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ON ALGEBRAS OF RELATIONS

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Throughout, by relation we mean a binary relation. Let $\mathrm{Rel}(X)$ be the set of all binary relations on the set X. An *algebra of relations* is a pair (Φ, Ω) where Ω is a set of operations on relations and $\Phi \subset \mathrm{Rel}(X)$ is a set of relations closed under the operations of Ω. Each algebra of relations can be considered as ordered by the set-theoretic inclusion $\subset$. Denote by $M\{\Omega\}$ the class of all algebras isomorphic to ones whose elements are relations and whose operations are members of Ω. The class $M\{\Omega, \subset\}$ is determined in the same way.

We will consider the following operations on relations: relation product $\circ$, relation inverse $^{-1}$, intersection $\cap$, diagonal relation Δ, and the unary operation * determined as follows: $\varrho^* = \varrho \cap \Delta$.

The class $M\{\circ, ^{-1}, \cap, \Delta\}$ was introduced and characterized in [6]. It is not finitely axiomatizable [5]. The classes $M\{\circ, ^{-1}, \Delta\}$ and $M\{\circ, ^{-1}, \Delta, \subset\}$ were characterized in [1, 8]. The class $M\{\circ, ^{-1}, \Delta\}$ is not finitely axiomatizable [2].

In this paper we find a system of axioms for the class $M\{\circ, ^{-1}, ^*, \Delta, \subset\}$ and use it to obtain some results about the class $M\{\circ, ^{-1}, \cap, \Delta\}$.

THEOREM 1. *An algebra* $(A, \cdot, ^{-1}, ^*, 1, \leq)$ *belongs to* $M\{\circ, ^{-1}, ^*, \Delta, \subset\}$ *iff it satisfies the following conditions*:

(1) $(A, \cdot, ^{-1}, 1)$ *is an involuted monoid*, *i.e.* $(xy)z = x(yz)$, $1x = x1 = x$, $(x^{-1})^{-1} = x$, $(xy)^{-1} = y^{-1}x^{-1}$.

(2) $\leq$ *is an order relation and all operations are monotonic*, *i.e.* $x \leq y$ *implies* $xz \leq yz$, $zx \leq zy$, $x^{-1} \leq y^{-1}$, $x^* \leq y^*$.

(3) *The following identities are satisfied*:

$$(3.1) \qquad (x^*)^* = x^*,$$

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$$x^* x^* = x^* , \tag{3.2}$$

$$x^* y^* = y^* x^* , \tag{3.3}$$

$$(x^* y^*)^* = x^* y^* , \tag{3.4}$$

$$(xx^{-1})^* x = x , \tag{3.5}$$

$$(xyy^{-1}x^{-1})^* = (x(yy^{-1})^* x^{-1})^* , \tag{3.6}$$

$$x^* \le 1 , \tag{3.7}$$

$$x^* \le x . \tag{3.8}$$

Suppose that an algebra $\underline{A} = (A, \cdot, ^{-1}, \wedge, 1) \in M\{\circ, ^{-1}, \cap, \Delta\}$. Then $\underline{A}$ is a semilattice ordered involuted monoid, i.e. $(A, \cdot, ^{-1}, 1)$ is an involuted monoid, $(A, \wedge, ^{-1})$ is an involuted semilattice and the identity $x(y \wedge z) \le xy \wedge xz$ holds ($\le$ is the natural order of the semilattice) [6]. It is known [3] that $\underline{A}$ also satisfies

$$xy \wedge z \le (x \wedge yz^{-1})y . \tag{4}$$

We say that a semilattice ordered involuted monoid $\underline{A}$ is *weakly representable* if there exists a mapping $F : A \to \mathrm{Rel}(X)$ for some X such that $F(ab) = F(a) \circ F(b)$, $F(a^{-1}) = F(a)^{-1}$, $F(1) = \Delta$, $F(a \wedge 1) = F(a) \cap \Delta$ and $a \le b$ iff $F(a) \subset F(b)$ for all $a, b \in A$.

THEOREM 2. *Suppose that a semilattice ordered involuted monoid satisfies* (4). *Then it is weakly representable.*

The following condition plays an important role in applications of algebras of relations to logic [4]:

$$(\exists u, v)(\forall a)\ u^{-1}u \le 1 \ \&\ v^{-1}v \le 1 \ \&\ a \le u^{-1}v . \tag{5}$$

Consider the condition

$$(\forall a)(\exists u, v)\ u^{-1}u \le 1 \ \&\ v^{-1}v \le 1 \ \&\ a \le u^{-1}v . \tag{6}$$

Obviously, (5) implies (6).

