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A NOTE ON CYLINDRIC LATTICES

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0. Introduction. Besides being of intrinsic interest, cylindric (semi-) lattices arise naturally from the study of dependencies in relational databases; the polynomials on a cylindric semilattice are closely related to the queries obtainable from project-join mappings on a relational database (cf. [D] for references).

This note is intended to initiate the study of these structures, and only a few, rather basic results will be given. Some problems at the end will hopefully stimulate further research. Related issues are discussed in [H], and for further background material the reader is invited to consult [N].

I should like to thank H. Andréka and I. Németi for stimulating discussions on the subject.

1. Definitions and notation. The main references are [HMT], [G] and [N], and any notion not explained in this note can be found there.

Let α be an ordinal. A *cylindric lattice of dimension* α (cl_α) is an algebraic structure $(S, \cdot, +, c_i, 0, 1)_{i<\alpha}$, where for all $x, y \in S$, $i, j < \alpha$,

C0 $(S, \cdot, +, 0, 1)$ is a bounded distributive lattice.

C1 $c_i 0 = 0$.

C2 $x \cdot c_i x = x$.

C3 $c_i(x \cdot c_i y) = c_i x \cdot c_i y$.

C4 $c_i c_j x = c_j c_i x$.

C5 If $x_n \in S$ for $n \in I$ and $\sum_n x_n$ exists, then $\sum_n c_i x_n$ does as well and $c_i(\sum_n x_n) = \sum_n c_i x_n$.

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The paper is in final form and no version of it will be published elsewhere.

The "·" free reduct of a cl_α is called a *cylindric semilattice of dimension* α (csl_α).

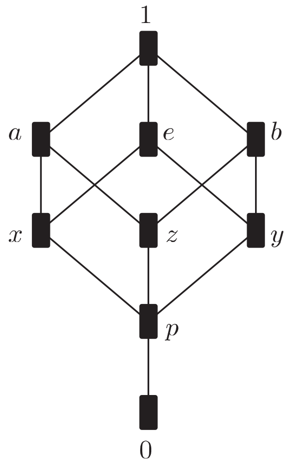
The operations c_i are called *cylindrifications*. It is not hard to see that $c_i c_i x = x$, and so with the aid of C2 we note that the cylindrifications are closure operators.

Note that in the absence of complementation we have to add the distributivity of the c_i over the join as an axiom.

Furthermore, we cannot replace C1–C3 by

1. $c_i 1 = 1$ [HMT, 1.2.2],
2. $c_i c_i x = x$ [HMT, 1.2.3],
3. $x \cdot c_i y = 0$ iff $y \cdot c_i x = 0$ [HMT, 1.2.5],

as is possible in cylindric algebras [HMT, I, p. 177]. The latter three conditions are strictly weaker in our setting: Consider S:



with

	a	b	z	p	x	y	e
c_0	a	1	a	x	x	1	1

Then, S satisfies 1.2.2, 1.2.3, and 1.2.5 of [HMT], but

$$c_0(y \cdot c_0 a) = c_0(y \cdot a) = c_0 p = x \neq a = c_0 y \cdot c_0 a\,.$$

If $n < \omega$, then we define

$$c_{(n)} x = c_{i_0} c_{i_1} \ldots c_{i_{n-1}} x\,.$$

Observe that by C4 the order in which we perform the cylindrifications is irrelevant, so $c_{(n)}$ is well defined.

Throughout this note, S is a distributive lattice and $S^+ = \{x \in S : x > 0\}$.

1.1. Examples. 1. A primary source for cylindric lattices are those cl_n which arise from n-ary relations, where $n < \omega$. Denote by $\mathrm{Re}_n\, U$ the set of all n-ary relations on some set U, i.e. $\mathrm{Re}_n\, U$ is the power set of the set ${}^nU$ of all functions $f : n \to U$. For $X \subseteq {}^nU$ and $i < n$ let

$$c_i X = \{f \in {}^nU : \exists\, g \in X\, \forall\, i \neq j\, [f(j) = g(j)]\}\,.$$

Thus, we obtain $c_i X$ from X by erasing the ith column in the sense that it contains no useful information as it contains all information. In a way, the cylindrifications correspond to projections of a database scheme onto (the complements of) single attributes.

