subsets of X containing X itself and closed under union and complementation with respect to X. Then A is also closed under intersection, and has the empty set 0 as a member. Unless confusion might result, we identify such a collection with the *algebra* (in the sense of universal algebra) $\langle A, \cup, \cap, \setminus, 0, X\rangle$. Here $\setminus$ is the operation of complementation with respect to X; many people denote it by $-$.

2. Cylindric-relativized set algebras. We begin with some purely set-theoretical notation. For any sets A and B, the set of all functions from A into B is denoted by ${}^AB$ (many people denote this by B^A). For any set V, $\mathcal{P}(V)$ is the collection of all subsets of V; for any function y and any i in its domain, y^i_u is the function which is like y except that its value at i is equal to u. For any function f and any a in its domain, the value of f at a will be indicated by fa, f_a, or other similar things.

Now we define the basic notions of cylindric-relativized set algebras. Let U and I be sets and $V \subseteq {}^IU$. For all $i, j \in I$ we set

$$D^{[V]}_{ij} = \{v \in V : v_i = v_j\};$$

this is a *diagonal set*. Furthermore, for each $i \in I$ we let $C^{[V]}_i$ be the mapping from $\mathcal{P}(V)$ into $\mathcal{P}(V)$ defined as follows: for any $X \subseteq V$,

$$C^{[V]}_i X = \{y \in V : y^i_u \in X \text{ for some } u \in U\}.$$

This is called the *V-relativized cylindrification in the direction i*. Usually here and in the literature one uses an ordinal α in place of I; the more general definition here is sometimes useful. Here is a general convention: **When no confusion is likely, we omit superscripts and subscripts from defined objects.** Thus, for example, we frequently write merely D_{ij} or C_i.

A *cylindric-relativized field of sets* is a field A of sets such that there exist sets I, U, V such that $V \subseteq {}^IU$, A is a field of subsets of V, $D^{[V]}_{ij} \in A$ for all $i, j \in I$, and A is closed under each operation $C^{[V]}_i$, $i \in I$. A *cylindric-relativized set algebra* is the associated algebra

$$\mathfrak{A} \stackrel{\text{def}}{=} \langle A, \cup, \cap, \setminus, 0, V, C^{[V]}_i, D^{[V]}_{ij}\rangle_{i,j\in I}.$$

Cylindric-relativized set algebras are the main things that we shall be discussing in these notes. There are three natural areas of investigation concerning them. First, there are intrinsic questions deriving from the very definitions: what happens to these algebras when the sets I, U, or V are changed; and what can one say about algebraic operations (homomorphisms, subalgebras, products, etc.) applied to them? Second, can one abstractly characterize such algebras up to isomorphism, like one does for permutation groups via the abstract notion of group, for example? Third, how do such algebras relate to other objects in mathematics, in particular to logic, which, as indicated at the beginning, is the main justification for their consideration? In these notes we will be concerned mainly with the first type of question, with some consideration of the second and third questions. We will right now say a few words about the third aspect of this subject.

3. The logical origin of cylindric algebras. Let L be a first-order language, and $\mathfrak{M}$ a model for L. The universe of any model $\mathfrak{M}$ is denoted by M. We assume that L has a countably infinite sequence of variables $\langle v_i : i \in \omega \rangle$. We take as well-known what it means for a sequence $x \in {}^{\omega}M$ to *satisfy* a formula ϕ in $\mathfrak{M}$. Set $\phi^{\mathfrak{M}} = \{x \in {}^{\omega}M : x \text{ satisfies } \phi \text{ in } \mathfrak{M}\}$. The set $A = \{\phi^{\mathfrak{M}} : \phi \text{ a formula of } L\}$ is a cylindric-relativized field of sets; the corresponding sets I, U, and V are, respectively, ω, M, and ${}^{\omega}M$. This is the main motivating source for the notion of cylindric-relativized field of sets and, indeed, for the whole topic of algebraic logic. The cylindric-relativized set algebra obtained from $\mathfrak{M}$ will be denoted by $\mathfrak{Cs}\,\mathfrak{M}$. By the above convention, $Cs\,\mathfrak{M}$ then denotes the indicated cylindric-relativized field of sets.

The cylindric-relativized field of sets obtained in this way has many special properties. Some of these will be described and studied later.

For now we want to indicate some important connections between logic and such algebras. We use $\cong$ to indicate isomorphism. The central logical notion of elementary equivalence is characterized algebraically as follows:

THEOREM 3.1. *If $\mathfrak{M}$ is elementarily equivalent to $\mathfrak{N}$, then $\mathfrak{Cs}\,\mathfrak{M} \cong \mathfrak{Cs}\,\mathfrak{N}$.*

In fact, let $\mathfrak{M}$ and $\mathfrak{N}$ be similar structures, and let $f = \{(\phi^{\mathfrak{M}}, \phi^{\mathfrak{N}}) : \phi \text{ a formula}\}$. Then the following conditions are equivalent:

(i) $\mathfrak{M}$ *is elementarily equivalent to* $\mathfrak{N}$;

(ii) f *is a function from $Cs\,\mathfrak{M}$ into $Cs\,\mathfrak{N}$ such that $f\phi^{\mathfrak{M}} = \phi^{\mathfrak{N}}$ for every formula ϕ*;

(iii) f *is an isomorphism from $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{N}$ such that $f\phi^{\mathfrak{M}} = \phi^{\mathfrak{N}}$ for every formula ϕ.*

Proof. (i)$\Rightarrow$(iii). That f is a function and is one-one is seen as follows (using $[\chi]$ temporarily to denote the universal closure of any formula χ): for any formulas ϕ and ψ, $\phi^{\mathfrak{M}} = \psi^{\mathfrak{M}}$ iff $\mathfrak{M} \models (\phi \leftrightarrow \psi)$ iff $\mathfrak{M} \models [\phi \leftrightarrow \psi]$ iff $\mathfrak{N} \models [\phi \leftrightarrow \psi]$ iff ... iff $\phi^{\mathfrak{N}} = \psi^{\mathfrak{N}}$. The other conditions in (iii) are clear from the definitions involved.

(iii)$\Rightarrow$(ii). Obvious.

(ii)$\Rightarrow$(i). For any sentence ϕ, $\mathfrak{M} \models \phi \Rightarrow \phi^{\mathfrak{M}} = {}^{\omega}M \Rightarrow f\phi^{\mathfrak{M}} = {}^{\omega}N \Rightarrow \phi^{\mathfrak{N}} = {}^{\omega}N \Rightarrow \mathfrak{N} \models \phi$. Applying this argument to $\neg\phi$ gives the other direction. ∎

On the other hand, it is also natural to try to characterize logically the isomorphism of structures $\mathfrak{Cs}\,\mathfrak{M}$. To do this, we need to discuss a special topic in logic, definitional equivalence. Given two first-order structures $\mathfrak{M}$ and $\mathfrak{N}$, not necessarily similar, we say that they are *definitionally equivalent* provided that $M = N$ and the following two conditions hold (we restrict ourselves to languages with only relation symbols, for simplicity):

(1) Each fundamental relation of $\mathfrak{M}$ is elementarily definable in $\mathfrak{N}$, i.e., if R is an m-ary fundamental relation of $\mathfrak{M}$, then there is a formula ϕ of the language of $\mathfrak{N}$ with free variables among $v_0, \ldots, v_{m-1}$ such that $R = \{x \in {}^mM : \mathfrak{N} \models \phi[x]\}$.

(2) Each fundamental relation of $\mathfrak{N}$ is elementarily definable in $\mathfrak{M}$.

Two standard examples of this sort of thing are: groups as structures with a single binary operation, or as structures with a binary operation and an inverse operation; Boolean algebras with lattice operations versus Boolean algebras with ring operations.

THEOREM 3.2. *$\mathfrak{M}$ and $\mathfrak{N}$ are definitionally equivalent iff $\mathfrak{Cs}\,\mathfrak{M} = \mathfrak{Cs}\,\mathfrak{N}$.*

Proof. $\Rightarrow$ Let ϕ be a function which assigns to each fundamental relation R of $\mathfrak{M}$ a formula ϕ_R as in the definition. Then we define a function ϕ' from formulas of the language of $\mathfrak{M}$ into formulas of the language of $\mathfrak{N}$:

$$\begin{aligned} \phi'(\mathbf{R}v_{i_0}\ldots v_{i_{m-1}}) &= \phi_R(v_{i_0},\ldots,v_{i_{m-1}}), & \phi'(v_i = v_j) &= (v_i = v_j)\,, \\ \phi'(\neg\chi) &= \neg\phi'(\chi), & \phi'(\chi\vee\theta) &= \phi'(\chi)\vee\phi'(\theta)\,, \\ \phi'(\chi\wedge\theta) &= \phi'(\chi)\wedge\phi'(\theta), & \phi'(\forall v_i\chi) &= \forall v_i\phi'(\chi)\,. \end{aligned}$$

Now a straightforward induction shows that $\chi^{\mathfrak{M}} = (\phi'(\chi))^{\mathfrak{N}}$ for every formula χ of the language of $\mathfrak{M}$. This proves that $Cs\,\mathfrak{M} \subseteq Cs\,\mathfrak{N}$. The converse is similar.

$\Leftarrow$ Let R be an m-ary fundamental relation of $\mathfrak{M}$. Then $R' \overset{\text{def}}{=} \{x \in {}^{\omega}M : x \restriction m \in R\} \in Cs\,\mathfrak{M}$, and hence it is also in $Cs\,\mathfrak{N}$, say $R' = \psi^{\mathfrak{N}}$. Now if $i \geq m$ then $C_iR' = R'$; hence $(\exists v_i\,\psi)^{\mathfrak{N}} = C_i\psi^{\mathfrak{N}} = \psi^{\mathfrak{N}}$ also. Hence without loss of generality we may assume that the free variables of ψ are among $v_0, \ldots, v_{m-1}$. Thus ψ defines R in $\mathfrak{N}$. By symmetry, this proves that $\mathfrak{M}$ and $\mathfrak{N}$ are definitionally equivalent. ∎

For the characterization of isomorphism of structures $\mathfrak{Cs}\,M$ we also need to use the following not so well-known fact about ordinary first-order logic:

FACT. *Every first-order formula is logically equivalent to a formula in which all non-equality atomic parts have the standard form*

$$\mathbf{R}v_0 \ldots v_{m-1},$$

thus with the first m variables following each m-ary relation symbol (*in a language with only relation symbols*).

Here is a sketch of the proof of this fact. Note the following logical equivalence:

$$\mathbf{R}v_{i_0}\ldots v_{i_{m-1}} \leftrightarrow \exists v_j\,(v_j = v_{i_0} \wedge \mathbf{R}v_jv_{i_1}\ldots v_{i_{m-1}}),$$

provided that j is different from each of $i_0, \ldots, i_{m-1}$. This is an elementary exercise. A similar result holds for a replacement of any variable instead of just the first one. So any atomic formula is equivalent to a more complicated expression involving existential quantifiers and equality formulas, and an atomic formula $\mathbf{R}v_{j_0} \ldots v_{j_{m-1}}$ in which all the indices are distinct and greater than m. Then the same procedure can be applied to "replace" these variables by $v_0, \ldots, v_{m-1}$ respectively.

THEOREM 3.3. *$\mathfrak{Cs}\,\mathfrak{M}$ is isomorphic to $\mathfrak{Cs}\,\mathfrak{N}$ iff $\mathfrak{M}$ is elementarily equivalent to a structure definitionally equivalent to $\mathfrak{N}$.*

Proof. $\Rightarrow$ Let f be an isomorphism from $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{N}$. We define a new structure $\mathfrak{P}$ similar to $\mathfrak{M}$ and with universe N. For each fundamental relation $\mathbf{R}$ of $\mathfrak{M}$, let

$$\mathbf{R}^{\mathfrak{P}} = \{x \in {}^m N : x \subseteq y \text{ for some } y \in f(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}}\}.$$

Now we claim that $f(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}} = (\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{P}}$. In fact, if $y \in f(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}}$, then $y \restriction m \in \mathbf{R}^{\mathfrak{P}}$, and hence $y \in (\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{P}}$. On the other hand, suppose that $y \in (\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{P}}$. Then $y \restriction m \subseteq z \in f(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}}$ for some z. Write $f(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}} = \phi^{\mathfrak{N}}$. If $i \geq m$, then $C_i(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}} = (\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}}$, and hence $C_i\phi^{\mathfrak{N}} = \phi^{\mathfrak{N}}$. So without loss of generality we may assume that the free variables of ϕ are among $v_0, \ldots, v_{m-1}$. Hence from $y \restriction m \subseteq z \in \phi^{\mathfrak{N}}$ it follows that $y \in \phi^{\mathfrak{N}} = f(\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}}$, as desired: this proves our claim. From the claim and the FACT it follows that $f\psi^{\mathfrak{M}} = \psi^{\mathfrak{P}}$ for every formula ψ of the language of $\mathfrak{M}$. Therefore by Theorem 3.1, f is an isomorphism from $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{P}$. Hence $\mathfrak{Cs}\,\mathfrak{N} = \mathfrak{Cs}\,\mathfrak{P}$, and the desired conclusion follows from previous theorems.

$\Leftarrow$ Clear from previous theorems. ∎

We now make some remarks about Boolean algebras. The abstract operations in a Boolean algebra corresponding to the set-theoretic operations $\cup$, $\cap$, $\setminus$, 0, and X in a field of sets (subsets of X) are denoted by $+$, $\cdot$, $-$, 0, and 1 respectively.

An important aspect of the theory of Boolean algebras is the description of the Lindenbaum–Tarski algebras of common first-order theories. Given a theory T, one defines an equivalence relation $\equiv$ on sentences of the given language by defining $\phi \equiv \psi$ iff $T \models \phi \leftrightarrow \psi$. Then the collection of equivalence classes forms a Boolean algebra under the operations $[\phi] + [\psi] = [\phi \vee \psi]$, $[\phi] \cdot [\psi] = [\phi \wedge \psi]$, $-[\phi] = [\neg\phi]$, $0 = [\mathbf{F}]$, $1 = [\mathbf{T}]$; this is the *Lindenbaum–Tarski* algebra of T ($\mathbf{F}$ and $\mathbf{T}$ are any fixed logically invalid and logically valid sentences, respectively). For what these algebras look like for common theories T, see the chapter by Myers in the Boolean algebra handbook [7]. For Boolean algebras, the description consists in describing a linear order L such that the Lindenbaum–Tarski algebra is isomorphic to the interval algebra on L.

The corresponding facit of the theory of cylindric algebras is to describe the cylindric set algebras $\mathfrak{Cs}\,\mathfrak{M}$ for important models $\mathfrak{M}$. This amounts to looking at complete theories only, which is customary in model theory. It is somewhat surprising that this aspect of the theory of cylindric algebras has been almost entirely neglected. A complete description of $\mathfrak{Cs}\,\mathfrak{M}$ is known only in the case in which $\mathfrak{M}$ has only one-place relations. There are many other simple structures where the description of $\mathfrak{Cs}\,\mathfrak{M}$ should not be difficult; for example, for $\mathfrak{M}$ the rationals under their natural ordering.

4. Elementary facts. We summarize some of the elementary arithmetic of cylindric-relativized set algebras in the following lemma. This lemma will be used later without specific citation of it.

LEMMA 4.1. (i) $X \cap C_iY = 0$ *iff* $C_iX \cap Y = 0$.
(ii) $X \subseteq C_iX$.
(iii) *If* $X \subseteq Y$ *then* $C_iX \subseteq C_iY$.
(iv) $C_i \bigcup X = \bigcup_{x\in X} C_ix$.

Proof. (i) Suppose that $x \in X \cap C_iY$. Then $x^i_u \in Y$ for some $u \in U$. $x = (x^i_u)^i_{xi}$, so $x^i_u \in C_iX \cap Y$; (i) follows by symmetry.
(ii)–(iv). Easy. ∎

Now we introduce some notation. Crs_I is the class of all cylindric-relativized set algebras with associated set I, called its *dimension*. When we say "a Crs_I", we mean "a member of Crs_I", and similarly for other classes of algebras introduced later. For any collection V of functions with domain I, the collection of all subsets of V forms a cylindric-relativized field of sets; the associated algebra is denoted by $\mathfrak{P}V$. If $\mathfrak{A}$ is any Crs_I, with notation as in Section 2, then the set V is called the *unit* of $\mathfrak{A}$. The *base* of the Crs_I and of V is the set $\bigcup_{p\in V} \mathrm{range}(p)$; this is the smallest set U such that $V \subseteq {}^IU$. For any Crs_I $\mathfrak{A}$, we denote by $\mathfrak{Bl}\mathfrak{A}$ the *Boolean reduct* of $\mathfrak{A}$; it consists of A together with the operations $\cup$, $\cap$, $\backslash$, 0, and V. For any a in a Crs_I, we define the *dimension set* of a to be

$$\Delta a = \{i \in I : C_ia \neq a\}.$$

An element a is *zero-dimensional* if its dimension set is 0. The 0 and unit of a Crs_I are always zero-dimensional. In an algebra $\mathfrak{Cs}\,\mathfrak{M}$ these are the only zero-dimensional elements. But if, for example, we take $V = {}^\omega\{0,1\} \cup {}^\omega\{2,3\}$ and consider the Crs_ω of all subsets of V, then both ${}^\omega\{0,1\}$ and ${}^\omega\{2,3\}$ are zero-dimensional, as well as the 0 and unit of the algebra.

We use "BA" to abbreviate "Boolean algebra".

LEMMA 4.2. *The collection of all zero-dimensional elements of a* Crs_I $\mathfrak{A}$ *forms a subalgebra of the BA* $\mathfrak{Bl}\mathfrak{A}$.

Proof. Let Z be the indicated collection. Clearly Z is closed under $\cup$. To show that it is closed under $\backslash$, suppose that $z \in Z$, $i \in I$, and $x \in C_i(V\backslash z)$; we want to show that $x \in V\backslash z$. We have $x^i_u \in V\backslash z$ for some u. If $x \in z$, then $x^i_u \in C_iz = z$, contradiction. ∎

A *subunit* of $\mathfrak{A}$ is an atom of the BA of zero-dimensional elements of $\mathfrak{P}V$ (where V is the unit of $\mathfrak{A}$). A *subbase* of $\mathfrak{A}$ is the base of some subunit of $\mathfrak{A}$. Note that it may be that some subunits of $\mathfrak{A}$ are not members of A. For any set U and any function p mapping I into U we denote by ${}^IU^p$ the set $\{q \in {}^IU : \{i \in I : pi \neq qi\}$ is finite$\}$.