THEOREM 3. *Suppose that a semilattice ordered involuted monoid $\underline{A}$ satisfies* (6). *Then $\underline{A}$ belongs to $M\{\circ, ^{-1}, \cap, \Delta\}$ iff it satisfies* (4).

Proof of Theorem 1. *Necessity*. Consider the ordered algebra of relations of the form $(\Phi, \circ, ^{-1}, ^*, \Delta, \subset)$. It is well known that $(\Phi, \circ, ^{-1}, \Delta)$ is an involuted monoid and the operations $\circ$ and $^{-1}$ are monotonic [1, 8]. Suppose that $\varrho, \pi \in \Phi$. Since $\varrho^*, \pi^* \subset \Delta$, we have $\varrho^* \circ \pi^* = \varrho^* \cap \pi^*$. It follows that (3.1)–(3.4) hold. If $\varrho \subset \pi$ then $\varrho^* = \varrho \cap \Delta \subset \pi \cap \Delta = \pi^*$, i.e. the operation * is monotonic. Since $(\varrho \circ \varrho^{-1})^* \subset \Delta$, we have $(\varrho \circ \varrho^{-1})^* \circ \varrho \subset \varrho$. Conversely, if $(x, y) \in \varrho$ then $(x, x) \in \varrho \circ \varrho^{-1} \cap \Delta = (\varrho \circ \varrho^{-1})^*$ and $(x, y) \in (\varrho \circ \varrho^{-1})^* \circ \varrho$, i.e. $\varrho \subset (\varrho \circ \varrho^{-1})^* \circ \varrho$. Since $(\pi \circ \pi^{-1})^* \subset \pi \circ \pi^{-1}$, we have $(\varrho \circ (\pi \circ \pi^{-1})^* \circ \varrho^{-1})^* \subset (\varrho \circ \pi \circ \pi^{-1} \circ \varrho^{-1})^*$. Conversely, if $(x, x) \in (\varrho \circ \pi \circ \pi^{-1} \circ \varrho^{-1})^*$ then there exist y, z such that $(x, y) \in \varrho$ and $(y, z) \in \pi$, hence $(y, y) \in \pi \circ \pi^{-1} \cap \Delta = (\pi \circ \pi^{-1})^*$ and $(x, y) \in (\varrho \circ (\pi \circ \pi^{-1})^* \circ \varrho^{-1})^*$. Therefore, $(\varrho \circ \pi \circ \pi^{-1} \circ \varrho^{-1})^* \subset (\varrho \circ (\pi \circ \pi^{-1})^* \circ \varrho^{-1})^*$.

Sufficiency. Suppose that $(A, \cdot, ^{-1}, ^{*}, 1, \leq)$ satisfies the conditions of Theorem 1. Put $E(A) = \{a \in A : a^* = a\}$.

LEMMA 1. *$(E(A), \cdot)$ is a semilattice*; $1 \in E(A)$; $(a^{-1})^* = a^*$; $(a^*)^{-1} = a^*$; $a^* \leq b^*$ *iff* $a^*b^* = a^*$; *if* $a \leq b^*$ *then* $a \in E(A)$.

It follows from (3.1)–(3.4) that $(E(A), \cdot)$ is a semilattice. Note that $1^{-1} = 1 \cdot 1^{-1} = (1 \cdot 1^{-1})^{-1} = 1$. Since $1 = (1 \cdot 1^{-1})^* \cdot 1 = 1^* \cdot 1 = 1^*$, we have $1 \in E(A)$. Since $(a^*)^{-1} = ((a^*)^{-1}((a^*)^{-1})^{-1})^*(a^*)^{-1} \leq (((a^*)^{-1})^{-1})^* = (a^*)^* = a^*$ and $a^* = ((a^*)^{-1})^{-1} \leq (a^*)^{-1}$, we have $(a^*)^{-1} = a^*$. Since $(a^*)^{-1} \leq a^{-1}$, we have $(a^{-1})^* = ((a^{-1})^*)^{-1} \leq (a^{-1})^{-1} = a$, hence $(a^{-1})^* \leq a^*$. Analogously, $a^* \leq (a^{-1})^*$, i.e. $(a^{-1})^* = a^*$.