2. If d is the identity function on S, then d is a cylindrification.

3. If s is the function on S with $s0 = 0$ and $s[S^+] = 1$, then s is a cylindrification.

In the rest of this note, d and s will always denote the functions of 2. and 3.

4. If $(S,^*)$ is a pseudocomplemented bounded distributive lattice, then the operation ** is a cylindrification if and only if $(S,^*)$ is a Stone algebra.

Let cls_α (csls_α) be the class of elements of cl_α (csl_α) which are isomorphic to subalgebras of $\mathrm{Re}({}^nU)$ with the appropriate operations; set $\mathrm{clr}_\alpha = \mathrm{ISP}(\mathrm{cls}_\alpha)$ and $\mathrm{cslr}_\alpha = \mathrm{ISP}(\mathrm{csl}_\alpha)$. An element of clr_α (cslr_α) will be called a *representable* cl_α (cls_α).

2. Structural properties. In this section let $n < \omega$ and $S \in \mathrm{cl}_n$. For $a < b \in S$ we denote by $\vartheta_L[a, b]$ the smallest lattice congruence which identifies a and b, and by $\vartheta[a, b]$ the smallest cl_n congruence with this property. It is well known that

$$\vartheta_L[a, b] = \{\langle x, y\rangle \in {}^2S : x \cdot a = y \cdot a,\ x + b = y + b\}\,.$$

Note for later use that for $b = 1$

$$\vartheta_L[a, 1] = \{\langle x, y\rangle \in {}^2S : x \cdot a = y \cdot a\}\,.$$

An element x of S is called *dense* if $y \cdot x > 0$ for all $y > 0$. The set D of all dense elements of S is a filter, appropriately named the *dense filter* (or *set*). If $c_i x = x$, then x is called c_i*-closed.* If x is c_i-closed for all $i < \alpha$, we call x simply *closed.* It is well known (see [HMT]) that the principal ideal generated by a c_i-closed element generates a lattice congruence which preserves c_i.

For things to come it is worthy to record the following slightly more general result:

2.1. Proposition. *Let $S \in \mathrm{cl}_n$, $a, b \in S$, and $a < b$. If*

1. *a and b are c_i-closed, or*
2. *$c_i = s$, and $a \cdot x = 0$ iff $b \cdot x = 0$ for all $x \in S$,*

then $\vartheta_L[a, b]$ preserves c_i.

Proof. Let $\vartheta = \vartheta_L[a, b]$, $x, y \in S$ and $x \equiv y$ (ϑ), i.e. $x \cdot a = y \cdot a$ and $x + b = y + b$.

1. $c_i x \cdot a = c_i x \cdot c_i a = c_i(x \cdot c_i a) = c_i(x \cdot a) = c_i(y \cdot a) = \ldots = c_i y \cdot a$.
   $c_i x + b = c_i x + c_i b = c_i(x + b) = c_i(y + b) = \ldots = c_i y + b$.

2. If $x, y > 0$ or $x = y = 0$, then the conclusion is obvious by $c_i = s$. Assume that $x > 0$ and $y = 0$; then $x \cdot a = y \cdot a = 0$, and thus $x \cdot b = 0$ by our hypothesis. On the other hand, $x + b = y + b = b$, which implies $0 < x < b$, a contradiction. ∎

If A is a nontrivial cylindric algebra of dimension n then the following statements are equivalent (cf. [HMT]):

1. A is simple.
2. A is subdirectly irreducible.
3. For all $x \in A$ with $x > 0$, $c_{(n)} x = 1$.

The next result shows that this is not true in cl_n:

2.2. Proposition. *In* cl_n, 1⇒2⇒3, *and none of these implications can be reversed.*

Proof. 1⇒2 is clear. Let S be subdirectly irreducible, and assume there is some $0 < x < 1$ such that $c_{(n)} x = x$. Let $\vartheta = \vartheta[u, v]$ be the smallest nontrivial congruence on S; we suppose w.l.o.g. that $u < v$. Since $c_{(n)} x = x$, the lattice congruence ψ on S which is induced by the principal ideal $(x]$ also preserves the cylindrifications. Now $\vartheta \subseteq \psi$ implies the existence of some $y \in S$ with $0 < y \le x$ and $u + y = v$. We now have

$$u \cdot y \equiv v \cdot y = y \ (\vartheta)\,,$$

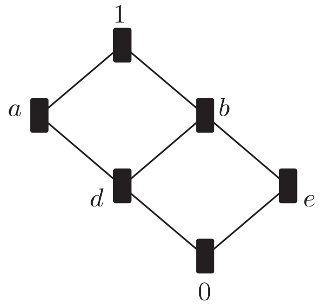
and, since $y \le x$, we may suppose that $u < v \le x$. Set $z = c_{(n)} v$ and note that $z < 1$ since $x < 1$; hence, $\varphi = \vartheta_L[z, 1]$ is not the identity. This congruence also preserves cylindrifications by 2.1.1 above. Since φ is not trivial, we have $\vartheta \subseteq \varphi$, and hence $u \equiv v \ (\varphi)$; but then $u = v$, a contradiction.

The three-element chain with $0 < a < 1$ and cylindrification s is subdirectly irreducible as a cl_1 and a csl_1, but it is not simple. The four-element chain $0 < a < b < 1$ with cylindrification s is a cl_1 which satisfies 3, but it is not subdirectly irreducible: Let ϑ have the classes $\{0\}$, $\{a\}$, $\{b, 1\}$ and ψ the classes $\{0\}$, $\{a, b\}$, $\{1\}$. Then both ϑ and ψ are congruences and their infimum is the identity. ∎

The classes cl_α do not behave well as congruences go, as the next result shows:

2.3. Proposition. cl_1 *is not congruence extensible.*

Proof. Let (S, c) be the distributive lattice on the top of the opposite page, with $c = s$, and let L be the subalgebra $\{0, d, b, a, 1\}$. Let ϑ be the equivalence relation on L with classes $\{a, b, d, 1\}$ and $\{0\}$. It is easily checked that ϑ is a proper cl_1 congruence on L. On the other hand, any congruence ψ on S which



identifies 1 and a is universal, since $0 = a \cdot e \equiv e > 0$ (ψ), and thus

$$0 = s0 \equiv se = 1 \ (\psi)\,.$$

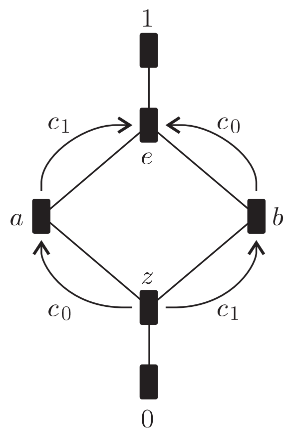
It follows that ϑ cannot be extended to S. ∎

Thus, cl_n is not a discriminator variety, in contrast to CA_n.

3. The classes cl_1 **and** cl_2**.** Proposition 2.2 shows that for a subdirectly irreducible $S \in \mathrm{cl}_1$, $c_0 = s$. This need not be true for a two-dimensional cl:

3.1. Example. *There is some $S \in \mathrm{cl}_2$ such that $c_0 \neq s$ and $c_1 \neq s$.*

Indeed, let S be the following cl_2 with the cylindrifications as indicated by the arrows:



It is straightforward to verify that (S, c_0, c_1) is a cl_2 and that no nontrivial congruence can separate e and 1. ∎

The simple algebras in CA_1 or Df_1 are the Boolean algebras with the cylindrification $c_0 = s$ [HMT, 2.3.15]. The example of the three-element chain in 2.2 shows that in cl_1 this is not enough. However, an additional purely lattice theoretic condition suffices:

3.2. Proposition. *Let $S \in \mathrm{cl}_1$; then S is simple if and only if $c_0 = s$ and for all $b \in S^+$ the only dense element of the sublattice $(b]$ of S is b.*

P r o o f. $\Rightarrow$ Suppose that S is simple; by 2.2 we have $c_0 = s$. Assume that $0 < d < b$ and d is dense in $(b]$. If $x \in S$, then $x \cdot d = 0$ if and only if $x \cdot b = 0$, since d is dense. Hence, $\vartheta_L[d, b]$ is a proper congruence on S by 2.1, a contradiction.