LEMMA 4.3. *Let $\mathfrak{A}$ be a Crs_I with unit V. Then V is the disjoint union of all subunits of $\mathfrak{A}$. Moreover, for each subunit W of $\mathfrak{A}$ there is a subbase Y of $\mathfrak{A}$ and some $p \in V$ such that $W \subseteq {}^IY^p$.*

Proof. For each $p \in V$ let

$$zd(p) = \bigcup\{C_{i_0} \ldots C_{i_m}\{p\} : m \in \omega,\ i \in {}^{m+1}I\}.$$

Clearly $zd(p)$ is a zero-dimensional element of $\mathfrak{P}V$. We claim that it is an atom of the BA of zero-dimensional elements of $\mathfrak{P}V$. To show this, suppose that a is any zero-dimensional element, and $zd(p) \cap a \neq 0$. Thus $C_{i_0} \ldots C_{i_m}\{p\} \cap a \neq 0$ for some $i_0, \ldots, i_m$, and hence $\{p\} \cap C_{i_m} \ldots C_{i_0} a \neq 0$, i.e. (since a is zero-dimensional), $p \in a$. Hence clearly $zd(p) \subseteq a$, as desired. This shows that $zd(p)$ is a subunit of $\mathfrak{A}$. If a is any subunit of $\mathfrak{A}$, choose $p \in a$; then clearly $zd(p) \subseteq a$, and hence $zd(p) = a$. So, every subunit has the form $zd(p)$. For any $p \in V$ we have $p \in zd(p)$. This proves that V is the disjoint union of all subunits of $\mathfrak{A}$.

Let W be any subunit of $\mathfrak{A}$. By the preceding paragraph, $W = zd(p)$ for some $p \in V$. Clearly, then, $W \subseteq {}^IY^p$, where Y is the base of W. ∎

Note that the sets ${}^IY^p$ may not be in the algebra $\mathfrak{P}V$, since the cylindrifications may lead outside of V, so to speak. For example, if $V = \{\langle i : i \in \omega\rangle\}$, then the base of V is ω, but of course for all p, ${}^\omega\omega^p \notin \mathfrak{P}V$.

5. Relativization. Let $\mathfrak{A}$ be a Crs_I with unit element V, and suppose that $W \subseteq V$. We define a mapping $\mathrm{rl}_W^{\mathfrak{A}}$ from $\mathfrak{A}$ into $\mathcal{P}(W)$ by setting, for any $X \in A$,

$$\mathrm{rl}_W^{\mathfrak{A}} X = W \cap X.$$

Thus $\mathrm{rl}_W^{\mathfrak{A}}$ (the *relativization* operation) maps into the Crs_I $\mathfrak{P}W$. It clearly preserves all of the Boolean operations (union, intersection, complementation, 0, unit) and takes $D_{ij}^{[V]}$ to $D_{ij}^{[W]}$. Also, for any $X \in A$ we have $C_i^{[W]}(\mathrm{rl}\, X) \subseteq \mathrm{rl}(C_i^{[V]}X)$. In fact, if $x \in C_i^{[W]}(\mathrm{rl}\, X)$, then $x \in W$, and say $x_u^i \in W \cap X$. Thus $x \in W$ and $x \in C_i^{[V]}X$, i.e., $x \in \mathrm{rl}(C_i^{[V]}X)$. The other inclusion does not in general hold, but we have the following important case in which it does:

PROPOSITION 5.1. *Let $\mathfrak{A}$ be a Crs_I with unit element V, and suppose that W is a zero-dimensional element of $\mathfrak{P}V$. Then $\mathrm{rl}_W^{\mathfrak{A}}$ is a homomorphism from $\mathfrak{A}$ into $\mathfrak{P}W$.*

Proof. By the remarks before the proposition, it suffices to show that for any $X \in A$ we have $\mathrm{rl}(C_i^{[V]}X) \subseteq C_i^{[W]}(\mathrm{rl}\, X)$. So, suppose that $x \in \mathrm{rl}(C_i^{[V]}X)$.

Thus $x \in W \cap C_i^{[V]} X$. Choose u so that $x^i_u \in X$. Since $x = (x^i_u)^i_{xi}$ we have $x^i_u \in C_i^{[V]} W = W$. So $x^i_u \in \mathrm{rl}\, X$, and hence $x \in C_i^{[W]}(\mathrm{rl}\, X)$, as desired. ∎

[Here is an example where the indicated inclusion does not hold: $I = \omega$, $V = {}^\omega\{0,1\}$, $x = \langle 0 : i \in \omega\rangle$, $W = \{x\}$, $f0 = 1$, $fi = 0$ for all $i \in \omega\backslash\{0\}$, $X = \{f\}$; then $x \in \mathrm{rl}_W(C_0^{[V]} X)\backslash C_0^{[W]}(\mathrm{rl}_W\, X)$.]

The Crs's obtained from logic also provide an important example where the function rl is a homomorphism—even an isomorphism. And we get an algebraic version of elementary substructure:

PROPOSITION 5.2. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are similar structures. Let $\mathfrak{A} = \mathfrak{Cs}\,\mathfrak{N}$, $\mathfrak{B} = \mathfrak{Cs}\,\mathfrak{M}$, $V = {}^\omega N$, and $W = {}^\omega M$.*

(i) *If $\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$, then $\mathrm{rl}^{\mathfrak{A}}_W$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.*

(ii) *Assume that $M \subseteq N$. Then the following conditions are equivalent*:
- (a) *$\mathfrak{M}$ is an elementary substructure of $\mathfrak{N}$*;
- (b) *$\mathrm{rl}^{\mathfrak{A}}_W$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ and $\mathrm{rl}^{\mathfrak{A}}_W\, \phi^{\mathfrak{N}} = \phi^{\mathfrak{M}}$ for every formula ϕ*;
- (c) *$\mathrm{rl}^{\mathfrak{A}}_W\, \phi^{\mathfrak{N}} = \phi^{\mathfrak{M}}$ for every formula ϕ.*

Proof. Since (i) obviously follows from (ii), we restrict the proof to (ii). For (a)⇒(b), note that the defining property of elementary substructure can be expressed as saying that $\mathrm{rl}\, \phi^{\mathfrak{N}} = {}^\omega M \cap \phi^{\mathfrak{N}} = \phi^{\mathfrak{M}}$ for every formula ϕ. So by Theorem 3.1, (b) follows. (b)⇒(c) is trivial, and (c)⇒(a) has essentially been proved now too. ∎

The converse of Proposition 5.2(i) does not hold. In fact, let $\mathfrak{M} = (\mathbb{Q}, >)$ (the rationals under $>$), and let $\mathfrak{N} = (\mathbb{R}, <)$ (the reals under $<$). Clearly $\mathfrak{M}$ is *not* an elementary substructure of $\mathfrak{N}$ (it is not even an ordinary substructure), but $\mathrm{rl}^{\mathfrak{A}}_W$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$—this follows from our next theorem, which logically characterizes when $\mathrm{rl}^{\mathfrak{A}}_W$ is an isomorphism:

PROPOSITION 5.3. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are first-order structures, not necessarily similar. Let $\mathfrak{A} = \mathfrak{Cs}\,\mathfrak{N}$, $\mathfrak{B} = \mathfrak{Cs}\,\mathfrak{M}$, $V = {}^\omega N$, and $W = {}^\omega M$. Then the following conditions are equivalent*:

(i) *$\mathfrak{M}$ is definitionally equivalent to an elementary substructure of $\mathfrak{N}$.*

(ii) *rl_W is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$.*

Proof. (i)⇒(ii). This is clear from previous theorems.

(ii)⇒(i). We define a structure $\mathfrak{P}$ with universe M: if $\mathbf{R}$ is an m-ary fundamental relation of $\mathfrak{N}$, let $\mathbf{R}^{\mathfrak{P}} = {}^m M \cap \mathbf{R}^{\mathfrak{N}}$. We claim that

(∗) for any formula ϕ of the language of $\mathfrak{N}$, $\mathrm{rl}_W\, \phi^{\mathfrak{N}} = \phi^{\mathfrak{P}}$.

The proof is by induction on ϕ:

$$\begin{aligned}\mathrm{rl}_W(\mathbf{R}v_{i_0}\dots v_{i_{m-1}})^{\mathfrak{N}} &= W\cap(\mathbf{R}v_{i_0}\dots v_{i_{m-1}})^{\mathfrak{N}}\\ &= \{x\in {}^{\omega}M : \mathfrak{N}\models \mathbf{R}v_{i_0}\dots v_{i_{m-1}}[x]\}\\ &= \{x\in {}^{\omega}M : x\circ i\in \mathbf{R}^{\mathfrak{N}}\} = \{x\in {}^{\omega}M : x\circ i\in \mathbf{R}^{\mathfrak{P}}\}\\ &= (\mathbf{R}v_{i_0}\dots v_{i_{m-1}})^{\mathfrak{P}};\end{aligned}$$

$$\mathrm{rl}_W(\phi\vee\psi)^{\mathfrak{N}} = W\cap(\phi\vee\psi)^{\mathfrak{N}} = W\cap(\phi^{\mathfrak{N}}\cup\psi^{\mathfrak{N}}) = (\phi\vee\psi)^{\mathfrak{P}};$$

similarly for $\neg$;

$$\mathrm{rl}_W(\exists v_i\,\phi)^{\mathfrak{N}} = \mathrm{rl}_W\, C_i\phi^{\mathfrak{N}} = C_i\,\mathrm{rl}_W\,\phi^{\mathfrak{N}} = C_i\phi^{\mathfrak{P}} = (\exists v_i\,\phi)^{\mathfrak{P}}.$$

So, $(*)$ holds. It follows that rl_W is an isomorphism from $\mathfrak{Cs}\,\mathfrak{N}$ onto $\mathfrak{Cs}\,\mathfrak{P}$. Thus $\mathfrak{Cs}\,\mathfrak{P} = \mathfrak{Cs}\,\mathfrak{M}$, and so the desired conclusion follows from previous theorems. ∎

One more question in this little circle of ideas is to discuss the logical meaning of $\mathrm{rl}^{\mathfrak{A}}_W$ merely being a homomorphism, not necessarily an isomorphism. Well, every non-trivial homomorphism defined on an algebra $\mathfrak{Cs}\,\mathfrak{M}$ is an isomorphism, since as we will see in a future section (or the reader can easily verify for herself now), every algebra $\mathfrak{Cs}\,\mathfrak{M}$ is simple.

Next, we want to give an algebraic version of the downward Löwenheim–Skolem–Tarski theorem. To this end we introduce some more terminology. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Crs_I's with unit elements V and W respectively, where $W\subseteq V$. If $\mathrm{rl}^{\mathfrak{A}}_W$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, then we say that $\mathfrak{A}$ is *ext-isomorphic* to $\mathfrak{B}$, and $\mathfrak{B}$ is *sub-isomorphic* to $\mathfrak{A}$; $\mathrm{rl}^{\mathfrak{A}}_W$ is an *ext-isomorphism*, and $(\mathrm{rl}^{\mathfrak{A}}_W)^{-1}$ is a *sub-isomorphism*.

THEOREM 5.4. *Let $\mathfrak{A}$ be a Crs_I with unit element V and base U. Let κ be an infinite cardinal such that $|A|\le\kappa\le|U|$. Assume that $S\subseteq U$ and $|S|\le\kappa$. Finally, assume that $\kappa^{|I|}=\kappa$. Then there is a W such that $S\subseteq W\subseteq U$, $|W|=\kappa$, and $\mathfrak{A}$ is ext-isomorphic to a Crs_I with unit element $V\cap {}^IW$.*

Proof. Let a well-ordering of U be given. Now we define by induction sets T_α for all $\alpha<\kappa$. Let T_0 be a subset of U such that $|T_0|=\kappa$, $S\subseteq T_0$, and $X\cap {}^IT_0\ne 0$ for all $X\in A$; clearly such a set exists. (Note that $|I|<\kappa$ since $\kappa^{|I|}=\kappa$.) Suppose that $0<\beta<\kappa$ and T_α has been defined for all $\alpha<\beta$. Let $M=\bigcup_{\alpha<\beta}T_\alpha$, and let

$$T_\beta = M\cup\{a\in U : \exists X\in A\,\exists i\in I\,\exists x\in {}^IM\cap V\,[a \text{ is the first element of } U \text{ such that } x^i_a\in X]\}.$$

Let $W=T_\kappa=\bigcup_{\alpha<\kappa}T_\alpha$. Set $Z=V\cap {}^IW$. It is clear by induction that $|T_\alpha|=\kappa$ for all $\alpha\le\kappa$; here again the assumption $\kappa^{|I|}=\kappa$ comes in. By the definition of T_0 it is clear that $\mathrm{rl}^{\mathfrak{A}}_Z$ is one-one. To prove that rl preserves C_i, by the comment before Proposition 5.1 it suffices to take any $X\in A$ and $x\in C^{[V]}_iX\cap Z$ and show that $x\in C^{[Z]}_i(X\cap Z)$. Thus $x\in {}^IW$. The assumption $\kappa^{|I|}=\kappa$ implies that $|I| < \operatorname{cf}\kappa$, and hence there is some $\beta < \kappa$ such that $x \in {}^I T_\beta$. From the construction it follows that there is an $a \in T_{\beta+1}$ such that $x^i_a \in X$. Thus $x^i_a \in Z$, and hence $x \in C_i^{[Z]}(X \cap Z)$, as desired. ■

This theorem has been considerably generalized in the literature, and we shall give one or two of these generalizations later; see [6], pp. 47ff, and [9].

6. Change of base. The procedure of relativization in general changes the base of a Crs_I, going from a base to a subset. Now we want to consider another way of changing the base, to an entirely new set. Let f be a one-one function from U into W, and let $\mathfrak{A}$ be a Crs_I with base U and unit V. We define a function $\widetilde{f}$ on A as follows: for any $a \in A$,

$$\widetilde{f}a = \{x \in {}^I W : f^{-1} \circ x \in a\}.$$

The operation $\widetilde{\ }$ is actually a general set-theoretic operation. It would perhaps be more natural to define it, for *any* function f, by

$$\widetilde{f}a = \{x : f \circ x \in a\},$$

but we take the above definition to be consistent with the basic references mentioned in the introduction.

PROPOSITION 6.1. *Let* $\mathfrak{A}$ *be a* Crs_I *with base* U, *and let* f *be a one-one function mapping* U *onto* W. *Then* $\widetilde{f}$ *is an isomorphism from* $\mathfrak{A}$ *onto a* Crs_I *with base* W.

Proof. From the form of the definition it is a straightforward matter to check that $\widetilde{f}$ preserves the Boolean operations and the D_{ij}'s. To prove that $\widetilde{f}C_i^{[V]}a \subseteq C_i^{[\tilde{f}V]}\widetilde{f}a$, suppose that $x \in \widetilde{f}C_i^{[V]}a$. Thus $x \in {}^I W$ and $f^{-1} \circ x \in C_i^{[V]}a$. So there is a u such that $(f^{-1} \circ x)^i_u \in a$. But $(f^{-1} \circ x)^i_u = f^{-1} \circ x^i_{fu}$, so $f^{-1} \circ x^i_{fu} \in a$ and hence $x^i_{fu} \in \widetilde{f}a$ and $x \in C_i^{[\tilde{f}V]}\widetilde{f}a$, as desired.

To prove that $C_i^{[\tilde{f}V]}\widetilde{f}a \subseteq \widetilde{f}C_i^{[V]}a$, suppose that $x \in C_i^{[\tilde{f}V]}\widetilde{f}a$. So $x \in \widetilde{f}V$ and $x^i_w \in \widetilde{f}a$ for some w. Therefore $x \in {}^I W$, $w \in W$, $f^{-1} \circ x \in V$, and $f^{-1} \circ x^i_w \in a$. Let $fu = w$. Then $(f^{-1} \circ x)^i_u = f^{-1} \circ x^i_w$, so $f^{-1} \circ x \in C_i^{[V]}a$ and $x \in \widetilde{f}C_i^{[V]}a$, as desired. ■

If $\mathfrak{A}$ is a Crs_I with base U, $\mathfrak{B}$ is a Crs_I with base W, and g is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, we call g a *base isomorphism* from $\mathfrak{A}$ onto $\mathfrak{B}$ if there is a one-one function f from U onto W such that $g = \widetilde{f}$.

Base isomorphisms in algebras roughly correspond to isomorphisms of structures; this is expressed in the following two results:

PROPOSITION 6.2. *Let* $\mathfrak{M}$ *and* $\mathfrak{N}$ *be similar structures*, *and let* f *be a one-one function from* M *onto* N. *Then the following conditions are equivalent*:

(i) f *is an isomorphism from* $\mathfrak{M}$ *onto* $\mathfrak{N}$.

(ii) $\widetilde{f}$ *is a base isomorphism from* $\mathfrak{Cs}\,\mathfrak{M}$ *onto* $\mathfrak{Cs}\,\mathfrak{N}$, *and* $\widetilde{f}\phi^{\mathfrak{M}} = \phi^{\mathfrak{N}}$ *for every formula* ϕ.

Proof. (i)⇒(ii). For any formula ϕ, $\mathfrak{M} \models \phi[x]$ iff $\mathfrak{N} \models \phi[f \circ x]$; this elementary logical fact clearly implies that $\widetilde{f}\phi^{\mathfrak{M}} = \phi^{\mathfrak{N}}$ for every formula ϕ. Then Theorem 3.1 says that also $\widetilde{f}$ is a base isomorphism from $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{N}$.

(ii)⇒(i). Easy. ■

PROPOSITION 6.3. *Suppose that* $\mathfrak{M}$ *and* $\mathfrak{N}$ *are first-order structures*, *not necessarily similar. Let* $\mathfrak{A} = \mathfrak{Cs}\,\mathfrak{M}$, $\mathfrak{B} = \mathfrak{Cs}\,\mathfrak{N}$. *Suppose that* f *is a one-one function mapping* M *onto* N. *Then the following conditions are equivalent*:

(i) f *is an isomorphism from* $\mathfrak{M}$ *onto a structure* $\mathfrak{P}$ *definitionally equivalent to* $\mathfrak{N}$.

(ii) $\widetilde{f}$ *is a base isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

Proof. (i)⇒(ii). By Proposition 6.1, $\widetilde{f}$ is an isomorphism from $\mathfrak{A}$ onto *some* Crs_I. Proposition 6.2 says that $\widetilde{f}\phi^{\mathfrak{M}} = \phi^{\mathfrak{P}}$ for every formula ϕ. Thus $\widetilde{f}$ maps onto $\mathfrak{Cs}\,\mathfrak{P}$, which is the same as $\mathfrak{B}$, as desired.