If $a^*b^* = a^*$ then $a^* = a^*b^* \leq b^*$. Conversely, if $a^* \leq b^*$ then $a^* = a^*a^* \leq a^*b^*$. Since $a^*b^* \leq a^*$, we have $a^*b^* = a^*$.

Suppose that $a \leq b^*$. Then $a^* \leq (b^*)^* = b^*$ and $a = (aa^{-1})^*a \leq (aa^{-1})^*b^* \leq (a(b^*)^{-1})^*b^* \leq a^*b^* \leq a^*$. Since $a^* \leq a$, we have $a^* = a$, i.e. $a \in E(A)$. This completes the proof of Lemma 1.

Define the unary operations R and L as follows: $Ra = (aa^{-1})^*$; $La = (a^{-1}a)^*$. Then $R(a^{-1}) = La$ and $Ra = a$, $La = a$ for each $a \in E(A)$. It follows from (3.5) that $Raa = a$ and $aLa = a$. It follows from (3.6) that $R(ab) = R(aRb)$ and $L(ab) = R((ab)^{-1}) = R(b^{-1}a^{-1}) = R(b^{-1}R(a^{-1})) = R(b^{-1}La) = R((Lab)^{-1}) = L(Lab)$.

LEMMA 2. *If $La = Rb$ then $R(ab) = Ra$ and $L(ab) = Lb$.*

Indeed, if $La = Rb$ then $R(ab) = R(aRb) = R(aLa) = Ra$ and $L(ab) = L(Lab) = L(Rbb) = Lb$.

For each $B, C \subset A$ we define $BC = \{bc : b \in B \,\&\, c \in C\}$ and $B \leq C$ iff $(\forall c \in C)(\exists b \in B)\ b \leq c$. Note that if $B \subset C$ then $C \leq B$.

Let $\mathbb{N} = \{0, 1, \ldots, n, \ldots\}$ and let f be a one-to-one mapping from $\mathbb{N}$ onto $\mathbb{N}^5$ ($f(k) = (n_k^1, n_k^2, \ldots, n_k^5)$). Define functions $\varphi, \psi, \alpha, \beta : \mathbb{N} \to \mathbb{N}$ as follows: $\varphi(k) = n_k^1$ if $n_k^1 \leq k$, and $\varphi(k) = k$ otherwise; $\psi(k) = n_k^2$ if $n_k^2 \leq k$, and $\psi(k) = k$ otherwise; $\alpha(k) = n_k^3$; $\beta(k) = n_k^4$. Clearly, for each $p \in \mathbb{N}$ and $(i, j, m, n) \in \mathbb{N}^4$ there exists $k \in \mathbb{N}$ such that $k \geq p$ and $\varphi(k) = i$, $\psi(k) = j$, $\alpha(k) = m$, $\beta(k) = n$.

Suppose that $b_0, \ldots, b_{2n} \in A$. Define subsets $B_{i,j}^k \subseteq A$ for $k = 1, \ldots, n+1$ and $i, j \leq k$ as follows:

(D1) $$B_{0,1}^1 = \{b_0\}, \quad B_{1,0}^1 = \{b_0^{-1}\}, \quad B_{0,0}^1 = \{Rb_0\}, \quad B_{1,1}^1 = \{Lb_0\};$$

(D2) $$B_{i,j}^{k+1} = \bigcup_{p=0}^{k} B_{i,p}^k B_{p,j}^k;$$

(D3) $$B_{i,k+1}^{k+1} = B_{i,\varphi(k)}^{k+1}\{b_{2k-1}\} \cup B_{i,\psi(k)}^{k+1}\{b_{2k}^{-1}\};$$
$$B_{k+1,i}^{k+1} = \{b_{2k-1}^{-1}\}B_{\varphi(k),i}^{k+1} \cup \{b_{2k}\}B_{\psi(k),i}^{k+1};$$

(D4) $$B_{k+1,k+1}^{k+1} = \{Lb_{2k-1}Rb_{2k}\}.$$

LEMMA 3. *$(B^k_{i,j})^{-1} = B^k_{j,i}$ and $B^i_{i,i} \subset B^m_{i,i}$, $B^m_{i,j} \leq B^k_{i,j}$ for $k \leq m$.*