$\Leftarrow$ Suppose that S fulfills the hypotheses, and assume that ϑ is a proper nontrivial congruence on S with $a < b$ and $a \equiv b\ (\vartheta)$. Since $c = s$ and ϑ is proper, we have $0 < a$. Since a is not dense in $(b]$, there is some $0 < x < b$ such that $a \cdot x = 0$. Then, $0 = a \cdot x \equiv b \cdot x\ (\vartheta)$, but $b \cdot x = x > 0$, a contradiction. ∎

Proposition 2.2 also implies that every cl_1 is representable:

3.3. PROPOSITION. $\mathrm{cl}_1 = \mathrm{clr}_1$.

P r o o f. If $(S, c_0) \in \mathrm{cl}_1$ and $c_0 x = 1$ iff $x > 0$, then clearly $S \in \mathrm{cls}_1$. Thus, by 2.2, every subdirectly irreducible $S \in \mathrm{cl}_1$ is in cls_1, and hence $\mathrm{cl}_1 = \mathrm{clr}_1$. ∎

It is well known that every cylindric algebra can be embedded into an algebra whose Boolean part is complete and atomic. At least for cylindric lattices of dimension one we can do the same:

3.4. PROPOSITION. *Every* cl_1 *can be embedded into a* Df_1.

P r o o f. This is similar to [HMT, 2.7.4]. Let $\langle L, c_0 \rangle \in \mathrm{cl}_1$, $S(L)$ be the set of all prime ideals of L, and B_L the power set algebra of $S(L)$. Let $h : L \to B_L$ be defined by

$$h(a) = \{P \in S(L) : a \notin P\}.$$

By the Birkhoff–Stone Theorem, h embeds L into B_L as a $0,1$-distributive lattice. For each $M \in B_L$ define

$$\underline{c}M = \{P \in S(L) : \text{there is some } Q \in M \text{ with } c_0^{-1}Q = c_0^{-1}P\}.$$

Note that this differs from 2.7.4 of [HMT]. It is easily checked that $\underline{c}$ is a completely additive closure operator and that $\underline{c}\emptyset = \emptyset$.

To show C3, let $X, Y \in B_L$ and $P \in \underline{c}X \cap \underline{c}Y$. Then there are $R \in X$, $Q \in Y$ such that $c_0^{-1}P = c_0^{-1}R$ and $c_0^{-1}P = c_0^{-1}Q$, hence, $c_0^{-1}R = c_0^{-1}Q$. It follows that $R \in X \cap \underline{c}Y$, and thus $P \in \underline{c}(X \cap \underline{c}Y)$.

It remains to show that h preserves c_0: Let $a \in L$ and $P \in \underline{c}(h(a))$. Then there is some $Q \in S(L)$ such that $a \notin Q$ and $c_0^{-1}P = c_0^{-1}Q$. From $a \notin Q$ it follows that $c_0 a \notin Q$ and thus $c_0 a \notin P$, i.e. $P \in h(c_0 a)$. Conversely, let $c_0 a \notin P \in S(L)$, and set $I := c_0^{-1}P$. By the additivity of c_0, I is an ideal of L. Now, let F be the filter of L generated by $\{a\} \cup \{c_0 x : x \in L,\ c_0 x \notin P\}$. Noting that the meet of c_0-closed elements is c_0-closed, we see that $F = \{y \in L : \text{there is some } x \in L \text{ such that } c_0 x \notin P \text{ and } a \cdot c_0 x \le y\}$. Assume that $b \in I \cap F$; then there are $x, y \in L$ such that $c_0 x \notin P$ and $a \cdot c_0 x \le b \le c_0 y \in P$. By C3, $c_0 a \cdot c_0 x \le c_0 b \le c_0 y$, thus, $c_0 a \cdot c_0 x \in P$. Since P is prime, $c_0 a \in P$ or $c_0 x \in P$, a contradiction in both cases. Let $Q \in S(L)$ such that $I \subseteq Q$ and $Q \cap F = \emptyset$, in particular, $a \notin Q$ and $c_0^{-1}P = c_0^{-1}Q$. It follows that $P \in \underline{c}(h(a))$. ∎