(ii)⇒(i). There is a unique way of defining a structure $\mathfrak{P}$ such that f is an isomorphism from $\mathfrak{M}$ onto $\mathfrak{P}$. Then Proposition 6.2 yields that $\widetilde{f}$ is a base isomorphism from $\mathfrak{A}$ onto $\mathfrak{Cs}\,\mathfrak{P}$. The desired result follows. ■

An algebraic version of elementary embeddings is captured in the following definition. Let $\mathfrak{A}$ be a Crs_I with unit V and base U, and let $\mathfrak{B}$ be a Crs_I with unit X and base W. An isomorphism f of $\mathfrak{A}$ onto $\mathfrak{B}$ is a *sub-base-isomorphism* provided there exist a base isomorphism h and a sub-isomorphism g such that $f = g \circ h$. The following equivalent version of this notion is sometimes useful.

PROPOSITION 6.4. *Let* $\mathfrak{A}$ *be a* Crs_I *with unit* V *and base* U, *and let* $\mathfrak{B}$ *be a* Crs_I *with unit* X *and base* W. *Let* f *be an isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$. *Then the following conditions are equivalent*:

(i) f *is a sub-base-isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$.

(ii) *There exist a base isomorphism* h' *and an ext-isomorphism* g' *such that* $f^{-1} = g' \circ h'$.

Proof. (i)⇒(ii). Let l be a one-one function from U onto some set S such that $f = (\mathrm{rl}_Z^{\mathfrak{B}})^{-1} \circ \widetilde{l}$, where $Z = \widetilde{l}V$; this is possible by the assumption (i). Say that $\widetilde{l}$ is a base isomorphism from $\mathfrak{A}$ onto $\mathfrak{D}$. Then purely set-theoretically it is possible to find a one-one function k with domain W and range some set $T \supseteq U$ such that $l^{-1} \subseteq k$. So $\widetilde{k}$ is a base isomorphism from $\mathfrak{B}$ onto some Crs_I $\mathfrak{C}$ with unit $Y \overset{\mathrm{def}}{=} \widetilde{k}V$ and base T. In pictures:

$$\begin{array}{ccc}
\mathfrak{C}, Y, T & \xleftarrow{\ \widetilde{k}\ } & \mathfrak{B}, X, W \\
\big\downarrow \mathrm{rl}_V^{\mathfrak{C}} & & \big\downarrow \mathrm{rl}_Z^{\mathfrak{B}} \\
\mathfrak{A}, V, U & \xrightarrow{\ \widetilde{l}\ } & \mathfrak{D}, Z, S
\end{array}$$

We claim that $\widetilde{l}\circ \mathrm{rl}^{\mathfrak{C}}_V \circ\widetilde{k} = \mathrm{rl}^{\mathfrak{B}}_Z$; this will establish (ii). To prove this claim, take any $b\in B$. Then

$$\begin{aligned}(\widetilde{l}\circ \mathrm{rl}^{\mathfrak{C}}_V \circ\widetilde{k})b &= \widetilde{l}\,\mathrm{rl}^{\mathfrak{C}}_V\{x\in {}^IT : k^{-1}\circ x\in b\}\\ &= \widetilde{l}\{x : x\in V,\ x\in {}^IT,\ k^{-1}\circ x\in b\}\\ &= \{z\in {}^IS : l^{-1}\circ z\in V,\ l^{-1}\circ z\in {}^IT,\ k^{-1}\circ l^{-1}\circ z\in b\}\\ &= b\cap Z,\end{aligned}$$

as desired.

(ii)⇒(i). Let k be a one-one function from W onto some set T such that $h'=\widetilde{k}$; say that h' is a base isomorphism from $\mathfrak{B}$ onto a Crs_I $\mathfrak{C}$ with base T and unit Y. Thus $g' = \mathrm{rl}^{\mathfrak{C}}_V$. Let $l = k^{-1}\restriction U$, and let S be the range of l. Then $\widetilde{l}$ is an isomorphism from $\mathfrak{A}$ onto some Crs_I $\mathfrak{D}$ with some unit Z and with base S. So we have the same picture as before. By steps similar to the above one can verify that $\mathrm{rl}^{\mathfrak{B}}_Z = \widetilde{l}\circ \mathrm{rl}^{\mathfrak{C}}_V\circ\widetilde{k}$, and this yields (i). ∎

The actual algebraic equivalence of elementary embeddings is given in the following result.

Proposition 6.5. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be (not necessarily similar) structures, and let f be a one-one function from M into N. Then the following conditions are equivalent*:

(i) *f is an elementary embedding of $\mathfrak{M}$ into a structure $\mathfrak{P}$ which is definitionally equivalent to $\mathfrak{N}$.*

(ii) *There is a sub-isomorphism g such that $g\circ\widetilde{f}$ is a sub-base-isomorphism of $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{N}$.*

Proof. (i)⇒(ii). Let $\mathfrak{Q}$ be a structure similar to $\mathfrak{M}$ (and $\mathfrak{P}$) such that f is an isomorphism from $\mathfrak{M}$ onto $\mathfrak{Q}$ and $\mathfrak{Q}$ is an elementary substructure of $\mathfrak{P}$. By Proposition 6.3, $\widetilde{f}$ is a base isomorphism from $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{Q}$. By Proposition 5.3, rl_W is an isomorphism of $\mathfrak{Cs}\,\mathfrak{P}$ onto $\mathfrak{Cs}\,\mathfrak{Q}$, where $W = {}^IQ$. By Theorem 3.2, $\mathfrak{Cs}\,\mathfrak{P} = \mathfrak{Cs}\,\mathfrak{N}$. So $\mathrm{rl}^{-1}\circ f$ is a sub-base-isomorphism of $\mathfrak{Cs}\,\mathfrak{M}$ onto $\mathfrak{Cs}\,\mathfrak{N}$.

(ii)⇒(i). Similar. ∎

7. Subalgebras. For the general notion of a Crs we have nothing to say about subalgebras except the following connection with logic. There are interesting results and questions concerning subalgebras in special classes of Crs's.

Theorem 7.1. *For any $\mathcal{L}$-structure $\mathfrak{M}$ and any Crs_ω $\mathfrak{A}$ the following conditions are equivalent*:

(i) $\mathfrak{A}\subseteq\mathfrak{Cs}\,\mathfrak{M}$.

(ii) *There exist a structure $\mathfrak{N}$ definitionally equivalent to $\mathfrak{M}$, say $\mathfrak{N}$ an $\mathcal{L}'$-structure, and a sublanguage $\mathcal{L}''$ of $\mathcal{L}'$ such that $\mathfrak{A} = \mathfrak{Cs}(\mathfrak{N}\restriction\mathcal{L}'')$.* ($\mathfrak{N}\restriction\mathcal{L}''$ is the reduct of $\mathfrak{N}$ to the language $\mathcal{L}''$.)

Proof. (i)⇒(ii). For each $x \in \mathfrak{Cs}\,\mathfrak{M}$ and each positive integer m such that $\Delta x \subseteq m$ we introduce an m-ary relation symbol $\mathbf{R}_{xm}$ in a language $\mathcal{L}'$; and we also choose ϕ_x with $\phi_x^{\mathfrak{M}} = x$ with free variables among $\{v_i : i \in \Delta x\}$. Define $N = M$ and

$$\mathbf{R}_{xm}^{\mathfrak{N}} = \{u \in {}^mN : \mathfrak{M} \vDash \phi_x[u]\}.$$

Let $\mathcal{L}''$ be the sublanguage of $\mathcal{L}'$ consisting of all of the relation symbols $\mathbf{R}_{xm}$ for $x \in A$. To check (ii) we first show that $\mathfrak{M}$ and $\mathfrak{N}$ are definitionally equivalent. Obviously every fundamental relation of $\mathfrak{N}$ is definable in $\mathfrak{M}$. Now take a fundamental relation $\mathbf{R}^{\mathfrak{M}}$ of $\mathfrak{M}$; say $\mathbf{R}$ is an m-ary relation symbol of the language of $\mathfrak{M}$. Let $x = (\mathbf{R}v_0 \ldots v_{m-1})^{\mathfrak{M}}$. Note that $\mathfrak{M} \vDash \phi_x \leftrightarrow \mathbf{R}v_0 \ldots v_{m-1}$. Hence

$$\{u \in {}^mM : \mathfrak{N} \vDash \mathbf{R}_{xm}v_0 \ldots v_{m-1}[u]\} = \mathbf{R}_{xm}^{\mathfrak{N}} = \{u \in {}^mN : \mathfrak{M} \vDash \phi_x[u]\} = \mathbf{R}^{\mathfrak{M}},$$

as desired. This proves that $\mathfrak{N}$ is definitionally equivalent to $\mathfrak{M}$.

Now we show that $\mathfrak{A} = \mathfrak{Cs}(\mathfrak{N} \restriction \mathcal{L}'')$. To do this, it suffices to show that if $x \in A$ and $\Delta x \subseteq m$, then $x = (\mathbf{R}_{xm}v_0 \ldots v_{m-1})^{\mathfrak{N}}$, since this shows that $\mathfrak{Cs}(\mathfrak{N} \restriction \mathcal{L}'')$ has A as a set of generators and hence must coincide with $\mathfrak{A}$. We have

$$\begin{aligned}(\mathbf{R}_{xm}v_0 \ldots v_{m-1})^{\mathfrak{N}} &= \{u \in {}^\omega N : u \restriction m \in \mathbf{R}_{xm}^{\mathfrak{N}}\} \\ &= \{u \in {}^\omega M : \mathfrak{M} \vDash \phi_x[u]\} = \phi_x^{\mathfrak{M}} = x.\end{aligned}$$

As to (ii)⇒(i), take any $a \in A$ and by (ii) write $a = \phi^{\mathfrak{N} \restriction L''}$ for some formula ϕ of $\mathcal{L}''$. Then

$$a = \phi^{\mathfrak{N} \restriction L''} = \phi^{\mathfrak{N}} \in Cs\,\mathfrak{N} = Cs\,\mathfrak{M}. \ \blacksquare$$

8. Homomorphisms. The basic result about homomorphisms is that a homomorphic image of a Crs is isomorphic to a Crs. The proof that we give for this (due to Andréka and Németi) depends on ultraproducts, and so it will be postponed to Section 10. Closure under homomorphic images is the difficult thing in proving that the class of isomorphs of Crs's is equational. There is another, involved, proof due to Resek and Thompson, based on an axiom system for Crs_I's, and a simple proof that this axiom system works is due to Andréka and Thompson independently; this simple proof has not been published, but is sketched in Resek, Thompson [8]. See also Section 9.

Concerning connections with logic, the basic result is that $\mathfrak{Cs}\,\mathfrak{M}$ is always simple, in the general algebraic sense. We prove this now, assuming only a basic knowledge of universal algebra.

THEOREM 8.1. *For any $\mathcal{L}$-structure $\mathfrak{M}$, the Crs_ω $\mathfrak{Cs}\,\mathfrak{M}$ is simple.*

Proof. Suppose that E is a congruence relation on $\mathfrak{Cs}\,\mathfrak{M}$ and $\phi^{\mathfrak{M}}$ and $\psi^{\mathfrak{M}}$ are distinct elements such that $\phi^{\mathfrak{M}} E \psi^{\mathfrak{M}}$; we want to show that $E = Cs\,\mathfrak{M} \times Cs\,\mathfrak{M}$. Say $\phi^{\mathfrak{M}} \not\subseteq \psi^{\mathfrak{M}}$. Let $\chi = \exists v_0 \ldots v_{m-1}(\phi \wedge \neg\psi)$, where m is such that all of the free variables of $\phi \wedge \neg\psi$ are among $v_0, \ldots, v_{m-1}$. Thus $\mathfrak{M} \vDash \chi$, and hence $\chi^{\mathfrak{M}} = {}^\omega M$.

Therefore

$$\begin{aligned}
&\phi^{\mathfrak{M}} E \psi^{\mathfrak{M}};\\
&\phi^{\mathfrak{M}} \cdot -\psi^{\mathfrak{M}} E 0;\\
&(\phi \wedge \neg\psi)^{\mathfrak{M}} E 0;\\
&C_0 \ldots C_{m-1}[(\phi \wedge \neg\psi)^{\mathfrak{M}}] E C_0 \ldots C_{m-1} 0 = 0;\\
&[\exists v_0 \ldots v_{m-1}(\phi \wedge \neg\psi)]^{\mathfrak{M}} E 0;\\
&{}^{\omega}M E 0;
\end{aligned}$$

hence for any $x, y \in Cs\,\mathfrak{M}$ we have $x = (x \cdot {}^{\omega}M) E (x \cdot 0) = 0$, and similarly $yE0$, so xEy, as desired. ∎

9. Products. The basic fact here is that a product of Crs's is isomorphic to a Crs:

THEOREM 9.1. *For $|K| > 1$, any product of Crs_K's is isomorphic to a Crs_K.*

Proof. Let $\langle \mathfrak{A}_i : i \in I \rangle$ be a system of Crs_K's. Say V_i is the unit element of $\mathfrak{A}_i$ for each $i \in I$. Without loss of generality, the bases of $\mathfrak{A}_i$ and $\mathfrak{A}_j$ are disjoint for distinct $i, j \in I$. Let $W = \bigcup_{i \in I} V_i$. Now we define $f : \prod_{i \in I} A_i \to \mathcal{P}(W)$ by setting $fx = \bigcup_{i \in I} x_i$ for any $x \in \prod_{i \in I} A_i$. Thus f maps into the Crs_K of all subsets of W. Clearly f preserves $+$, $-$, and d_{kl} for $k, l \in K$. Moreover, $x \neq 0 \Rightarrow fx \neq 0$, so f is one-one. Finally, f preserves c_k for each $k \in K$:

$$\begin{aligned}
a \in f c_k x \quad &\text{iff} \quad \exists i \in I\,(a \in C_k^{[V_i]} x_i)\\
&\text{iff} \quad \exists i \in I\,(a \in V_i \text{ and } \exists u\,(a_u^k \in x_i))\\
&\text{iff} \quad \exists u\, \exists i \in I\,(a \in V_i \text{ and } a_u^k \in x_i)\\
&\text{iff} \quad a \in W \text{ and } \exists u\,(a_u^k \in fx)\\
&\text{iff} \quad a \in C_k^W fx,
\end{aligned}$$

as desired. Note that the next to the last equivalence uses the fact that $|K| > 1$ and that the bases are disjoint. ∎

Theorem 9.1 does not extend to the case $|K| \leq 1$; but we shall not go into this. For the rest of the present remarks assume that $|K| > 1$. According to Theorem 9.1 and preceding sections, the class $\mathbf{K}$ of isomorphs of Crs_K's is closed under subalgebras, homomorphisms, and products. Hence by the well-known theorem of Birkhoff, $\mathbf{K}$ is a variety, i.e., it is characterized by a set of equations. One of the major results in the theory of cylindric algebras is that $\mathbf{K}$ is not finitely axiomatizable if K has at least 3 elements; this is a result of Andréka and Németi. For K infinite the result is somewhat trivial, but there is a stronger, non-trivial result: $\mathbf{K}$ is not definable by a finite schema. We shall prove the first result here, but in order not to digress too much we omit the definition of "finite schema" and the proof of the second result.

LEMMA 9.2. *The following inequality holds in every* Crs_K, *for any* $m \in \omega$ *and any distinct* $j, k, l \in K$:

$$(c_j c_k)^{m+1}(d_{kl} \cdot x) \cdot d_{kl} \leq c_j x.$$

Proof. Let $\mathfrak{A}$ be a Crs_K. Suppose that a is in the left side of the indicated inequality. Then there exist $u_0, u_1, \ldots, u_{2m+1}$ such that

$$b \stackrel{\text{def}}{=} ((\ldots((a^j_{u_0})^k_{u_1})\ldots)^j_{u_{2m}})^k_{u_{2m+1}} \in D_{kl} \cap x.$$

Hence it suffices to show that $a^j_{u_{2m}} = b$. Since these two functions clearly agree except possibly at k, we just check k: $(a^j_{u_{2m}})_k = a_k = a_l = b_l = b_k$, as desired. ■

THEOREM 9.3. *If* K *has at least* 3 *elements*, *then the class* $\mathbf{L}$ *of isomorphs of* Crs_K*'s is not finitely axiomatizable. Specifically*, *there is a system* $\langle \mathfrak{A}_m : m \in \omega \rangle$ *of algebras similar to* Crs_K*'s such that no* $\mathfrak{A}_m$ *is isomorphic to a* Crs_K, *while* $\prod_{m\in\omega} \mathfrak{A}_m/F$ *is isomorphic to a* Crs_K *for every non-principal ultrafilter* F *on* ω.

Proof. For notational convenience we assume that K is an ordinal α. Let $a = \langle 1, 0, 0, \ldots \rangle$ (a sequence of length α), $c = \{a\}$, and for each $m \in \omega$ let $b^m = \langle 2m+1, 2m+2, 0, 0, \ldots \rangle$ (a sequence of length α), $d^m = \{b^m\}$. Now for each $m \in \omega$ we set

$$V_m = \{f \in {}^\alpha\omega : \text{for some } n \leq m \text{ we have } f0 = 2n+1,$$
$$f1 \in \{2n, 2n+2\}, \text{ and } f\kappa = 0 \text{ for all } \kappa \in \alpha\backslash\{0,1\}\},$$

$$d^m_{\kappa\kappa} = V_m,$$

$$d_{0\kappa} = d_{\kappa 0} = 0 \text{ if } 0 < \kappa < \alpha,$$

$$d_{1\kappa} = d_{\kappa 1} = \{a, b^m\} \text{ if } 1 < \kappa < \alpha,$$

$$d_{\kappa\lambda} = V_m \text{ if } \kappa, \lambda \in \alpha\backslash\{0,1\},$$

$$\mathfrak{A}_m = \langle \mathcal{P}(V_m), \cup, \cap, \backslash, 0, V_m, C^{[V_m]}_\kappa, d^m_{\kappa\lambda} \rangle_{\kappa,\lambda\in\alpha}$$

($\backslash$ is complementation relative to V_m). First we apply Lemma 9.2 to see that no algebra $\mathfrak{A}_m$ is isomorphic to a Crs_K. We claim that b^m is in the left side of the inequality of Lemma 9.2 but not in the right, for $j = 0$, $k = 1$, $l = 2$, $x = \{a\}$. In fact,

$$((\ldots(((b^m)^0_{2m+1})^1_{2m})^0_{2m-1}\ldots)^0_1)^1_0 = a,$$

and by construction all of the elements

$$(b^m)^0_{2m+1},\ ((b^m)^0_{2m+1})^1_{2m},\ (((b^m)^0_{2m+1})^1_{2m})^0_{2m-1},\ \ldots$$

are in V_m. Hence b^m is clearly in the left side, and it also clearly fails to be in the right side.

Now let F be any non-principal ultrafilter on ω. Set $\mathfrak{B} = \prod_{m\in\omega} \mathfrak{A}_m/F$. The rest of the proof is devoted to showing that $\mathfrak{B}$ is isomorphic to a Crs_K.