According to (D1), $(B^1_{i,j})^{-1} = B^1_{j,i}$. Suppose that $(B^k_{i,j})^{-1} = B^k_{j,i}$. Then

$$\begin{aligned}
(B^{k+1}_{i,j})^{-1} &= \Big(\bigcup_{p=0}^{k} B^k_{i,p}B^k_{p,j}\Big)^{-1} = \bigcup_{p=0}^{k}(B^k_{i,p}B^k_{p,j})^{-1} \\
&= \bigcup_{p=0}^{k}(B^k_{p,j})^{-1}(B^k_{i,p})^{-1} = \bigcup_{p=0}^{k} B^k_{j,p}B^k_{p,i} = B^{k+1}_{i,j}\,, \\
(B^{k+1}_{i,k+1})^{-1} &= (B^{k+1}_{i,\varphi(k)}\{b_{2k-1}\} \cup B^{k+1}_{i,\psi(k)}\{b^{-1}_{2k}\})^{-1} \\
&= \{b^{-1}_{2k-1}\}B^{k+1}_{\varphi(k),i} \cup \{b_{2k}\}B^{k+1}_{\psi(k),i} = B^{k+1}_{k+1,i}\,, \\
(B^{k+1}_{k+1,k+1})^{-1} &= B^{k+1}_{k+1,k+1}\,.
\end{aligned}$$

Suppose that $B^i_{i,i} \subset B^m_{i,i}$. Then $B^i_{i,i} = B^i_{i,i}B^i_{i,i} \subset B^m_{i,i}B^m_{i,i} \subset B^{m+1}_{i,i}$. If $B^m_{i,j} \leq B^k_{i,j}$ then $B^{m+1}_{i,j} \leq B^m_{i,i}B^m_{i,j} \leq B^i_{i,i}B^m_{i,j} \leq \{1\}B^m_{i,j} = B^m_{i,j}$.

Since $M\{\circ, ^{-1}, ^{*}, \Delta, \subset\}$ is a quasivariety [7], without loss of generality we may suppose that A is countable, i.e. $A = \{a_1, a_2, \ldots, a_n, \ldots\}$.

Using induction for each $d \in A$ we define sequents $b_0, \ldots, b_{2n-1}, b_{2n}, \ldots$ and $r_0, \ldots, r_n, \ldots$ of elements of A and $E(A)$ respectively such that $b_0 = d$ and for all n the following conditions hold:

(a) $B^n_{\varphi(n),\psi(n)} \leq \{a_{\alpha(n)}a_{\beta(n)}\}$;

(b) $Rb = r_i$ and $Lb = r_j$ for each $b \in B^n_{i,j}$ and $B^i_{i,i} = \{r_i\}$.

Base of induction. Put $b_0 = d$ and $r_0 = Rb_0$, $r_1 = Lb_0$.

Inductive step. Suppose that $b_0, \ldots, b_{2m-3}, b_{2m-2}$ and $r_0, \ldots, r_m$ have already been defined and (a), (b) are satisfied for $n = 1, \ldots, m$ and $i, j \leq m$. Put

$$b_{2m-1} = r_{\varphi(m)}a_{\alpha(m)}R(a_{\beta(m)}r_{\psi(m)})\,,$$
$$b_{2m} = L(r_{\varphi(m)}a_{\alpha(m)})a_{\beta(m)}r_{\psi(m)}\,, \qquad r_{m+1} = Lb_{2m-1}$$

if $B^m_{\varphi(m),\psi(m)} \leq \{a_{\alpha(m)}a_{\beta(m)}\}$, and $b_{2m-1} = b$ for some $b \in B^m_{\varphi(m),\psi(m)}$, $b_{2m} = r_{\psi(m)}$, $r_{m+1} = r_{\psi(m)}$, otherwise.

If $b_{2m-1} = b$ for some $b \in B^m_{\varphi(m),\psi(m)}$, $b_{2m} = r_{\psi(m)}$, $r_{m+1} = r_{\psi(m)}$ then $b = bLb = br_{\psi(m)} = b_{2m-1}b_{2m}$, i.e. $B^m_{\varphi(m),\psi(m)} \leq b_{2m-1}b_{2m}$ and $Lb_{2m-1} = Rb_{2m} = r_{m+1}$, hence $B^{m+1}_{m+1,m+1} = \{Lb_{2m-1}Rb_{2m}\} = \{r_{m+1}\}$.