Note that 3.4 also implies that every cl_1 is representable. For cl_2 we can give the following condition:

3.5. PROPOSITION. *Let $L \in \mathrm{cl}_2$ be subdirectly irreducible and define conditions* $(*)$ *and* $(**)$ *by*

$(*)$ *If $c_0x, c_1y < 1$ then $c_0x + c_1y < 1$.*

$(**)$ *If $x, y, u, v \in L$ such that $c_0y \cdot c_1x \le c_0u + c_1v$, then $c_0y \le c_0u$ or $c_1x \le c_1v$.*

Then L is representable if and only if L satisfies $(*)$ *and* $(**)$.

Proof. $\Rightarrow$ Suppose that L is representable; since it is subdirectly irreducible, it is in fact in cls_2. Thus, we may suppose that L is a subalgebra of the cl_2 of all binary relations on some set U with the cylindrifications as defined in Example 1.1.1. Let $x, y \in L$ such that $y = c_0y < {}^2U$, $x = c_1x < {}^2U$. Then there are $M, N \subseteq U$ such that $M, N \neq U$ and $y = U \times M$, $x = N \times U$. If $a \in U \setminus N$ and $b \in U \setminus M$, then $\langle a, b\rangle \notin (N \times U) \cup (U \times M) = x + y$, and thus $x + y < {}^2U$. Now let x and y be as above and $u = c_0u = U \times A$, $v = c_1v = B \times U$, and $x \cdot y \le u + v$, i.e. $N \times M \subseteq (B \times U) \cup (U \times A)$. Assume that $t \in M \setminus A$ and $s \in N \setminus B$; then $\langle s, t\rangle \in N \times M$, but $\langle s, t\rangle \notin (B \times U) \cup (U \times A)$. Thus, $M \subseteq A$ or $N \subseteq B$, i.e. $y \le u$ or $x \le v$.

$\Leftarrow$ We show that under these conditions L can be embedded as a cl_2 into a simple complete atomic Df_2. By [HMT, 5.1.47] this Df_2 is representable, whence the result follows.

Since L is subdirectly irreducible, $c_0c_1x = c_1c_0x = 1$ for all $x \in L^+$. Let B_L, c_0, c_1 be as in 3.4; all we have to show is that $\underline{c}_0\underline{c}_1M = S(L)$ for all atoms M of B_L (since C4 will then be satisfied), and that $\langle B_L, \underline{c}_0, \underline{c}_1\rangle$ is simple. The latter condition is easily seen to be fulfilled; thus, let $P, Q \in S(L)$. We need to find some $R \in S(L)$ such that $c_0^{-1}R = c_0^{-1}P$ (then $R \in \underline{c}_0\{P\}$) and $c_1^{-1}R = c_1^{-1}Q$ (then $Q \in \underline{c}_1^{-1}\{R\}$). Let I the ideal of L generated by $c_0^{-1}P \cup c_1^{-1}Q$; by $(*)$, I is proper. Let F be the filter of L generated by the c_1-closed elements of L which are not in Q and the c_0-closed elements of L which are not in P. If $c_0y, c_1x \in F$ and $c_0y \cdot c_1x = 0$, then

$$0 = c_0(c_0y \cdot c_1x) = c_0y \cdot c_0c_1x = c_0y \notin P\,,$$

a contradiction; thus, F is proper. Assume that $b \in I \cap F$; then, there are $x, y, u, v \in L$ such that $c_0y \notin P$, $c_1x \notin Q$, $c_0u \in P$, $c_1v \in Q$ and

$$0 < c_0y \cdot c_1x \le b \le c_0u + c_1v\,.$$

By $(**)$, $c_0y \le c_0u$ or $c_1x \le c_1v$, a contradiction in both cases.

Let $R \in S(L)$ such that $I \subseteq R$ and $R \cap F = \emptyset$. Then R is the desired prime ideal of L. ■

The results and discussions above suggest, among others, the following problems:

1. Is $\mathrm{cl}_2 = \mathrm{clr}_2$?

2. For which $\alpha > 1$ is clr_α a variety?
3. For which α is the equational theory of cls_α decidable?

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