To prove this, we first develop some notation for the algebras $\mathfrak{A}_m$. Each such algebra is an atomic Boolean algebra with additional operations. If u is an atom of $\mathfrak{A}_m$, then there is a unique $n \le m$ such that u has the form $\{\langle 2n+1, \ldots\rangle\}$; we denote this n by $\operatorname{int} u$. In case $u \in A_m$ is not an atom we let $\operatorname{int} u = 0$.

Another ultraproduct will play an important role in the rest of the proof. Let $\mathfrak{C} = \prod_{m\in\omega}(m+1, <)/F$. Let $\overline{0} = \langle 0 : m \in \omega\rangle/F$ and $\overline{\infty} = \langle m : m \in \omega\rangle/F$. Thus $\mathfrak{C}$ is a linearly ordered structure with least element $\overline{0}$ and greatest element $\overline{\infty}$. Moreover, every element except $\overline{\infty}$ has an immediate successor, and every element except $\overline{0}$ has an immediate predecessor. Therefore the order type of $\mathfrak{C}$ consists of ω followed by 2^ω copies of $\mathbb{Z}$ in some order not of interest in this proof, followed by ω^*. (It is well known that C has power 2^ω.) For any element x of C and any $n \in \mathbb{Z}$ we denote by $x + n$ the nth successor of x (meaning $(-n)$th predecessor if $n < 0$), if it exists (which is only problematical for the initial ω and terminal ω^*). Two elements u, v of C are said to be *equivalent* if u is the nth successor of v or v is the nth successor of u for some $n \in \omega$.

If x/F is an atom of $\mathfrak{B}$, then we say that x/F is of

- *type* 1 if $\{m \in \omega : \exists n\,(x_m = \{\langle 2n+1, 2n, 0, 0, \ldots\rangle\})\} \in F$;
- *type* 2 if $\{m \in \omega : \exists n\,(x_m = \{\langle 2n+1, 2n+2, 0, 0, \ldots\rangle\})\} \in F$.

Note that every atom is either of type 1 or of type 2. For any atom x/F of $\mathfrak{B}$ we set $\operatorname{int}(x/F) = \langle \operatorname{int} x_m : m \in \omega\rangle/F$; clearly this is a well-defined function from the set of atoms of $\mathfrak{B}$ into C. Then we call atoms u, v of B *equivalent* if $\operatorname{int} u$ and $\operatorname{int} v$ are equivalent.

(1) For any $i \in \{1, 2\}$ and any $n \in \prod_{m\in\omega}(m+1)$ there is at most one atom u of $\mathfrak{B}$ of type i such that $\operatorname{int} u = n/F$.

In fact, suppose that x/F and y/F are atoms of $\mathfrak{B}$ of the same type, and $\operatorname{int}(x/F) = \operatorname{int}(y/F) = n/F$. By symmetry we assume that the type is 1. Then each of the following sets is in F, and hence so is their intersection, which we call X:

$$\begin{aligned}
&\{m \in \omega : \exists n\,(x_m = \{\langle 2n+1, 2n, 0, 0, \ldots\rangle\})\},\\
&\{m \in \omega : \exists n\,(y_m = \{\langle 2n+1, 2n, 0, 0, \ldots\rangle\})\},\\
&\{m \in \omega : \operatorname{int} x_m = n_m\},\\
&\{m \in \omega : \operatorname{int} y_m = n_m\}.
\end{aligned}$$

Then it is clear that for any $m \in X$ we have $x_m = y_m$, as desired.

Next, let $c' = \langle c : m \in \omega\rangle$ and $d' = \langle d^m : m \in \omega\rangle$. The following rules for calculation of cylindrifications will be useful; the rules are clear on the basis of (1):

(2) $c_0(c'/F) = c'/F$ and $c_0(d'/F) = d'/F$.

(3) If x/F is an atom of $\mathfrak{B}$, then $c_1(x/F) = x/F + y/F$, where y/F is the other atom such that $\operatorname{int}(y/F) = \operatorname{int}(x/F)$.

(4) If x/F is an atom of type 1 and $x/F \ne c'/F$, then $c_0(x/F) = x/F + y/F$, where y/F is the unique atom of type 2 such that $\operatorname{int}(y/F)$ is the immediate predecessor of $\operatorname{int}(x/F)$.

(5) If x/F is an atom of type 2 and $x/F \neq d'/F$, then $c_0(x/F) = x/F + y/F$, where y/F is the unique atom of type 1 such that $\mathrm{int}(y/F)$ is the immediate successor of $\mathrm{int}(x/F)$.

(6) If $\kappa > 1$, then $c_\kappa u = u$ for any $u \in B$.

Next we define a function G mapping the set of atoms of $\mathfrak{B}$ into ${}^\alpha C$ by defining its restriction to each equivalence class k under the above equivalence relation.

Case 1: $c'/F \in k$. Let x/F be any member of k. Then $\mathrm{int}(x/F)$ is, for some $n \in \omega$, the nth successor of $\overline{0}$ in $\mathfrak{C}$. Then we set

$$G(x/F) = \begin{cases} \langle \overline{0} + 2n + 1, \overline{0} + 2n, \overline{0}, \overline{0}, \ldots \rangle & \text{if } x/F \text{ is of type 1,} \\ \langle \overline{0} + 2n + 1, \overline{0} + 2n + 2, \overline{0}, \overline{0}, \ldots \rangle & \text{otherwise.} \end{cases}$$

Case 2: $d'/F \in k$. Let y/F be any member of k. Then $\mathrm{int}(y/F)$ is, for some $n \in \omega$, the nth predecessor of $\overline{\infty}$ in $\mathfrak{C}$. Then we set

$$G(y/F) = \begin{cases} \langle \overline{\infty} - (2n+1), \overline{\infty} - 2n, \overline{\infty}, \overline{\infty}, \ldots \rangle & \text{if } y/F \text{ is of type 2,} \\ \langle \overline{\infty} - (2n+1), \overline{\infty} - (2n+2), \overline{\infty}, \overline{\infty}, \ldots \rangle & \text{otherwise.} \end{cases}$$

Case 3: $c'/F, d'/F \notin k$. Fix an element s of the equivalence class of $\mathrm{int}(u/F)$, where u/F is any element of k. Now for any $z/F \in k$ write $\mathrm{int}(z/F) = s + n$ with $n \in \mathbb{Z}$ and define

$$G(z/F) = \begin{cases} \langle s + 2n + 1, s + 2n, \overline{0}, \overline{0}, \ldots \rangle & \text{if } z/F \text{ is of type 1,} \\ \langle s + 2n + 1, s + 2n + 2, \overline{0}, \overline{0}, \ldots \rangle & \text{otherwise.} \end{cases}$$

This finishes the definition of G. Note that G is one-one.

Finally, we define $H : B \to \mathcal{P}({}^\alpha C)$, which will turn out to be the desired isomorphism. For any $x \in B$, let

$$Hx = \{Gy : y \leq x \text{ and } y \text{ is an atom of } \mathfrak{B}\}.$$

We claim that H is an isomorphism from $\mathfrak{B}$ onto a Crs_α with unit element $Z \stackrel{\text{def}}{=} H1$. Clearly H is a Boolean isomorphism. Now we check the diagonals. In $\mathfrak{B}$ we have $d_{\kappa\kappa} = 1$ for any $\kappa < \alpha$, and $D_{\kappa\kappa} = 1$ in any Crs_α, so there is no problem with that. For $0 < \kappa < \alpha$ we have $d_{0\kappa} = 0$ in $\mathfrak{B}$. Now Z is simply the range of G, and clearly $(Gy)_0 \neq (Gy)_\kappa$ for all atoms y of $\mathfrak{B}$, so $D_{0\kappa} = 0$ also. For $1 < \kappa < \alpha$ we clearly have, in $\mathfrak{B}$, $d_{1\kappa} = \{c'/F, d'/F\}$. So, using the notation introduced in the definition of G,

$$Hd_{1\kappa} = \{\langle \overline{0} + 1, \overline{0}, \overline{0}, \ldots \rangle, \langle \overline{\infty} - 1, \overline{\infty}, \overline{\infty}, \ldots \rangle\}.$$

This is clearly equal to $D_{1\kappa}$. Finally, for $\kappa, \lambda > 1$ we have $d_{\kappa\lambda} = 1$ in $\mathfrak{B}$, and clearly also $D_{\kappa\lambda} = 1$, as desired.

Finally, we have to check the cylindrifications. First note

(7) For any $x, y \in B$ with x an atom, and any $\kappa < \alpha$, $x \leq c_\kappa y$ iff there is an atom $u \leq y$ such that $x \leq c_\kappa u$.

We omit the proof, which is straightforward.

To check preservation of cylindrifications, note that $c_\kappa x = x$ for all x if $\kappa > 1$, in both algebras considered, so it is only necessary to check c_0 and c_1. Here there are many little cases to be considered. To illustrate the ideas, we take one typical case and leave the rest to the reader. Suppose that $Gz \in Hc_0 y$; we want to show that $Gz \in C_0 Hy$. By (7) there is an atom u such that $u \le y$ and $z \le c_0 u$. Without loss of generality, $u \ne z$. We now consider one of two possibilities for the type of u: assume that u has type 1. Then by (4), z is of type 2 and $\operatorname{int} z$ is the immediate predecessor of $\operatorname{int} u$. Now we consider one of three possibilities for the equivalence class of u: assume that u is equivalent to c'/F. Let $n = \operatorname{int} u$. Then the definition of G gives

$$Gu = \langle \overline{0} + 2n + 1, \overline{0} + 2n, \overline{0}, \overline{0}, \ldots \rangle, \qquad Gz = \langle \overline{0} + 2n - 1, \overline{0} + 2n, \overline{0}, \overline{0}, \ldots \rangle,$$

so $Gz \in C_0 Hy$, as desired. ∎

Although Theorem 9.3 discourages the idea of abstractly characterizing the class of isomorphs of Crs_I's, it turns out that it *is possible* to give a rather simple description of an infinite set of equations which characterizes this class. This description is due to Resek and Thompson. We need some simple notation in order to conveniently formulate their description. Let $s^i_j x = c_i(d_{ij} \cdot x)$ if $i \ne j$ and $s^i_i x = x$. We use $[i/j]$ for the function with domain I which sends i to j and fixes all other elements of I (here I is to be understood from the context). For any function f, $f[K] = \{fx : x \in K\}$. Now for any set I let Σ_I be the following set of equations, where we use $u \le v$ to mean that $u \cdot v = u$:

(1) Equations characterizing Boolean algebras (for $+, \cdot, -, 0, 1$).
(2) $c_i 0 = 0$.
(3) $c_i(x + y) = c_i x + c_i y$.
(4) $x \le c_i x$.
(5) $c_i c_i x = c_i x$.
(6) $c_i(-c_i x) = -c_i x$.
(7) $d_{ii} = 1$.
(8) $d_{ij} = d_{ji}$.
(9) $d_{ij} \cdot d_{jk} \le d_{ik}$.
(10) $c_i(x \cdot d_{ij}) \cdot d_{ij} \le x$ if $i \ne j$.
(11) $s^{i_n}_{j_n} c_{k_n} \ldots s^{i_1}_{j_1} c_{k_1} x \cdot \prod_{l \in K} d_{l\tau(l)} \le c_i x$, where $K = \{i_1, \ldots, i_n, k_1, \ldots, k_n\} \setminus \{i\}$, $\tau = [i_n/j_n] \circ \ldots \circ [i_1/j_1]$ and $k_{m+1} \notin ([i_m/j_m] \circ \ldots \circ [i_1/j_1])[K]$ for all $m < n$.

The result of Resek and Thompson is then that Σ_I characterizes the isomorphs of members of Crs_I for every I with at least two elements. A simple proof of this result is due to Hajnal Andréka, and we will now give the essential part of her proof, which establishes the following theorem. For this theorem, for convenience we work with an ordinal rather than our general set I.

THEOREM 9.4. *Let α be an ordinal greater than* 1. *Then every* Crs_α *is a model of* Σ_α. *Moreover*, *every atomic model of* Σ_α *is isomorphic to a* Crs_α.

R e m a r k. From results in Section 2.7 of [4] it then follows easily that every model of Σ_α is isomorphic to a Crs_α, giving the indicated result.

P r o o f. First we prove that any Crs_α is a model of Σ_α. So, let $\mathfrak{A}$ be a Crs_α with unit V and base U. All of the parts of Σ_α except (11) are completely routine, and will be left to the reader. Now let f be in the left-hand side of (11). For $1 \le \gamma \le n$ let $\mathcal{F}^f_\gamma$ be the member of ${}^\alpha U$ defined by setting $(\mathcal{F}^f_\gamma)l = f[i_n/j_n]\dots[i_\gamma/j_\gamma]l$ for all $l \in \alpha$. Let $\phi(g,\gamma)$ be the statement that there exist $q \in \omega$, $x_1,\dots,x_q \in \{i, i_1,\dots,i_{\gamma-1}\}$, and $u_1,\dots,u_q \in U$ such that

$(a)_\gamma$ for all $u = 1,\dots,q$ and all $\varepsilon = 1,\dots,\gamma-1$, if $x_u = j_\varepsilon \ne i_\varepsilon$, then there is some δ with $\varepsilon < \delta < \gamma$ such that $x_u = i_\delta \ne j_\delta$;

$(\mathrm{b})_\gamma$ $g = (\mathcal{F}^f_\gamma)^{x_1\dots x_q}_{u_1\dots u_q}$.

Now we will define by downward induction functions $g_n,\dots,g_1$ and $h_n,\dots,h_1$ so that for each $\gamma = 1,\dots,n$ the following conditions hold:

(1) $\phi(g_\gamma,\gamma+1)$ and $g_\gamma \in s^{i_\gamma}_{j_\gamma} c_{k_\gamma} \dots s^{i_1}_{j_1} c_{k_1} x$;

(2) $\phi(h_\gamma,\gamma)$ and $h_\gamma \in c_{k_\gamma} s^{i_{\gamma-1}}_{j_{\gamma-1}} \dots s^{i_1}_{j_1} c_{k_1} x$.

To start with, we let $g_n = f$; condition (1) for $\gamma = n$ is clear. Now assume that g_γ has been defined; we define h_γ. Assume the notation of (1), $(\mathrm{a})_{\gamma+1}$, and $(\mathrm{b})_{\gamma+1}$. If $i_\gamma = j_\gamma$, let $h_\gamma = g_\gamma$; clearly (2) holds for γ. Now assume that $i_\gamma \ne j_\gamma$. Then we have

$$(g_\gamma)^{i_\gamma}_{(g_\gamma)j_\gamma} \in c_{k_\gamma} s^{i_{\gamma-1}}_{j_{\gamma-1}} \dots c_{k_1} x,$$

and we let $h_\gamma = (g_\gamma)^{i_\gamma}_{(g_\gamma)j_\gamma}$. To see that (2) holds for γ, first note that $j_\gamma \ne x_u$ for all $u = 1,\dots,q$. Hence $(g_\gamma)j_\gamma = \mathcal{F}^f_{\gamma+1} j_\gamma$. If any of the x_u's are equal to i_γ, delete them, forming thereby subsequences $\langle y_1,\dots,y_p\rangle$ of $\langle x_1,\dots,x_q\rangle$ and $\langle v_1,\dots,v_p\rangle$ of $\langle u_1,\dots,u_q\rangle$. Then it is clear that

$$h_\gamma = (\mathcal{F}^f_\gamma)^{y_1\dots y_p}_{v_1\dots v_p},$$

as desired.

Finally, suppose that h_γ has been defined, where $\gamma > 1$; we want to define $g_{\gamma-1}$. Assume the notation of (2), $(\mathrm{a})_\gamma$, and $(\mathrm{b})_\gamma$. There is a $v \in U$ such that

$$(h_\gamma)^{k_\gamma}_v \in s^{i_{\gamma-1}}_{j_{\gamma-1}} \dots c_{k_1} x,$$

and so we can let $g_{\gamma-1} = (h_\gamma)^{k_\gamma}_v$; thus

$$g_{\gamma-1} = (\mathcal{F}^f_\gamma)^{x_1\dots x_q k_\gamma}_{u_1\dots u_q v},$$

as desired.

So the construction is complete. Applying it to h_1, we see that $h_1 \in c_{\kappa_1} x$ and h_1 has the form

$$h_1 = (\mathcal{F}^f_1)^{x_1\dots x_q}_{u_1\dots u_q},$$

where $x_u = i$ for all u (but possibly $q = 0$). Note that $\mathcal{F}_1^f = f \circ \tau$. Hence from $h_1 \in c_i x$ and $f \in \prod_{l \in K} d_{l\tau(l)}$ we get $f \in c_i x$, as desired.

We now turn to the second part of the proof. Suppose that $\mathfrak{A}$ is an atomic model of Σ_α, and denote by At the set of all atoms of $\mathfrak{A}$. We shall define a function rep from At into $\mathcal{P}(V)$ (for some set V of functions with domain α) so that the following conditions will hold for all $a, b \in At$ and $i, j \in \alpha$:

(I) $rep(a) \cap rep(b) = 0$ if $a \neq b$.
(II) $rep(a) \neq 0$.
(III) $rep(a) \subseteq D_{ij}^{[V]}$ if $a \leq d_{ij}^{\mathfrak{A}}$, and $rep(a) \cap D_{ij}^{[V]} = 0$ if $a \not\leq d_{ij}^{\mathfrak{A}}$.
(IV) $rep(a) \subseteq C_i^{[V]} rep(b)$ if $a \leq c_i^{\mathfrak{A}} b$.
(V) $rep(a) \cap C_i^{[V]} rep(b) = 0$ if $a \not\leq c_i^{\mathfrak{A}} b$.
(VI) $\bigcup_{a \in At} rep(a) = V$.

If we manage to do this, then rep can be extended to all of A by defining, for any $x \in A$,

$$rep(x) = \bigcup_{a \in At,\ a \leq x} rep(a).$$

Then it is routine to check that rep is the desired isomorphism from $\mathfrak{A}$ onto a Crs_α with unit V.

For every α-sequence f let $ker(f) = \{(i,j) \in \alpha \times \alpha : f_i = f_j\}$, and for every $a \in At$ let $ker(a) = \{(i,j) \in \alpha \times \alpha : a \leq d_{ij}^{\mathfrak{A}}\}$. Both of these are equivalence relations on α, using for $ker(a)$ the axioms (7)–(9) from Σ_α. Now (III) is equivalent to

(III′) If $s \in rep(a)$ then $ker(s) = ker(a)$.

We also notice that (IV) is equivalent to

(IV′) If $s \in rep(a)$ and $a \leq c_i^{\mathfrak{A}} b$, then $s_u^i \in rep(b)$ for some u.