If $B^m_{\varphi(m),\psi(m)} \leq \{a_{\alpha(m)}a_{\beta(m)}\}$, i.e. $b \leq a_{\alpha(m)}a_{\beta(m)}$ for some $b \in B^m_{\varphi(m),\psi(m)}$, then

$$\begin{aligned}
b &= r_{\varphi(m)}br_{\psi(m)} \leq r_{\varphi(m)}a_{\alpha(m)}a_{\beta(m)}r_{\psi(m)} \\
&= r_{\varphi(m)}a_{\alpha(m)}L(r_{\varphi(m)}a_{\alpha(m)})R(a_{\beta(m)}r_{\psi(m)})a_{\beta(m)}r_{\psi(m)} \\
&= r_{\varphi(m)}a_{\alpha(m)}R(a_{\beta(m)}r_{\psi(m)})L(r_{\varphi(m)}a_{\alpha(m)})a_{\beta(m)}r_{\psi(m)} = b_{2m-1}b_{2m}\,,
\end{aligned}$$

i.e. $B^m_{\varphi(m),\psi(m)} \leq b_{2m-1}b_{2m}$.

Since $b \le b_{2m-1}b_{2m}$ for some $b \in B^m_{\varphi(m),\psi(m)}$, we have

$$r_{\varphi(m)} = Rb \le R(b_{2m-1}b_{2m}) \le R(b_{2m-1}Rb_{2m}) \le R(b_{2m-1}) .$$

On the other hand,

$$\begin{aligned} Rb_{2m-1} &= R(r_{\varphi(m)}a_{\alpha(m)}R(a_{\beta(m)}r_{\psi(m)})) \\ &= R(r_{\varphi(m)}R(a_{\alpha(m)}R(a_{\beta(m)}r_{\psi(m)}))) \le R(r_{\varphi(m)}) = r_{\varphi(m)} , \end{aligned}$$

hence $Rb_{2m-1} = r_{\varphi(m)}$. Analogously, $Rb_{2m} = r_{\psi(m)}$. Since $r_{m+1} = Lb_{2m-1}$, we have

$$\begin{aligned} r_{m+1} = Lb_{2m-1} &= L(r_{\varphi(m)}a_{\alpha(m)}R(a_{\beta(m)}r_{\psi(m)})) \\ &= L(L(r_{\varphi(m)}a_{\alpha(m)})R(a_{\beta(m)}r_{\psi(m)})) = R(L(r_{\varphi(m)}a_{\alpha(m)})R(a_{\beta(m)}r_{\psi(m)})) \\ &= R(L(r_{\varphi(m)}a_{\alpha(m)})a_{\beta(m)}r_{\psi(m)}) = Rb_{2m} . \end{aligned}$$

Therefore, using Lemma 2 and the definition (D1)–(D4) we conclude that (b) is satisfied for $i, j \le m+1$.

Put $B_{i,j} = \bigcup\{B^n_{i,j} : n \in \mathbb{N}\}$. Then $B_{i,k}B_{k,j} \le B_{i,j}$.

LEMMA 4. $\{b_0\} \le B_{0,1}$ *and* $\{r_i\} \le B_{i,i}$.

Note that if $b \in B_{i,j}$ then b can be represented as a product of elements $b_1, \ldots, b_m, \ldots$ and $b_0^{-1}, \ldots, b_m^{-1}, \ldots$ This product constructed according to (D1)–(D4) will be called the *canonical form* of b. Let $Q_k(b)$ be the number of occurrences of elements a_{2k-1}, a_{2k} in the canonical form of b and $Q(b) = \max\{k : Q_k(b) > 0\}$.