Now we shall construct the set V and the function rep step-by-step. Let $W = \{(a,b,i) : a, b \in At,\ a \leq c_i b,\ i \in \alpha\}$. We claim:

($\star$) There is an infinite cardinal κ and a function $\sigma : \kappa \to W \times \kappa$ such that for all $w \in W$ and $\lambda < \kappa$ there is a ν such that $\lambda < \nu < \kappa$ and $\sigma(\nu) = (w, \lambda)$.

To prove ($\star$), take κ to be any infinite cardinal at least as big as $|W|$, let g be any function from κ onto $W \times \kappa$ and let τ be a one-one function from κ onto $\kappa \times \kappa$. If $x \in \kappa \times \kappa$, we write $x = (x_0, x_1)$. Define $\sigma(\nu) = g(\tau(\nu))_0$. Clearly this works for ($\star$).

Now we really begin the construction. Let $rep_0(a) = 0$ for all $a \in At$, and also let $V_0 = 0$.

Assume that $\nu < \kappa$, and V_ν, $rep_\nu : At \to \mathcal{P}(V_\nu)$, and p_ξ for all $\xi < \nu$ have been defined. Write $\sigma(\nu) = (a, b, i, \lambda)$. First we define p_ν.

Case 1: $\lambda < \nu$ and $p_\lambda \in rep_\nu(a)$. If $b \leq d_{ij}^{\mathfrak{A}}$ for some $j \neq i$, choose the smallest such j and let $u = p_\lambda(j)$; if $b \not\leq d_{ij}^{\mathfrak{A}}$ for all $j \neq i$, then let u be a new object, not in the range of any of the functions p_ξ for $\xi < \nu$. Under either possibility define p_ν to be $(p_\lambda)^i_u$.

Case 2: $\lambda \geq \nu$ or $p_\lambda \notin rep_\nu(a)$. In this case let p_ν be a sequence with the same kernel as b and with range consisting of entirely new objects, not in the range of any of the functions p_ξ for $\xi < \nu$.

This defines p_ν. Then we define

$rep_{\nu+1}(b) = rep_\nu(b) \cup \{p_\nu\}$;
$rep_{\nu+1}(a') = rep_\nu(a')$ for any atom $a' \neq b$;
$V_{\nu+1} = V_\nu \cup \{p_\nu\}$.

That describes the step from ν to $\nu+1$. Now if $\nu \leq \kappa$ is a limit ordinal and rep_ξ has been defined for all $\xi < \nu$, we set

$rep_\nu(a) = \bigcup_{\xi<\nu} rep_\xi(a)$ for every atom a;
$V_\nu = \bigcup_{\xi<\nu} V_\xi$.

Finally, let $rep = rep_\kappa$ and $V = V_\kappa$.

Now we start checking the conditions (I)–(VI).

(VI) This is obvious from the definitions.

(III′) Suppose that $s \in rep(a)$. Then for some $\nu < \kappa$, s was constructed as p_ν in the passage from ν to $\nu+1$, with "a" in the role of "b". It is straightforward to check that $ker(p_\nu) = ker(a)$.

(II) Given an atom a, let ν be such that $\sigma(\nu) = (a, a, 0, 0)$. Then Case 2 in the definition applies, and we get $p_\nu \in rep(a)$.

(IV′) Suppose that $s \in rep(a)$ and $a \leq c_i b$, where a and b are atoms. By the construction, $s = p_\lambda$ for some $\lambda < \kappa$. Choose $\nu < \kappa$ with $\lambda < \nu$ such that $s(\nu) = (a, b, i, \lambda)$. Then by construction, $p_\nu \in rep(b)$ and p_ν has the form $(p_\lambda)^i_u$ for some u, as desired.

That takes care of the easy ones—the ones that really were forced to be true by the construction. It remains to show that (I) and (V) hold; this amounts to showing that in the construction no unwanted connections arose between representatives of atoms. Before proceeding with the proofs of (I) and (V) we need an auxiliary statement $(*)$, whose formulation depends on the following definition.

Let $s, z \in V$ and $a, b \in At$. We say that $\langle s_0, s_1, \ldots, s_n\rangle$, $\langle a_0, a_1, \ldots, a_n\rangle$, $\langle i_1, \ldots, i_n\rangle$ is a *chain* (*of length* n) *leading from* s, a *to* z, b provided that the following conditions hold:

(a) $s = s_0$, $z = s_n$, $a = a_0$, and $b = a_n$.

(b) For all $m < n$, s_{m+1} differs from s_m exactly at i_{m+1}, i.e., $s_{m+1} = (s_m)^{i_{m+1}}_u$ for some $u \neq s_m(i_{m+1})$.

(c) $a_{m+1} \leq c_{i_{m+1}} a_m$, $s_m \in rep(a_m)$, and $Rng(s) \cap Rng(z) \subseteq Rng(s_m)$. (For any function g, $Rng(g)$ is the range of g.)

Here is the statement $(*)$:

$(*)$ Suppose that $s \in rep(a)$, $a \in rep(b)$, and $Rng(s) \cap Rng(z) \neq 0$. Then there is a chain leading from s, a to z, b.

Proof of $(*)$. For each $\nu < \kappa$, let $(*)_\nu$ denote the statement we obtain from $(*)$ by replacing rep by rep_ν in it and in the corresponding definition of a chain leading from s, a to z, b (where rep is mentioned once). Then $(*)$ is $(*)_\kappa$, and we shall prove $(*)_\nu$ for all $\nu \leq \kappa$ by induction on κ. Clearly $(*)_0$ holds and $(*)_\nu$ is preserved in limit steps.

Let $\nu < \kappa$ and assume that $(*)_\nu$ holds; also, assume the hypothesis of $(*)_{\nu+1}$. If $s, z \in V_\nu$, then we are through by our induction hypothesis $(*)_\nu$, since $rep_\nu(a') = rep_{\nu+1}(a') \cap V_\nu$ for any $a' \in At$. If both $s, z \notin V_\nu$, then $s = z = p_\nu$ and $a = b$, since only one element is added at the $(\nu+1)$-st stage, and it is determined by $\sigma(\nu)$. But then we are done, since there is a chain of length 0 from s, a to s, a. Thus we may assume that one of s, z is in V_ν and the other not. Now the statement to be proved is symmetric in s, z, since there is a chain leading from s, a to z, b iff there is one leading from z, b to s, a. Here one needs to use the fact that $[a \leq c_i b$ iff $b \leq c_i a]$ for all $a, b \in At$ and all $i \in \alpha$, which follows from (2)–(6).

So, assume without loss of generality that $s \in V_\nu$ and $z \in V_{\nu+1} \backslash V_\nu$. Now $Rng(s) \cap Rng(z) \neq 0$, so our construction lands in Case 1. Thus there exist a', i and $\lambda < \nu$ such that $\sigma(\nu) = (a', b, i, \lambda)$, $p_\lambda \in rep_\nu(a')$, and $z = p_\nu = (p_\lambda)^i_u \neq p_\lambda$ for some u such that either $u \in Rng(p_\lambda)$ or $u \notin Rng(s)$. Therefore $Rng(s) \cap Rng(z) \subseteq Rng(p_\lambda)$. Hence by the induction hypothesis there is a chain $\langle s, s_1, \ldots, p_\lambda \rangle$, $\langle z, a_1, \ldots, a' \rangle$, $\langle i_1, \ldots, i_n \rangle$ leading from s, a to p_λ, a'. So $\langle s, \ldots, p_\lambda, z \rangle$, $\langle a, \ldots, a', b \rangle$, $\langle i_1, \ldots, i_m, i \rangle$ is a chain leading from s, a to z, b, as desired. This finishes the proof of $(*)$.

Now we are ready for the proofs of (V) and (I).

Proof of (V). Suppose that $s \in rep(a)$, $z \in rep(b)$, and $z = s^i_u$ for some u. We have to show that $a \leq c^{\mathfrak{A}}_i b$. From $\alpha \geq 2$ it follows that $Rng(s) \cap Rng(z) \neq 0$. By $(*)$ then, let $\langle s_0, \ldots, s_n \rangle$, $\langle a_0, \ldots, a_n \rangle$, $\langle i_1, \ldots, i_n \rangle$ be a chain leading from s, a to z, b. We will define $j_1, \ldots, j_n, k_1, \ldots, k_n$ such that

$$b \leq s^{i_n}_{j_n} c_{k_n} \ldots s^{i_1}_{j_1} c_{k_1} a \cdot \prod_{l \in K} d_{l\tau(l)},$$

where $i_1, \ldots, i, \tau, K$ satisfy the conditions in our equation (11) in Σ_α. Then $b \leq c^{\mathfrak{A}}_i a$ by (11) and hence $a \leq c^{\mathfrak{A}}_i b$, and we will be done.

Let $K = \{i_1, \ldots, i_n\} \backslash \{i\}$ and $K^+ = K \cup \{i\}$. Note that $|K^+| > |K|$. We will define j_m and k_m for $1 \leq m \leq n$ by induction on m so that by letting

$$\tau_m = [i_m / j_m] \circ \ldots \circ [i_1 / j_1]$$

we will have for all $m < n$ the following:

$$s_0(l) \leq s_m(\tau_m(l)) \quad \text{for all } l \in K,$$
$$a_{m+1} \leq s^{i_{m+1}}_{j_{m+1}} c_{k_{m+1}} a_m \quad \text{and} \quad k_{m+1} \in K^+ \backslash \tau_m[K].$$

Let $m < n$ and assume that j_t and k_t have been defined for all t with $1 \le t \le m$ so that the above properties hold ($m = 0$ is allowed).

Case 1: $s_m(i_{m+1}) \in Rng(s_{m+1})$, say $s_m(i_{m+1}) = s_{m+1}(j)$. Since $s_{m+1}(i_{m+1}) \neq s_m(i_{m+1})$, we have $j \neq i_{m+1}$. Therefore $s_m(j) = s_{m+1}(j) = s_m(i_{m+1})$. Hence by (III$'$) we get $a_m \le d^{\mathfrak{A}}_{i_{m+1}j}$, and hence $a_{m+1} \le s^{i_{m+1}}_j a_m$. We let $j_{m+1} = j$, and we let k_{m+1} be any member of $K^+ \setminus \tau_m[K]$ (recall that $|K^+| > |K|$, so that $K^+ \setminus \tau_m[K] \neq 0$).

Case 2: $s_m(i_{m+1}) \notin Rng(s_{m+1})$. This time we let $j_{m+1} = k_{m+1} = i_{m+1}$. Note that for any $l \in K$ we have $l \neq i$ and hence

$$s_m(\tau_m(l)) = s_0(l) = z(l) \in Rng(s) \cap Rng(z) \subseteq Rng(s_{m+1}),$$

and hence $i_{m+1} \neq \tau_m(l)$.

In either of these two cases it is easy to see that the above requirements are satisfied for $m+1$. It follows that $b \le s^{i_n}_{j_n} c_{k_n} \dots s^{i_1}_{j_1} c_{k_1} a$. Also, $z(l) = s(l) = s_0(l) = s_n(\tau_n(l)) = z(\tau(l))$ for all $l \in K$. Then it follows from (III$'$) that $b \le d_{l\tau(l)}$ for all $l \in K$. This is as desired, finishing the proof of (V).

Proof of (I). Let $a, b \in At$ and assume that $s \in rep(a) \cap rep(b)$; we want to show that $a = b$. By $(*)$, there is a chain $\langle s_0, \dots, s_n \rangle$, $\langle a_0, \dots, a_n \rangle$, $\langle i_1, \dots, i_n \rangle$ leading from s, a to s, b. If $n = 0$, then $a = b$ and we are done. Assume that $n > 0$. Let $a' = a_{n-1}$, $i = i_n$, and $z = s_{n-1}$. Then the facts that z and s differ exactly on i, $s \in rep(a)$, and $z \in rep(a')$ imply by (V) that $a \le c_i a'$. Then by use of (2)–(6) we derive from $a' \le c_i b$ that $a \le c_i b$. Next, since $z(i) \neq s(i)$ and $Rng(s) \subseteq Rng(z)$ (by virtue of one of the conditions on the chain from s, a to s, b), it follows that $s(i) = z(j) = s(j)$ for some $j \neq i$. Hence $a \le d^{\mathfrak{A}}_{ij}$ and $b \le d^{\mathfrak{A}}_{ij}$ by (III$'$). Thus by (10), $a \le d^{\mathfrak{A}}_{ij} \cdot c^{\mathfrak{A}}_i (d^{\mathfrak{A}}_{ij} \cdot b) \le b$. Since a and b are atoms, it follows that $a = b$. This finishes the proof of (I) and hence of the Theorem. ∎

10. Ultraproducts. As is to be expected, discussion of ultraproducts of Crs_I's requires some involved notation. Let F be an ultrafilter on a set J, $U = \langle U_j : j \in J \rangle$ a system of sets, and I any set. By an (F, U, I)-*choice function* we mean a function ch mapping $I \times \prod_{j \in J} U_j / F$ into $\prod_{j \in J} U_j$ such that for all $i \in I$ and all $y \in \prod_{j \in J} U_j / F$ we have $\mathrm{ch}(i, y) \in y$.

If ch is an (F, U, I)-choice function, then we define ch^+ mapping ${}^I(\prod_{j \in J} U_j / F)$ into $\prod_{j \in J} {}^I U_j$ by setting, for all $q \in {}^I(\prod_{j \in J} U_j / F)$ and all $j \in J$,

$$(\mathrm{ch}^+ q)_j = \langle \mathrm{ch}(i, q_i)_j : i \in I \rangle.$$

Lemma 10.1. *Let $A = \langle A_j : j \in J \rangle$ be a system of sets such that $A_j \subseteq \mathcal{P}({}^I U_j)$ for all $j \in J$, and let* ch *be an (F, U, I)-choice function. Then there is a function*

r mapping $\prod_{j\in J} A_j/F$ into $\mathcal{P}({}^I(\prod_{j\in J} U_j/F))$ such that for any $a \in \prod_{j\in J} A_j$,

$$r(a/F) = \Big\{q \in {}^I\Big(\prod_{j\in J} U_j/F\Big) : \{j \in J : (\mathrm{ch}^+ q)_j \in a_j\} \in F\Big\}.$$

P r o o f. To show that there is such a function, suppose that $a/F = b/F$ and $q \in {}^I(\prod_{j\in J} U_j/F)$. Then $\{j \in J : a_j = b_j\} \in F$, and so

$$\{j \in J : (\mathrm{ch}^+ q)_j \in a_j\} \in F \quad \text{iff} \quad \{j \in J : (\mathrm{ch}^+ q)_j \in b_j\} \in F,$$

as desired. ∎

The function given in Lemma 10.1 will be denoted by $\mathrm{Rep}_{FUIA\mathrm{ch}}$, where we will usually leave off all of the subscripts, or most of them. The basic result on ultraproducts of cylindric set algebras, corresponding to Łoś's theorem in logic, is the following somewhat technical result:

LEMMA 10.2. *Let F be an ultrafilter on a set J, $U = \langle U_j : j \in J\rangle$ a system of non-empty sets, and I a set. Let ch be an (F, U, I)-choice function. Further, let $\mathfrak{A} \in {}^J\mathsf{Crs}_I$, where each $\mathfrak{A}_j$ has base U_j and unit element V_j, and set $V = \langle V_j : j \in J\rangle$.*

Then $\mathrm{Rep}_{\mathrm{ch}}$ is a homomorphism from $\prod_{j\in J} \mathfrak{A}_j/F$ into a Crs_I. Furthermore, for every non-zero $x \in \prod_{j\in J} A_j/F$ there is an (F, U, I)-choice function ch such that $\mathrm{Rep}_{\mathrm{ch}} x \neq 0$. Namely, if $x = a/F$, $Z \in F$, $s \in \prod_{j\in J} V_j$, $s_j \in a_j$ for all $j \in Z$,

$$w = \langle\langle s_j i : j \in J\rangle : i \in I\rangle, \qquad q = \langle w_i/F : i \in I\rangle,$$

and $\mathrm{ch}(i, w_i/F) = w_i$ for all $i \in I$, then $q \in \mathrm{Rep}_{\mathrm{ch}} x$.

P r o o f. Let $f = \mathrm{Rep}_{\mathrm{ch}}$, $X = \prod_{j\in J} U_j/F$, and $T = f(V/F)$. Clearly f preserves $+$. Next we show that f preserves $-$. Clearly $f(-x) \subseteq T\backslash fx$. Now let $x = a/F$ and suppose that $q \in T\backslash fx$. Thus

$$\{j \in J : (\mathrm{ch}^+ q)_j \in V_j\} \in F \quad \text{and} \quad \{j \in J : (\mathrm{ch}^+ q)_j \in a_j\} \notin F,$$

i.e., $\{j \in J : (\mathrm{ch}^+ q)_j \in V_j\backslash a_j\} \in F$. Therefore $q \in f(-x)$, as desired.

So, f is a Boolean homomorphism. Next we show that f preserves d_{kl}. Since $d_{kl} \le V/F$ we have $fd_{kl} \subseteq T$. Now let $q \in T$. Then $\{j \in J : (\mathrm{ch}^+ q)_j \in V_j\} \in F$, and

$$\begin{aligned} q \in fd_{kl} \quad &\text{iff} \quad \{j \in J : (\mathrm{ch}^+ q)_j \in D_{kl}^{[V_j]}\} \in F \\ &\text{iff} \quad \{j \in J : ((\mathrm{ch}^+ q)_j)_k = ((\mathrm{ch}^+ q)_j)_l\} \in F \\ &\text{iff} \quad \{j \in J : \mathrm{ch}(k, q_k)_j = \mathrm{ch}(l, q_l)_j\} \in F \\ &\text{iff} \quad q_k = q_l \quad \text{iff} \quad q \in D_{kl}^{[T]}, \end{aligned}$$

as desired.

Next we check preservation of cylindrifications. Suppose $i \in I$. First suppose that $q \in f(c_i a/F)$. Hence $M \overset{\mathrm{def}}{=} \{j \in J : (\mathrm{ch}^+ q)_j \in C_i^{[V_j]} a_j\}$ is in F. So, there is an $s \in \prod_{j\in J} U_j$ such that $[(\mathrm{ch}^+ q)_j]^i_{s_j} \in a_j$ for all $j \in M$. Let $u = s/F$; we show that $q^i_u \in f(a/F)$, thus finishing this inclusion. Since $\mathrm{ch}(i,u) \in u$, the set $Z \stackrel{\mathrm{def}}{=} \{j \in J : s_j = \mathrm{ch}(i,u)_j\}$ is in F. Now for any $j \in J$, if $k \in I\backslash\{i\}$, then

$$\mathrm{ch}(k,(q^i_u)_k)_j = \mathrm{ch}(k,q_k)_j = ((\mathrm{ch}^+ q)_j)_k,$$

and

$$\mathrm{ch}(i,(q^i_u)_i)_j = \mathrm{ch}(i,u)_j;$$

hence for any $j \in Z \cap M$ we have

$$(\mathrm{ch}^+ q^i_u)_j = \langle \mathrm{ch}(k,(q^i_u)_k)_j : k \in I\rangle = [(\mathrm{ch}^+ q)_j]^i_{\mathrm{ch}(i,u)_j} = [(\mathrm{ch}^+ q)_j]^i_{s_j} \in a_j,$$

as desired.

Second, suppose that $q \in C^{[T]}_i f(a/F)$. Thus $q \in T$ and there is a $u \in X$ such that $q^i_u \in f(a/F)$. Let $M = \{j \in J : (\mathrm{ch}^+ q^i_u)_j \in a_j\}$; thus $M \in F$. Also, since $q \in T$, the set $Z \stackrel{\mathrm{def}}{=} \{j \in J : (\mathrm{ch}^+ q)_j \in V_j\}$ is in F. Now let $j \in M \cap Z$. Then $(\mathrm{ch}^+ q)_j \in V_j$ and $(\mathrm{ch}^+ q)_j \restriction (I\backslash\{i\}) \subseteq (\mathrm{ch}^+ q^i_u)_j \in a_j$, proving that $(\mathrm{ch}^+ q)_j \in C^{[V_j]}_i a_i$. Thus $q \in f(c_i a/F)$, since $M \cap Z \in F$. This finishes the first part of the proof.

For the "Furthermore" part, assume everything mentioned in the hypothesis of "Namely". Let $f = \mathrm{Rep}_{\mathrm{ch}}$. For any $j \in Z$ we have

$$\begin{aligned}(\mathrm{ch}^+ q)_j &= \langle \mathrm{ch}(i,q_i)_j : i \in I\rangle = \langle \mathrm{ch}(i,w_i/F)_j : i \in I\rangle \\ &= \langle (w_i)_j : i \in I\rangle = s_j \in a_j,\end{aligned}$$

so $q \in f(a/F)$, as desired. ∎

With the aid of this lemma we can prove the following basic theorem alluded to earlier:

THEOREM 10.3. *For $|I| > 1$, any homomorphic image of a Crs_I is isomorphic to a Crs_I.*

Proof. Let $\mathfrak{A}$ be a Crs_I, and let f be a homomorphism from $\mathfrak{A}$ onto some algebra $\mathfrak{B}$ (of course, $\mathfrak{B}$ is not necessarily a Crs_I, but is merely similar to a Crs_I, in the sense of universal algebra). By Theorem 9.1 it suffices to take any element x of A such that $fx \neq 0$ and find a homomorphism g from $\mathfrak{A}$ into a Crs_I such that $gx \neq 0$ and $gy = 0$ for all y such that $fy = 0$.

We are going to set up things to apply Lemma 10.2, in particular its last part. Let $J = \{y \in A : fy = 0\}$. Let F be an ultrafilter on J such that $\{y \in J : z \subseteq y\} \in F$ for all $z \in J$; clearly such an ultrafilter exists. Let U be the base of $\mathfrak{A}$. Now $x \notin J$, so for all $z \in J$ we have $x \not\subseteq z$, and so we can choose $s_z \in x\backslash z$. Let $w = \langle\langle s_z i : z \in J\rangle : i \in I\rangle$. Let ch be an $(F, \langle U : z \in J\rangle, I)$-choice function such that $\mathrm{ch}(i, w_i/F) = w_i$ for all $i \in I$. For each $y \in A$ let $\overline{y} = \langle y : z \in J\rangle$, and set $hy = \mathrm{Rep}(\overline{y}/F)$, where

$$\mathrm{Rep} = \mathrm{Rep}_{F\langle U:z\in J\rangle I\langle A:z\in J\rangle\mathrm{ch}}.$$

Let $q = \langle w_i/F : i \in I\rangle$. We take it as a matter of universal algebra that the mapping $y \mapsto \overline{y}/F$ is an isomorphism from $\mathfrak{A}$ into ${}^I\mathfrak{A}/F$. Hence by Lemma 10.2,

h is a homomorphism of $\mathfrak{A}$ into some $\mathsf{Crs}_I\ \mathfrak{D}$, and $q \in hx$. Let $V = h1$ and

$$W = \bigcup\{C^{[V]}_{u_0} \ldots C^{[V]}_{u_{m-1}}\{q\} : m \in \omega,\ u_0, \ldots, u_{m-1} \in I\}.$$

Then $C^{[V]}_i W = W$ for all $i \in I$, so W is a zero-dimensional element of $\mathcal{P}(V)$. Hence by Proposition 5.1, rl_W is a homomorphism from $\mathfrak{D}$ onto some $\mathsf{Crs}_I\ \mathfrak{C}$. Let $g = \mathrm{rl}_W \circ h$. So g is a homomorphism from $\mathfrak{A}$ onto $\mathfrak{C}$, and $gx \neq 0$. It remains only to take any z such that $fz = 0$ and show that $gz = 0$. Let $m \in \omega$ and $u_0, \ldots, u_{m-1} \in I$; we want to show that $hz \cap C^{[V]}_{u_0} \ldots C^{[V]}_{u_{m-1}}\{q\} = 0$. It suffices to show that $\{q\} \cap C^{[V]}_{u_{m-1}} \ldots C^{[V]}_{u_0} hz = 0$, i.e., that $q \notin C^{[V]}_{u_{m-1}} \ldots C^{[V]}_{u_0} hz$. Now $t \stackrel{\mathrm{def}}{=} c_{u_{m-1}} \ldots c_{u_0} z \in J$, so $\{v \in J : t \subseteq v\} \in F$. If $t \subseteq v \in J$, then $s_v \notin v$, hence $s_v \notin t$. Thus $\{v \in J : s_v \notin t\} \in F$. Now for any $v \in J$ such that $s_v \notin t$ we have

$$(\mathrm{ch}^+ q)_v = \langle \mathrm{ch}(i, q_i)_v : i \in I\rangle = \langle (w_i)_v : i \in I\rangle = \langle s_v i : i \in I\rangle = s_v \notin t.$$

Thus $q \notin \mathrm{Rep}(\overline{t}/F) = ht$, as desired. ■

There are many other useful and interesting facts about ultraproducts of cylindric set algebras; see the basic references mentioned in the introduction.

11. Cylindric set algebras. We finally come to the actual topic of these lectures: *cylindric set algebras*, a specialization of cylindric-relativized set algebras. A *cylindric set algebra* is a cylindric-relativized set algebra whose unit element has the form ${}^I U$. So, these have already been discussed, without having a special name for them. For any structure $\mathfrak{M}$, the algebra $\mathfrak{Cs}\,\mathfrak{M}$ is a cylindric set algebra. Let Cs_I be the collection of all cylindric set algebras with dimension set I. This class forms a closer algebraic approximation to the class of all algebras $\mathfrak{Cs}\,\mathfrak{M}$. For example, the simple law $c_0c_1x = c_1c_0x$ holds in all Cs_I's, but not in the larger class Crs_I. For example, let $I = \omega$, $V = \{\langle 0,0,0,0,\ldots\rangle, \langle 0,1,0,0,\ldots\rangle, \langle 1,1,0,0,\ldots\rangle\}$, $x = \{\langle 0,0,0,0,\ldots\rangle\}$. Then $\langle 1,1,0,0,\ldots\rangle \in C^{[V]}_0 C^{[V]}_1 x \backslash C^{[V]}_1 C^{[V]}_0 x$.

All of the theory developed in the preceding sections can be specialized to the class Cs_I, and some natural new questions and results arise. Some of these will be developed in the next few sections. We mention the main facts about cylindric set algebras:

I. The cylindric set algebras derivable from logic can be characterized from among all cylindric set algebras of dimension ω by two additional set-theoretical conditions: regularity and local finiteness.

II. The class of isomorphs of cylindric set algebras of a given infinite dimension is not even an elementary class, contrasting strongly with the case of cylindric-relativized set algebras.

III. The variety generated by Cs_I is not finitely axiomatizable when $|I| > 2$, much like the case of cylindric-relativized set algebras.

IV. This variety can be characterized set-theoretically by means of certain generalized cylindric set algebras.

V. If we restrict ourselves to cylindric set algebras of a fixed infinite dimension with infinite bases, then the unfortunate situation of II no longer holds: we get a variety, just like the case of cylindric-relativized set algebras.

VI. An equation holds in all cylindric set algebras of dimension ω iff it holds in all algebras $\mathfrak{Cs}\,\mathfrak{M}$, $\mathfrak{M}$ a first-order structure.

Results I, IV, and VI are due to Henkin and Tarski; result V is due to Henkin and Monk; results II and III are due to Monk. Important versions of all of these results will be proved in these notes.

We first mention the following obvious consequence of Theorem 5.4 and its proof.

THEOREM 11.1. *Let $\mathfrak{A}$ be a Cs_I with base U (and hence unit element ${}^IU$). Let κ be an infinite cardinal such that $|A| \le \kappa \le |U|$. Assume that $S \subseteq U$ and $|S| \le \kappa$. Finally, assume that $\kappa^{|I|} = \kappa$. Then there is a W such that $S \subseteq W \subseteq U$, $|W| = \kappa$, and $\mathfrak{A}$ is ext-isomorphic to a Cs_I with base W.* ∎

While the class of Crs_I's is a variety according to Section 9, the class Cs_I is not even elementary for I infinite (this is the result II mentioned above). To see this, let $\mathfrak{A}$ be the Cs_I of all subsets of ${}^I2$. Let J be a set with more than $2^{2^{|I|}}$ elements, and let F be an ultrafilter on J such that $|{}^JA/F| \ge |J|$. We claim that ${}^J\mathfrak{A}/F$ is not isomorphic to a Cs_I. For, suppose that f is an isomorphism from ${}^J\mathfrak{A}/F$ onto a Cs_I $\mathfrak{B}$. Say that $\mathfrak{B}$ has base U. Now in $\mathfrak{A}$ the equation $c_0c_1c_2(-d_{01}\cdot -d_{02}\cdot -d_{12}) = 0$ holds, so it holds in $\mathfrak{B}$, too. But this means that $|U| \le 2$, and hence $|B| \le 2^{2^{|I|}}$, a contradiction.

The same example shows that Theorem 9.1 does not extend to Cs_I's for I having at least three elements. Now we consider the variety RCA_I generated by Cs_I; members of RCA_I are called *representable.* Theorem 9.3 does extend to this variety. This is an old result of the author, and is more important than Theorem 9.3 itself since the notion of cylindric set algebra is more natural than that of a cylindric-relativized set algebra. We now give a proof of this result, due to Andréka [1] (the first version of her proof was developed in 1986). Her theorem is actually stronger. This time the proof in the infinite-dimensional case is easier; in my opinion this case is more important anyway, and we give only this case. The original proof of the author remains of interest in showing a connection with combinatorial structures which has been further worked on by Comer and Maddux. This theorem is the major part of the result III mentioned above.

THEOREM 11.2. *Let I be infinite. Then RCA_I cannot be axiomatized by a set Σ of quantifier-free formulas such that only finitely many variables appear in Σ.*

Proof. For simplicity of notation we assume that I is an infinite ordinal α. For each positive integer k we shall construct an algebra $\mathfrak{A}_k$ with the following two properties:

(1) $\mathfrak{A}_k \notin \mathsf{RCA}_\alpha$;

(2) Every k-generated subalgebra of $\mathfrak{A}_k$ is in RCA_α.

(An algebra $\mathfrak{B}$ is *k-generated* if it has a set of generators with at most k elements.) An easy argument shows that the theorem follows from (1) and (2). Fix k in order to do a construction yielding (1) and (2); and fix an integer $m \geq 2^k$. Let $\langle U_i : i \in \alpha\rangle$ be a system of pairwise disjoint sets each with m elements. Let $U = \bigcup_{i\in\alpha} U_i$ and fix $q \in \prod_{i\in\alpha} U_i$. (Here and further on, $\prod$ denotes the Cartesian product of sets.) Further, let

$$R = \Big\{z \in \prod_{i\in\alpha} U_i : \{i \in \alpha : z_i \neq q_i\} \text{ is finite}\Big\}.$$

Another way of putting this definition, using the notation ${}^I U^q$ from the end of Section 4, is: $R = (\prod_{i\in\alpha} U_i) \cap {}^I U^q$. Finally, let $\mathfrak{A}'$ be the subalgebra of $\mathfrak{P}({}^\alpha U)$ generated by the element R. Observe now that R is an atom of $\mathfrak{A}'$. To see this, note:

(3) If $s, z \in R$, then there is a permutation σ of U such that $\sigma \circ s = z$ and $R = \{\sigma \circ p : p \in R\}$.

In fact, there is a permutation σ such that $\sigma s_i = z_i$ and $\sigma z_i = s_i$ for all $i \in \alpha$ and $\sigma k = k$ for all $k \notin \{s_i, z_i : i \in \alpha\}$. Clearly σ is as desired in (3).

Note the following fact about permutations of U:

(4) If σ is a permutation of U and $R = \{\sigma\circ p : p \in R\}$, then $a = \{\sigma\circ p : p \in a\}$ for all $a \in A'$.

In fact, the collection of a such that the conclusion of (4) holds has R as an element and is closed under all of the operations of $\mathfrak{A}'$, so (4) holds.

Now we prove that R is an atom of $\mathfrak{A}'$. Suppose $a \in A'$ and $0 \neq a \cap R$. Fix $s \in a \cap R$. To show that $R \subseteq a$, let $z \in R$ be arbitrary. By (3) let σ be a permutation of U such that $\sigma \circ s = z$. Since $s \in a$, it follows from (4) that $z \in a$, as desired.

Of course $\mathfrak{A}'$ is not the algebra we want, since it is a Cs_α. We now extend A' to yield the desired algebra. There clearly is a BA $\mathfrak{A}$ obtained from $\mathfrak{Bl}\,\mathfrak{A}'$ by replacing R by $m+1$ new atoms R_j, $j \leq m$; thus $R = \sum_{j\leq m} R_j$. We expand $\mathfrak{A}$ to an algebra similar to Crs_α's as follows. Let the cylindrifications of $\mathfrak{A}$ be denoted with small letters to distinguish them from the "real" cylindrifications of $\mathfrak{A}'$, which are denoted by big letters as in the first part of these notes. For any $x \in A$ we define $c_i x$ as follows:

$$c_i x = \begin{cases} C_i x & \text{if } R\cdot x = 0 \text{ (then } x \in A'); \\ C_i(R + x) & \text{if } R \cdot x \neq 0 \text{ (always } R + x \in A'). \end{cases}$$

The diagonal elements of $\mathfrak{A}$ are defined to be the same as those of $\mathfrak{A}'$. (Note that $R \cap D_{ij} = 0$ for all distinct $i, j < \alpha$.) So, this defines $\mathfrak{A}$ fully, as a structure similar to Cs_α's. We mention for later reference some elementary properties of $\mathfrak{A}$:

(5) $\quad x \leq c_i x$.

(6) $c_i(x+y) = c_ix + c_iy$.
(7) If $x \in A'$, then $c_ix = C_ix$.
(8) $c_ix \in A'$.
(9) $c_ic_ix = c_ix$.
(10) If $x \le y$ then $c_ix \le c_iy$.

(5) and (8) are obvious. (6) is easily shown by considering cases. (7) is pretty immediate from the definition since R is an atom of $\mathfrak{A}'$, and (9) follows from (7) and (8). Finally, (10) is shown like this:

$$c_iy = c_i(x+y) = c_ix + c_iy.$$

Note from (7) that $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$.

Now we prove (1). We need some special notation: $s^i_jx = c_i(d_{ij} \cdot x)$ for $i \ne j$, $s^i_ix = x$. Consider the following term $\tau(x)$:

$$\prod_{i\le m} s^0_i c_1 \dots c_m x \cdot \prod_{i<j\le m} -d_{ij}.$$

We want to see the meaning of $\tau(R)$ in $\mathfrak{A}$. To this end, note, in $\mathfrak{A}'$,

$$\begin{aligned} C_1 \dots C_m R &= (U_0 \times {}^mU \times U_{m+1} \times \dots) \cap {}^\omega U^q; \\ s^0_i C_1 \dots C_m R &= ({}^{i-1}U \times U_0 \times {}^{m-i}U \times U_{m+1} \times \dots) \cap {}^\omega U^q \qquad (i \le m); \\ \prod_{i\le m} s^0_i C_1 \dots C_m R &= ({}^{m+1}U_0 \times U_{m+1} \times \dots) \cap {}^\omega U^q. \end{aligned}$$

Now since $|U_0| = m$, it follows that $\tau(R) = 0$ in $\mathfrak{A}'$. Since $\mathfrak{A}'$ is a subalgebra of $\mathfrak{A}$, also $\tau(R) = 0$ in $\mathfrak{A}$.

Suppose that $\mathfrak{A} \in \mathsf{RCA}_\alpha$. Then there is a homomorphism h of $\mathfrak{A}$ into a Cs_α $\mathfrak{B}$ such that $hR \ne 0$. Choose $t \in hR$. Now for each $i \le m$ we have $R \le c_0R_i$, and so $hR \subseteq C_0hR_i$, and so there is a u_i such that $t^0_{u_i} \in hR_i$. Since the R_i's are pairwise disjoint, the u_i's are pairwise distinct. Also note that $c_1 \dots c_m R_i = C_1 \dots C_m R$ for any $i \le m$. Hence

$$\langle u_0, u_1, \dots, u_m, t_{m+1}, t_{m+2}, \dots\rangle \in \tau(hR) = h\tau(R) = 0$$

in $\mathfrak{B}$, a contradiction. Thus (1) holds.

We turn to the proof of (2). Let $G \subseteq A$ with $|G| \le k$. Now we define

$$i \equiv j \text{ iff } i, j \le m \text{ and } \forall g \in G\,(R_i \le g \text{ iff } R_j \le g).$$

Clearly $\equiv$ is an equivalence relation on $m+1$. We claim that it has at most 2^k equivalence classes. To see this, let $f(i/\equiv) = \{g \in G : R_i \le g\}$ for all $i \le m$. Clearly f is well defined, mapping the set of equivalence classes into $\mathcal{P}(G)$. And f is clearly one-one by the definition of $\equiv$; this proves the claim. Let p be the number of equivalence classes. Recall also that $2^k \le m$. Now define

$$B = \{a \in A : \forall i, j \le m\,(\text{if } i \equiv j \text{ then } (R_i \le a \text{ iff } R_j \le a)\}.$$

We now show that B is closed under the operations of $\mathfrak{A}$. Clearly it is closed under the Boolean operations. Since $R_j \not\leq d_{il}$ for all distinct $i, l < \alpha$ and all $j \leq m$, it follows that $d_{il} \in B$. Since R is an atom of $\mathfrak{A}'$, it follows that $A' \subseteq B$; since $c_i a \in A'$ for all $a \in A$, it follows that $c_i b \in B$ for all $b \in B$. Thus, indeed, B is closed under the operations of $\mathfrak{A}$. Note also that we have shown that $A' \subseteq B$.