Suppose that $b \in B_{0,1}$. If $Q(b) = 0$ then according to (D1)–(D4), $b = b_0(b_0^{-1}b_0)^m$ for some m and we have $b_0 = b_0Lb_0 = b_0(b_0^{-1}b_0)^* \le b_0b_0^{-1}b_0 \le \ldots \le b_0(b_0^{-1}b_0)^m = b$. Assume that for every $b \in B_{0,1}$ if $Q(b) \le k$ and $Q_k(b) = p$ then $b_0 \le b$. Suppose that $Q_k(b) = p+1$. According to (D1)–(D4) the following cases are possible:

1) $b = c_1b_{2k-1}b_{2k}c_2$ where $c_1 \in B_{0,\varphi(k)}$ and $c_2 \in B_{\psi(k),1}$. Since $B^k_{\varphi(k),\psi(k)} \le \{b_{2k-1}b_{2k}\}$, i.e. $c \le \{b_{2k-1}b_{2k}\}$ for some $c \in B^k_{\varphi(k),\psi(k)}$, using the inductive assumption we have $b_0 \le c_1cc_2 \le c_1b_{2k-1}b_{2k}c_2 = b$.

2) $b = c_1b_{2k}^{-1}b_{2k-1}^{-1}c_2$ where $c_1 \in B_{0,\psi(k)}$ and $c_2 \in B_{\varphi(k),1}$. This case is analogous to Case 1.

3) $b = c_1b_{2k-1}b_{2k-1}^{-1}c_2$ where $c_1 \in B_{0,\varphi(k)}$ and $c_2 \in B_{\varphi(k),1}$. Since $Lc_1 = Rb_{2k-1} = r_{\varphi(k)}$, using the inductive assumption we have $b_0 \le c_1c_2 = c_1Lc_1c_2 = c_1Rb_{2k-1}c_2 = c_1(b_{2k-1}b_{2k-1}^{-1})^*c_2 \le c_1b_{2k-1}b_{2k-1}^{-1}c_2 = b$.

4) $b = c_1b_{2k}^{-1}b_{2k}c_2$ where $c_1 \in B_{0,\psi(k)}$ and $c_2 \in B_{\psi(k),1}$. This case is analogous to Case 3.

Suppose that $b \in B_{i,i}$. If $Q(b) = 0$ then according to (D1)–(D4), $i = 0$ or $i = 1$. If $i = 0$ then $b = r_0$ or $b = (b_0b_0^{-1})^m$ for some m and we have $r_0 = (r_0)^m = (Rb_0)^m = ((b_0b_0^{-1})^*)^m \le (b_0b_0^{-1})^m$. The case $i = 1$ is analogous. Assume that for every $b \in B_{i,i}$ if $Q(b) \le k$ and $Q_k(b) = p$ then $r_i \le b$. Suppose that $Q_k(b) = p+1$. According to (D1)–(D4) the following cases are possible:

1) $b = c_1 b_{2k-1} b_{2k} c_2$ where $c_1 \in B_{i,\varphi(k)}$ and $c_2 \in B_{\psi(k),i}$. Since $B^k_{\varphi(k),\psi(k)} \le \{b_{2k-1}b_{2k}\}$, i.e. $c \le \{b_{2k-1}b_{2k}\}$ for some $c \in B^k_{\varphi(k),\psi(k)}$, using the inductive assumption we have $r_i \le c_1 c c_2 \le c_1 b_{2k-1} b_{2k} c_2 = b$.

2) $b = c_1 b_{2k}^{-1} b_{2k-1}^{-1} c_2$ where $c_1 \in B_{i,\psi(k)}$ and $c_2 \in B_{\varphi(k),i}$. This case is analogous to Case 1.

3) $b = c_1 b_{2k-1} b_{2k-1}^{-1} c_2$ where $c_1 \in B_{i,\varphi(k)}$ and $c_2 \in B_{\varphi(k),i}$. Since $Lc_1 = Rb_{2k-1} = r_{\varphi(k)}$, using the inductive assumption we have $r_i \le c_1 c_2 = c_1 L c_1 c_2 = c_1 R b_{2k-1} c_2 = c_1 (b_{2k-1} b_{2k-1}^{-1})^* c_2 \le c_1 b_{2k-1} b_{2k-1}^{-1} c_2 = b$.

4) $b = c_1 b_{2k}^{-1} b_{2k} c_2$ where $c_1 \in B_{i,\psi(k)}$ and $c_2 \in B_{\psi(k),i}$. This case is analogous to Case 3.

This completes the proof of Lemma 3.

Define the mapping $F_d : A \to \mathrm{Rel}(\mathbb{N})$ as follows:

$$F_d(a) = \{(i,j) : B_{i,j} \le \{a\}\}\,.$$

Since $(B_{i,j})^{-1} = B_{j,i}$, we have $F_d(a^{-1}) = (F_d(a))^{-1}$. Obviously, $a \le b$ implies $F_d(a) \subset F_d(b)$.