We let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ with universe B. Clearly $G \subseteq B$, so it suffices to show that $\mathfrak{B} \in \mathsf{RCA}_\alpha$. We shall, in fact, show that $\mathfrak{B}$ is isomorphic to a Cs_α with base U (see the beginning of the construction).

Let $e_0, \ldots, e_{p-1}$ be all of the equivalence classes under $\equiv$. For each $j < p$ let $y_j = \sum\{R_k : k \in e_j\}$. Then $\langle y_j : j < p\rangle$ is a partition of R in B, $c_i y_j = C_i R$ for all $i < \alpha$ and all $j < p$, every element of B is a join of certain y_j's and elements of A', and the y_j's are atoms of $\mathfrak{B}$. We now consider m (which is $\{0, 1, \ldots, m-1\}$) along with addition $+$ modulo m; actually any group operation on m with identity 0 will do. For each $i < \alpha$ let f_i be a one-one function mapping U_i onto m such that $f_i q_i = 0$. For each $j < m$ let

$$R'_j = \Big\{z \in R : \sum_{i<\alpha} f_i z_i = j\Big\}.$$

(Note that for $z \in R$, $f_i z_i = 0$ except for finitely many $i < \alpha$.) Clearly the R'_j's are pairwise disjoint and $C_i R'_j = C_i R$ for all $i < \alpha$ and all $j < m$. Next we define

$$S_j = R'_j \ \text{ if } \ j < p-1, \qquad S_{p-1} = \bigcup_{p-1 \leq j < m} R'_j.$$

Now we define the desired isomorphism h: for all $b \in B$,

$$hb = (b \cdot -R) \cup \bigcup\{S_j : j < p,\ y_j \leq b\}.$$

Clearly h preserves the Boolean operations and the D_{ij}'s, and h is one-one. To show that h preserves c_i, first note the following two facts:

(11) $C_i hb = C_i(b \cdot -R) \cup \bigcup_{y_j \leq b} C_i S_j$;

(12) $hc_i b = (c_i b \cdot -R) \cup \bigcup_{y_j \leq c_i b} S_j$.

(11) and (12) follow from the definition of h. Now we consider two cases.

Case 1: $y_j \leq b$ for some j. Then by (10) and the definition of c_i, $C_i R = c_i y_j \leq c_i b$; and $C_i S_j = C_i R$. So by (11) and (12),

$$C_i hb = C_i(b \cdot -R) \cup C_i R = C_i(b \cup R) = c_i b;$$
$$hc_i b = (c_i b \cdot -R) \cup R = c_i b,$$

as desired.

Case 2: $y_j \not\leq b$ for all $j < p$. Then $b \cdot R = 0$, so by (11), $C_i hb = C_i(b \cdot -R) = c_i b$. Now we take two subcases. *Subcase* 2.1: $y_j \leq c_i b$ for some j. Then $R \subseteq C_i R = c_i y_j \leq c_i b$, so by (12), $hc_i b = (c_i b \cdot -R) \cup R = c_i b$, as desired. *Subcase* 2.2: $y_j \not\leq c_i b$ for all $j < p$. Then $c_i b \cdot R = 0$, so by (12) again, $hc_i b = c_i b \cdot -R = c_i b$, finishing the proof. ∎

12. Local finite-dimensionality and regularity. A Crs_I $\mathfrak{A}$ is *locally finite-dimensional* if Δx is finite for all $x \in A$. Thus each algebra $\mathfrak{Cs}\,\mathfrak{M}$ is locally finite-dimensional; this is a consequence of each formula in a first-order language being of finite length. Also note that if I is finite, then $\mathfrak{A}$ is automatically locally finite-dimensional. An element $a \in A$ is *finite-dimensional* if Δa is finite. Now let $\mathfrak{A}$ be a Cs_I with base U. We call $\mathfrak{A}$ *regular* provided that for all $a \in A$, all $f \in a$ and all $g \in {}^IU$, if $f \restriction \Delta x = g \restriction \Delta x$ then $g \in a$. It is easy to see that each algebra $\mathfrak{Cs}\,\mathfrak{M}$ is regular. At first glance, one might think that every Cs_I is regular. If I is finite, then it is easy to check that this is the case. But for I infinite we now give a counterexample. Let $\mathfrak{A}$ be the Cs_I of all subsets of ${}^I2$, and let

$$a = \{x \in {}^I2 : \{i \in I : x_i \neq 0\} \text{ is finite}\}.$$

Then $\Delta a = 0$, from which it is clear that $\mathfrak{A}$ is not regular. We understand in an obvious sense an element $a \in A$ being *regular*.

We now give another algebraic form of the downward Löwenheim–Skolem theorem.

THEOREM 12.1. *Let $\mathfrak{A}$ be a regular Cs_I with base U and unit element $Z = {}^IU$. Define λ to be the least infinite cardinal greater than each $|\Delta a|$, $a \in A$. Let κ be an infinite cardinal such that $|A| \le \kappa \le |U|$ and $\kappa = \sum_{\mu<\lambda} \kappa^\mu$. Assume that $S \subseteq U$ and $|S| \le \kappa$. Then there is a set W such that $S \subseteq W \subseteq U$, $|W| = \kappa$, and, with $V = {}^IW$, $\mathrm{rl}_V^{\mathfrak{A}}$ is an isomorphism from $\mathfrak{A}$ onto a regular Cs_I $\mathfrak{B}$ with base W.*

Proof. The proof, while basically similar to that of Theorem 5.4, has to be modified from that one. Let well-orderings of U and ${}^IU$ be given. Fix $u \in U$. For each $a \in A\backslash\{0\}$, let k_a be the first element of a such that $k_a i = u$ for all $i \in I\backslash\Delta a$; that there is such a k_a follows from the regularity of $\mathfrak{A}$. Note that the range of k_a has fewer than λ elements. Hence there is a set T_0 such that $|T_0| = \kappa$, $S \subseteq T_0$, $u \in T_0$, and $k_a \in T_0$ for all $a \in A\backslash\{0\}$. Now suppose that $0 < \beta < \kappa$ and T_α has been defined for all $\alpha < \beta$. Let $M = \bigcup_{\alpha<\beta} T_\alpha$, and let

$$\begin{aligned} T_\beta = M \cup \{v \in U : {} & \text{there exist } a \in A,\ i \in \Delta a,\ x \in {}^{\Delta a}M, \\ & \text{such that } v \text{ is the first element of } U \text{ with the property that} \\ & y \in a \text{ for some } y \in {}^IU \text{ such that } x_v^i \subseteq y\}. \end{aligned}$$

Finally, let $W = T_\kappa = \bigcup_{\alpha<\kappa} T_\alpha$. Note that in forming T_β, at most one element is added to M for each choice of the following: an element $a \in A$; an element $i \in \Delta a$; and a function $x \in {}^{\Delta a}M$. Thus if we assume that $|M| = \kappa$, we get that also $|T_\beta| = \kappa$. Hence it follows by induction that $|T_\alpha| = \kappa$ for all $\alpha \le \kappa$. By the definition of T_0 it is clear that $\mathrm{rl}_V^{\mathfrak{A}}$ is one-one. To prove that rl preserves C_i, by the comment before Proposition 5.1 it suffices to take any $a \in A$ and $z \in C_i^{[Z]}a \cap V$ and show that $z \in C_i^{[V]}(a \cap V)$. We may assume that $i \in \Delta a$. Now $z_v^i \in a$ for some $v \in U$. Let $x = z \restriction \Delta a$. Now $|\Delta a| < \lambda$, hence $\kappa^{|\Delta a|} = \kappa$, hence $|\Delta a| < \mathrm{cf}\,\kappa$, hence there is a $\beta < \kappa$ such that $x \in {}^{\Delta a}T_\beta$. It follows that there is a $w \in T_{\beta+1} \subseteq W$

such that $y \in a$ for some $y \in {}^I U$ with $x^i_w \subseteq y$. So $z^i_w \restriction \Delta a = y \restriction \Delta a$, hence by the regularity of $\mathfrak{A}$, $z^i_w \in a \cap V$, so $z \in C_i^{[V]}(a \cap V)$, as desired.

So $\mathrm{rl}_V^{\mathfrak{A}}$ is an isomorphism from $\mathfrak{A}$ onto some Cs_I $\mathfrak{B}$ with base W. As to the regularity of $\mathfrak{B}$, suppose that $a \in A$, $x \in a \cap V$, $y \in {}^I W$, and $x \restriction \Delta a = y \restriction \Delta a$. Then $y \in a$ by the regularity of $\mathfrak{A}$, and hence $y \in a \cap V$, as desired. ∎

We can now give the result I about cylindric set algebras mentioned above.

THEOREM 12.2. *Let $\mathfrak{A}$ be a Cs_ω. Then $\mathfrak{A}$ has the form $\mathfrak{Cs}\,\mathfrak{M}$ for some first-order structure $\mathfrak{M}$ iff $\mathfrak{A}$ is locally finite-dimensional and regular.*

Proof. We have already observed that $\mathfrak{Cs}\,\mathfrak{M}$ is always locally finite and regular. Now assume that $\mathfrak{A}$ is a locally finite and regular Cs_ω, say with base M. For each $a \in A$ let r_a be the smallest natural number such that $\Delta a \subseteq r_a$. Let $\mathcal{L}$ be the first-order language having, for each $a \in A$, an r_a-ary relation symbol $\mathbf{R}_a$. We make M into an $\mathcal{L}$-structure $\mathfrak{M}$ by setting, for each $a \in A$,

$$\mathbf{R}_a^{\mathfrak{M}} = \{x \in {}^{r_a}M : x \subseteq y \text{ for some } y \in a\}.$$

We claim that $\mathfrak{Cs}\,\mathfrak{M} = \mathfrak{A}$. Obviously both are Cs_ω's with base M, so it suffices to show that their universes are the same. Given $a \in A$, we show that $a = (\mathbf{R}_a v_0 \ldots v_{r_a-1})^{\mathfrak{M}}$; this will show $\supseteq$. In fact, for any $x \in {}^\omega M$ we have

$$\begin{aligned} x \in (\mathbf{R}_a v_0 \ldots v_{r_a-1})^{\mathfrak{M}} \quad &\text{iff} \quad \mathfrak{M} \models \mathbf{R}_a v_0 \ldots v_{r_a-1}[x] \\ &\text{iff} \quad \langle x_0, \ldots, x_{r_a-1}\rangle \in \mathbf{R}_a^{\mathfrak{M}} \\ &\text{iff} \quad \langle x_0, \ldots, x_{r_a-1}\rangle \subseteq y \text{ for some } y \in a \\ &\text{iff} \quad x \in a; \end{aligned}$$

in the last equivalence we use the regularity of $\mathfrak{A}$.

For the other inclusion it suffices to show that $\phi^{\mathfrak{M}} \in A$ for every formula ϕ, by induction on ϕ. We may assume that the atomic parts of ϕ have the standard form mentioned in the FACT formulated prior to Theorem 3.3. Then the atomic case is easy. All of the inductive steps are easy exercises, too. ∎

The following simple result will be needed later. The proof is straightforward.

LEMMA 12.3. *If $\mathfrak{A}$ is a Cs_I generated by a collection of regular finite-dimensional elements, then $\mathfrak{A}$ is regular and locally finite-dimensional.* ∎

13. Generalized cylindric set algebras. Let $\mathfrak{A}$ be a Crs_I with unit element V. We call $\mathfrak{A}$ a *generalized cylindric set algebra* provided that V has the form $\bigcup_{j\in J} {}^I Y_j$, where $Y_j \neq 0$ for each $j \in J$ and $Y_j \cap Y_k = 0$ for all distinct $j, k \in J$. And we denote by Gs_I the class of all generalized cylindric set algebras of dimension I.

Here is some general algebraic notation: for any class $\mathbf{K}$ of similar algebras,

$$\mathbf{IK} = \{\mathfrak{A} : \mathfrak{A} \text{ is isomorphic to some } \mathfrak{B} \in \mathbf{K}\};$$
$$\mathbf{HK} = \{\mathfrak{A} : \mathfrak{A} \text{ is the homomorphic image of some } \mathfrak{B} \in \mathbf{K}\};$$
$$\mathbf{SK} = \{\mathfrak{A} : \mathfrak{A} \text{ is a subalgebra of some } \mathfrak{B} \in \mathbf{K}\};$$
$$\mathbf{PK} = \{\mathfrak{A} : \mathfrak{A} \text{ is the product of some system of elements of } \mathbf{K}\}.$$

LEMMA 13.1. $\mathbf{I}\mathsf{Gs}_I = \mathbf{SP}\mathsf{Cs}_I$ *for all I with at least two elements.*

Proof. The proof is essentially contained in the proof of Theorem 9.1. Thus suppose that $\langle \mathfrak{A}_j : j \in J\rangle$ is a system of Cs_I's; say the base of $\mathfrak{A}_j$ is U_j for each $j \in J$. We may assume that $U_j \cap U_k = 0$ for all distinct $j, k \in J$. Let $W = \bigcup_{j\in J} {}^I U_j$. Then define $f : \prod_{j\in J} A_j \to \mathcal{P}(W)$ by setting $fx = \bigcup_{j\in J} x_j$ for any $x \in \prod_{j\in J} A_j$. Now, apart from notation, the proof proceeds as in the proof of Theorem 9.1. This proves that $\mathbf{SP}\mathsf{Cs}_I \subseteq \mathbf{I}\mathsf{Gs}_I$.

For the other inclusion, suppose that $\mathfrak{A}$ is a Gs_I, say with unit element $V = \bigcup_{j\in J} {}^I U_j$, where each U_j is non-empty and $U_j \cap U_k = 0$ for all distinct $j, k \in J$. Define g mapping A into $\prod_{j\in J} \mathcal{P}({}^I U_j)$ by setting, for any $a \in A$ and $j \in J$, $(ga)_j = a \cap {}^I U_j$. The details that g is an isomorphism from $\mathfrak{A}$ into a product of Cs_I's are very similar to the details in the proof of Theorem 9.1, and are left to the reader. ■

The following lemma holds for I finite with at least two elements as well as for I infinite, but we restrict ourselves to the case of I infinite. In its proof we need the following notation. For any finite subset K of I,

$$C_K x = C_{k_0} \ldots C_{k_{m-1}} x,$$

where $K = \{k_0, \ldots, k_{m-1}\}$. We depend on the context to determine whether C_K refers to this generalized cylindrification for a subset K of I or just to the ordinary cylindrification, usually using big letters for the former, and small ones for the latter. For a Gs_I with I infinite, the order of enumeration of K is easily seen to be unimportant in this definition.

LEMMA 13.2. $\mathbf{H}\mathsf{Gs}_I \subseteq \mathbf{I}\mathsf{Gs}_I$ *for I infinite.*

Proof. Let $\mathfrak{A}$ be a Gs_I and let $f : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. To prove the theorem it suffices to take any $a \in A$ such that $fa \neq 0$ and find a homomorphism h from $\mathfrak{A}$ onto some Cs_I such that $ha \neq 0$ but $hx = 0$ whenever $fx = 0$.

Since $\mathfrak{A}$ is a Gs_I, its unit element V has the form $\bigcup_{j\in J} {}^I U_j$ where $U_j \neq 0$ for all $j \in J$, and $U_j \cap U_k = 0$ for all distinct $j, k \in J$. Let

$$M = \{x \in A : fx = 0\} \times \{K \subseteq I : K \text{ is finite}\}.$$

Let F be an ultrafilter on M such that if $(x, K) \in M$ then

$$T_{xK} \stackrel{\text{def}}{=} \{(y, L) \in M : x \leq y,\ K \subseteq L\} \in F.$$

Clearly there is such an ultrafilter. Let $W = \bigcup_{j\in J} U_j$, and set $X = {}^M W/F$. Let a well-ordering of ${}^M W$ be given. Since $fa \neq 0$, there is an $r \in {}^M V$ such that for all $(x, K) \in M$ we have $r(x, K) \in a \cap -C_K x$. Then there is a function $j \in {}^M J$ such that for all $(x, K) \in M$ we have $r(x, K) \in {}^I U_{j(x,K)}$. Let $Q = \{k/F : k \in \prod_{m\in M} U_{jm}\}$ and $w = \langle\langle(rm)_i : m \in M\rangle : i \in I\rangle$. So $w_i/F \in Q$ for all $i \in I$. Now we define $\mathrm{ch} : I \times {}^M W/F \to {}^M W$ as follows. (a) $\mathrm{ch}(i, w_i/F) = w_i$ for all $i \in I$. (b) If $y \in {}^M W/F$, $i \in I$, and $y \neq w_i/F$, let y' be the first member of y which is in $\prod_{m\in M} U_{jm}$ if $y \in Q$, otherwise just the first member of y, and for each $(x, K) \in M$ let

$$\mathrm{ch}(i, y)_{xK} = \begin{cases} y'_{xK} & \text{if } i \in K, \\ r(x, K)_i & \text{if } i \notin K. \end{cases}$$

Note that always $\mathrm{ch}(i, y)/F = y$. This is obvious if $y = w_i/F$, while otherwise

$$T_{0\{i\}} \subseteq \{(x, K) : \mathrm{ch}(i, y)_{xK} = y'_{xK}\},$$

and hence $\mathrm{ch}(i, y)/F = y$. Thus ch is an $\langle F, \langle W : m \in M\rangle, I\rangle$-choice function. And note the following three properties of ch, for any $i \in I$:

(1) $\mathrm{ch}(i, w_i/F) = w_i$;
(2) if $y \in Q$, then $\mathrm{ch}(i, y) \in \prod_{m\in M} U_{jm}$;
(3) if $(x, K) \in M$ and $i \notin K$, then $\mathrm{ch}(i, y)_{xK} = r(x, K)_i$.

Let $g = \mathrm{Rep}_{F\langle W:m\in M\rangle I\langle A:m\in M\rangle\mathrm{ch}}$. For any $x \in A$ let $\overline{x} = \langle x : m \in M\rangle$. Finally, let $hx = g(\overline{x}/F)$ for all $x \in A$.

By Lemma 10.2, h is a homomorphism from $\mathfrak{A}$ onto some $\mathsf{Crs}_I\ \mathfrak{B}$, and $ha \neq 0$. Now suppose that $fy = 0$; we show $hy = 0$. It suffices to show:

($\star$) For all $q \in {}^I X$ and all $m \in T_{y0}$, $(\mathrm{ch}^+ q)_m \notin y$.

To prove ($\star$), say $m = (z, K)$, where $y \leq z$ and $fz = 0$. By (3), $\mathrm{ch}(i, q_i)_m = (rm)_i$ for all $i \in I\backslash K$. Thus $(\mathrm{ch}^+ q)_m \restriction (I\backslash K) = rm \restriction (I\backslash K)$. Since $rm \notin C_K z$, it follows that $(\mathrm{ch}^+ q)_m \notin C_K z$, and hence $(\mathrm{ch}^+ q)_m \notin y$. This proves ($\star$).

It remains only to show that $hV = {}^I Q$. To do this, we first note:

(4) For any $q \in {}^I X$ and any $m \in M$, $(\mathrm{ch}^+ q)_m \in V$ iff $(\mathrm{ch}^+ q)_m \in {}^I U_{jm}$.

For, write $m = (z, K)$; then by (3), $(\mathrm{ch}^+ q)_m \restriction (I\backslash K) = rm \restriction (I\backslash K)$, and (4) follows.

Now suppose $q \in {}^I Q$. By (2), $(\mathrm{ch}^+ q)_m \in {}^I U_{jm}$ for all $m \in M$, and so by (4), $(\mathrm{ch}^+ q)_m \in V$ for all $m \in M$, and hence $q \in hV$. On the other hand, suppose that $q \in hV$. Then the set $Z \stackrel{\mathrm{def}}{=} \{m \in M : (\mathrm{ch}^+ q)_m \in V\}$ is in F. By (4), $(\mathrm{ch}^+ q)_m \in {}^I U_{jm}$ for each $m \in Z$, i.e., $\mathrm{ch}(i, q_i)_m \in U_{jm}$ for all $i \in I$ and all $m \in Z$. Thus $q \in {}^I Q$, as desired. ■

Combining Lemmas 13.1 and 13.2 we obtain the following theorem, which gives an important part of the result IV mentioned in Section 11:

THEOREM 13.3. *For I infinite, the variety generated by Cs_I is equal to $\mathbf{I}\mathsf{Gs}_I$.* ■

The following lemma leads immediately to the result VI mentioned in Section 11.

LEMMA 13.4. *If $\mathfrak{A}$ is any Cs_ω, then $\mathfrak{A}$ is in the variety generated by the class of regular locally finite-dimensional Cs_ω's.*

Proof. It suffices to take any Cs_ω $\mathfrak{A}$ and any non-zero element $a \in A$ and find a homomorphism f of $\mathfrak{A}$ into an ultraproduct $\mathfrak{B}$ of regular locally finite-dimensional Cs_ω's such that $fa \neq 0$. Let U be the base of $\mathfrak{A}$. Fix $x \in a$.

For a while we will work with a fixed but arbitrary finite subset K of ω. For each $y \in {}^IU$ let $y^* = (y \restriction K) \cup (x \restriction (I \backslash K))$. For all $b \in A$ let $f_K b = \{y \in {}^IU : y^* \in b\}$. Clearly f_K is a homomorphism from $\mathfrak{Bl}\mathfrak{A}$ into the BA of all subsets of ${}^IU$. Since $x^* = x$, it is clear that $f_K a \neq 0$. It is also clear that $f_K D_{ij} = D_{ij}$ for all $i, j \in K$. We claim that also $f_K C_i b = C_i f_K b$ for all $i \in K$. In fact, suppose that $y \in f_K C_i b$. Thus $y^* \in C_i b$, so there is a $u \in U$ such that $(y^*)^i_u \in a$. Since $i \in K$, we have $(y^*)^i_u = (y^i_u)^*$. Hence $y^i_u \in f_K b$, and so $y \in C_i f_K b$. The converse is similar. For any $b \in A$, $f_K b$ is a finite-dimensional element of $\mathfrak{P}({}^IU)$; in fact, $\Delta f_K b \subseteq K$. And it is easy to check that $f_K b$ is regular. It follows from Lemma 12.3 that the Cs_ω $\mathfrak{B}_K$ generated by $f_K[A]$ is regular and locally finite-dimensional.

Now let $J = \{K : K \text{ is a finite subset of } \omega\}$, and let F be an ultrafilter on J such that $\{L \in J : K \subseteq L\} \in F$ for all $K \in J$. For each $b \in A$ let $gb = \langle f_K b : K \in J\rangle / F$. It is easy to check that g is an isomorphism from $\mathfrak{A}$ into $\prod_{K \in J} \mathfrak{B}_K / F$, as desired. ■

COROLLARY 13.5. *An equation holds in all cylindric set algebras of dimension ω iff it holds in all algebras $\mathfrak{Cs}\,\mathfrak{M}$, $\mathfrak{M}$ a first-order structure.* ■

14. Cylindric set algebras with infinite bases. In this section we prove the result V mentioned at the beginning of Section 11. The proof depends on the notion of a weak cylindric set algebra, which is one of the important notions concerning set algebras. But we are not going to develop the theory of these set algebras much, merely proving what is needed for the result V.

Recall the definition of ${}^IU^p$ from Section 4. A *weak cylindric set algebra* is a cylindric-relativized set algebra $\mathfrak{A}$ whose unit element has the form ${}^IU^p$. Note that U is the base of $\mathfrak{A}$ (see Section 4).

PROPOSITION 14.1. *Let $\mathfrak{A}$ be a Gs_I with unit element $V = \bigcup_{j \in J} {}^IU_j$, such that $U_j \cap U_k = 0$ for distinct $j, k \in J$. Then we can write $V = \bigcup_{k \in K} {}^IW_k^{p_k}$, where ${}^IW_k^{p_k} \cap {}^IW_l^{p_l} = 0$ for distinct $k, l \in K$. Moreover, for all $k \in K$ there is a $j \in J$ such that $W_k = U_j$.*

Proof. Fix $j \in J$. We define $p \equiv q$ iff $p, q \in {}^IU_j$ and $\{i \in I : p_i = q_i\}$ is finite. Clearly $\equiv$ is an equivalence relation on ${}^IU_j$. Let $\mathcal{K}_j$ consist of exactly one element from each $\equiv$-class. Then $V = \bigcup_{j\in J} \bigcup_{p\in \mathcal{K}_j} {}^I U_j^p$, and ${}^I U_j^p \cap {}^I U_k^q = 0$ if $j \neq k$ or $p \neq q$. ∎

Another notion we will need is well known in set theory. An ultrafilter F on a set X is *regular* if there is an $a \in {}^X F$ such that $\bigcap_{x\in M} a_x = 0$ for every infinite subset M of X. Another way of saying this is that there is an $h \in {}^X\{M : M \subseteq X, M \text{ finite}\}$ such that $\{x : y \in hx\} \in F$ for all $y \in X$. [To see the existence of h, let $hx = \{y : x \in a_y\}$ for all $x \in X$. Assuming that such an h exists, to see the existence of a, let $a_y = \{x : y \in hx\}$ for all $y \in X$.] It is known that for every infinite set X there is a regular ultrafilter on X (in fact, "most" ultrafilters are regular). Moreover, for any infinite set A, $|{}^X A/F| \geq 2^{|X|}$. For more on regular ultrafilters see Chang, Keisler [2] and Comfort, Negrepontis [3].

The following version of the upward Löwenheim–Skolem–Tarski theorem is crucial in the proof of V.

THEOREM 14.2. *Suppose that* $|I| \geq 2$. *Let* $\mathfrak{A}$ *be a weak cylindric set algebra with infinite base* U. *Let* κ *be a cardinal such that* $\max(|A|, |U|) \leq \kappa$ *and* $\kappa^{|I|} = \kappa$. *Then* $\mathfrak{A}$ *is sub-isomorphic to a* Cs_I *with base of power* κ.

Proof. Let $\mathfrak{A}$ have unit element $V \stackrel{\text{def}}{=} {}^I U^p$, and let $\lambda = \max(|I|, \kappa)$. It is convenient to assume that $I \subseteq \lambda$. Let F be a λ-regular ultrafilter on λ. So there is an $h \in {}^\lambda\{\Gamma \subseteq I : \Gamma \text{ finite}\}$ such that $\{\alpha : i \in h\alpha\} \in F$ for all $i \in I$. For each $a \in A$ let $\delta a = \langle a : \alpha < \lambda\rangle/F$. Thus δ is an isomorphism from $\mathfrak{A}$ into ${}^I\mathfrak{A}/F$. Also, for each $u \in U$ let $\varepsilon u = \langle u : \alpha < \lambda\rangle/F$. Let $X = {}^\lambda U/F$. Now we define a function $c : I \times X \to {}^\lambda U$ as follows: for any $i \in I$, $x \in X$, and $\alpha < \lambda$, write $x = y/F$ with $y = \langle u : \alpha < \lambda\rangle$ if $x = \varepsilon u$, and let

$$c(i,x)_\alpha = \begin{cases} p_i & \text{if } i \notin h\alpha; \\ y_\alpha & \text{otherwise.} \end{cases}$$

Since $\{\alpha : i \in h\alpha\} \in F$, it follows that $c(i,x)/F = y/F = x$, so c is an (F, U, I)-choice function. Let $f = \mathrm{Rep}(c)$. So by Lemma 10.2, $f \circ \delta$ is a homomorphism from $\mathfrak{A}$ onto some Crs_I.

(1) $f\delta V = {}^I X$.

In fact, $\subseteq$ is true by the definition of f. Now let $q \in {}^I X$; we want to show that $\{\alpha < \lambda : (c^+ q)_\alpha \in V\} \in F$. In fact, $(c^+ q)_\alpha \in V$ for all $\alpha < \lambda$. For, if $i \in I$, then $((c^+ q)_\alpha)_i = c(i, q_i)_\alpha \in U$, so $(c^+ q)_\alpha \in {}^I U$. And if $i \notin h\alpha$, then $((c^+ q)_\alpha)_i = c(i, q_i)_\alpha = p_i$, so $\{i \in I : ((c^+ q)_\alpha)_i \neq p_i\} \subseteq h\alpha$, which is finite, so $(c^+ q)_\alpha \in {}^I U^p = V$, as desired in (1).

(2) If $u \in {}^I U^p$ then there is a $\Gamma \in F$ such that $(c^+(\varepsilon \circ u))_\alpha = u$ for all $\alpha \in \Gamma$.

In fact, let M be a finite subset of I such that $u_i = p_i$ for all $i \in I \backslash M$. Let $\Gamma = \{\alpha < \lambda : M \subseteq h\alpha\}$. So $\Gamma \in F$. Then $i \in h\alpha$ implies that $c(i, \varepsilon u_i)_\alpha = u_i$, and $i \notin h\alpha$ implies that $c(i, \varepsilon u_i)_\alpha = p_i = u_i$. So $((c^+(\varepsilon \circ u))_\alpha)_i = c(i, \varepsilon u_i)_\alpha = r_i$ for all $i \in I$, and (2) follows.

(3) $f \circ \delta$ is one-one.

For, let a be a non-zero element of A; say $u \in a$. Taking Γ as in (2), we get $(c^+(\varepsilon \circ u))_\alpha = u \in a$ for all $\alpha \in \Gamma$, and hence $\varepsilon \circ u \in f\delta a$, as desired in (3).

Let $Z = {}^I(\varepsilon[U])^{\varepsilon \circ p}$. Then the following statement is clear:

(4) $\widetilde{\varepsilon}V = Z$.

(5) $\mathrm{rl}_Z \circ f \circ \delta = \widetilde{\varepsilon}$.

To prove (5), suppose $a \in A$ and $q \in Z$. Say $q = \varepsilon \circ u$ with $u \in {}^IU^p$. Choose Γ in accordance with (2). Then

$$\begin{aligned} q \in \mathrm{rl}_Z\, f\delta a \quad &\text{iff} \quad q \in f\delta a \\ &\text{iff} \quad \{\alpha < \lambda : (c^+q)_\alpha \in a\} \in F \\ &\text{iff} \quad \{\alpha \in \Gamma : (c^+q)_\alpha \in a\} \in F \\ &\text{iff} \quad \{\alpha < \lambda : u \in a\} \in F \\ &\text{iff} \quad u \in a \\ &\text{iff} \quad \varepsilon^{-1} \circ q \in a \\ &\text{iff} \quad q \in \widetilde{\varepsilon}a, \end{aligned}$$

as desired.

By (5), $f \circ \delta$ is a sub-base-isomorphism. By Proposition 6.4, there is a base isomorphism h' and an ext-isomorphism g' such that $(f \circ \delta)^{-1} = g' \circ h'$. Say h' is a base isomorphism of $\mathfrak{B}$ onto $\mathfrak{C}$. Clearly then $\mathfrak{C}$ is a Cs_I with a base T such that $|T| = |X|$. Moreover, $g' = \mathrm{rl}_V$ is an ext-isomorphism from $\mathfrak{C}$ onto $\mathfrak{A}$. Note that $|T| = |X| = 2^\lambda$. Thus $|A| \le \kappa \le 2^\lambda = |T|$. And $U \subseteq T$ with $|U| \le \kappa$. Therefore by Theorem 11.1 there is a W such that $U \subseteq W \subseteq T$, $|W| = \kappa$, and rl_W is an ext-isomorphism from $\mathfrak{C}$ onto a Cs_I with base W. Clearly then rl_U is an isomorphism from $\mathfrak{C}$ onto $\mathfrak{A}$, as desired. ■

Let ${}_\infty\mathsf{Cs}_I$ be the class of all cylindric set algebras of dimension I with infinite base, and let ${}_\infty\mathsf{Gs}_I$ be the class of all generalized cylindric set algebras of dimension I with unit of the form $\bigcup_{j\in J} {}^IY_j$, the Y_j's infinite and pairwise disjoint. The result V now reads as follows:

Theorem 14.3. *For I infinite*, $\mathbf{HSP}({}_\infty\mathsf{Cs}_I) = \mathbf{I}({}_\infty\mathsf{Gs}_I) = \mathbf{I}({}_\infty\mathsf{Cs}_I)$.

Proof. First note that $\mathbf{HSP}({}_\infty\mathsf{Cs}_I) = \mathbf{I}({}_\infty\mathsf{Gs}_I)$ by reading over the proofs of Lemmas 13.1 and 13.2. So we just have to show that every ${}_\infty\mathsf{Gs}_I$ is isomorphic to an ${}_\infty\mathsf{Cs}_I$. Let $\mathfrak{A}$ be an ${}_\infty\mathsf{Gs}_I$. By Proposition 14.1 we can write the unit element of $\mathfrak{A}$ in the form $\bigcup_{j\in J} V_j$, where $V_j = {}^IU_j^{p_j}$, each U_j infinite, $V_j \cap V_k = 0$ for distinct j,k. Choose $j \in {}^AJ$ so that $a \cap V_{j_a} \ne 0$ for all $a \in A\backslash\{0\}$. For all $a \in A$ let $\mathfrak{B}_a$ be the Crs_I of all subsets of V_{j_a}; so $\mathfrak{B}_a$ is a weak cylindric set algebra. Let $h_a = \mathrm{rl}^{\mathfrak{A}}_{V_{j_a}}$. By Proposition 5.1, h_a is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}_a$, and

$h_a a \neq 0$. Let

$$\kappa = |I| \cup \bigcup_{j\in J} |U_j| \cup \bigcup_{a\in A\setminus\{0\}} |B_a|.$$

Let $\langle W_a : a \in A\rangle$ be such that $2^\kappa = \bigcup_{a\in A} W_a$, $|W_a| = 2^\kappa$ for all $a \in A$, and $W_a \cap W_b = 0$ for all distinct a, b. By Theorem 14.2, $\mathfrak{B}_a$ is isomorphic to a Cs_I $\mathfrak{C}_a$ with base W_a for each $a \in A$; let k_a be an isomorphism from $\mathfrak{B}_a$ onto $\mathfrak{C}_a$. Choose $z \in {}^A({}^I(2^\kappa))$ so that $z_a \in k_a h_a a$ for each $a \in A\setminus\{0\}$. For each $a \in A\setminus\{0\}$ let $X_a = {}^I(2^\kappa)^{z_a}$, and let $X_0 = {}^I(2^\kappa)\setminus\bigcup_{a\in A\setminus\{0\}} X_a$. Note that each X_a is a zero-dimensional element in the Cs_I of all subsets of ${}^I(2^\kappa)$. Since $|I| \le \kappa$, for every $a \in A\setminus\{0\}$ there is a one-one function f_a from W_a onto 2^κ such that $f_a z_a i = z_a i$ for all $i \in I$. Let f_0 be any one-one function from W_0 onto 2^κ. Finally, for all $a \in A$ let

$$ga = \bigcup_{b\in A} \mathrm{rl}_{X_b}\, \widetilde{f}_b k_b h_b a.$$

We claim that g is an isomorphism from $\mathfrak{A}$ onto a Cs_I with infinite base. It is straightforward to check everything except one-one-ness and preservation of C_i. If $a \neq 0$, then $z_a \in \mathrm{rl}_{X_a}\, \widetilde{f}_a k_a h_a a$, showing that g is one-one. To check that g preserves C_i, suppose that $t \in C_i ga$. Say $\alpha \in 2^\kappa$ and $t^i_\alpha \in ga$. Choose $b \in A$ such that $t^i_\alpha \in \mathrm{rl}_{X_b}\, \widetilde{f} k_b h_b a$. In particular, $t^i_\alpha \in X_b$. From the form of the definition of X_b it follows that also $t \in X_b$. Hence $t \in C_i^{[X_b]}\, \mathrm{rl}_{X_b}\, \widetilde{f} k_b h_b a$. Then the fact that all of the functions rl_{X_b}, $\widetilde{f}$, k_b, and h_b are homomorphisms easily yields that $t \in gC_i a$. The converse is similar, so the proof is finished. ■

References

[1] H. Andréka, *Complexity of equations valid in algebras of relations*, thesis for D.Sc., Hungar. Acad. Sci., 1991, 102 pp.

[2] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, 1991.

[3] W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer, 1974.

[4] L. Henkin, J. D. Monk and A. Tarski, *Cylindric Algebras*, *Part I*, with an introductory chapter: *General theory of algebras*, North-Holland, 1971.

[5] —, —, —, *Cylindric Algebras*, *Part II*, North-Holland, 1985.

[6] L. Henkin, J. D. Monk, A. Tarski, H. Andréka and I. Németi, *Cylindric Set Algebras*, Lecture Notes in Math. 883, Springer, 1981.

[7] J. D. Monk (ed.), *Handbook of Boolean Algebras*, North-Holland, 1989.

[8] D. Resek and R. J. Thompson, *Characterizing relativized cylindric algebras*, in: Proc. 1988 Algebraic Logic Conf. Budapest, Colloq. Math. Soc. János Bolyai 54, North-Holland, Amsterdam 1991, 519–538.

[9] I. Sain, *Concerning some cylindric algebra versions of the downward Löwenheim–Skolem theorem*, Notre Dame J. Formal Logic 29 (3) (1988), 332–344.