We show that $F_d(ab) = F_d(a) \circ F_d(b)$. If $(i,j) \in F_d(a) \circ F_d(b)$, i.e. $(i,k) \in F_d(a)$ and $(k,j) \in F_d(b)$ for some k, then $B_{i,k} \le \{a\}$ and $B_{k,j} \le \{b\}$, hence $B_{i,j} \le B_{i,k} B_{k,j} \le \{ab\}$, i.e. $(i,j) \in F_d(ab)$. Conversely, suppose that $(i,j) \in F_d(ab)$, i.e. $B_{i,j} \le \{ab\}$. Then $B^p_{i,j} \le \{ab\}$ for some p and there exists $k \ge p$ such that $\varphi(k) = i$, $\psi(k) = j$, $a = a_{\varphi(k)}$, $b = a_{\psi(k)}$. Since $B^k_{\varphi(k),\psi(k)} \le B^p_{\varphi(k),\psi(k)} = B^p_{i,j} \le \{ab\} = \{a_{\alpha(k)} a_{\beta(k)}\}$, we have $b_{2m-1} = r_{\varphi(m)} a_{\alpha(m)} R(a_{\beta(m)} r_{\psi(m)})$, $b_{2m} = L(r_{\varphi(m)} a_{\alpha(m)}) b_{\beta(m)} r_{\psi(m)}$, $r_{m+1} = L b_{2m-1}$, hence $b_{2k-1} \le a$ and $b_{2k} \le b$. Since $b_{2k-1} = r_i b_{2k-1} \in B^k_{i,\varphi(k)} b_{2k-1} \subset B^{k+1}_{i,k+1}$, we have $(i,k+1) \in F_d(b_{2k-1}) \subset F_d(a)$. Analogously, $(k+1,j) \in F_d(b_{2k}) \subset F_d(b)$. Thus, $(i,j) \in F_d(a) \circ F_d(b)$.

We show that $F_d(a^*) = F_d(a) \cap F_d(1)$. Since $a^* \le a$ and $a^* \le 1$, we have $F_d(a^*) \subset F_d(a)$ and $F_d(a^*) \subset F_d(1)$. Conversely, suppose that $(i,j) \in F_d(a) \cap F_d(1)$; then $B_{i,j} \le \{a\}$ and $B_{i,j} \le \{1\}$, hence $B_{i,i} \le B_{i,j}(B_{i,j})^{-1} \le \{a1^{-1}\} = \{a\}$. Since $\{r_i\} \le B_{i,i}$, we have $r_i \le a$, hence $r_i = r_i^* \le a^*$. Since $r_i \in B_{i,i}$, we have $(i,i) \in F_d(r_i) \subset F_d(a^*)$. It now follows from $(i,i) \in F_d(a^*)$ and $(i,j) \in F_d(1)$ that $(i,j) \in F_d(a^*) \circ F_d(1) = F_d(a^* 1) = F_d(a^*)$.

Put $X_d = X \times \{d\}$ and $X = \bigcup\{X_d : d \in A\}$, $F^0_d(a) = \{((i,d),(j,d)) : (i,j) \in F_d(a)\}$ and $F(a) = \bigcup\{F^0_d(a) : d \in A\}$. Obviously $F(ab) = F(a) \circ F(b)$, $F(a)^{-1} = F(a)^{-1}$, $F(a^*) = F(a) \cap F(1)$, and $a \le b$ implies $F(a) \subset F(b)$. Suppose that $F(a) \subset F(b)$; then $F_a(a) \subset F_a(b)$. Since $(0,1) \in F_a(a)$, we have $(0,1) \in F_a(b)$, i.e. $B_{0,1} \le \{b\}$. Then using Lemma 4, we obtain $\{a\} \le B_{0,1} \le \{b\}$, i.e. $a \le b$. Therefore, F is an isomorphism of $(A, \cdot, ^{-1}, \le)$ into $(\mathrm{Rel}(X), \circ, ^{-1}, \subset)$ and $F(a^*) = F(a) \cap F(1)$.

It is clear that $\varepsilon = F(1)$ is an equivalence relation on X. Let $Y = X/\varepsilon$ and let η be the natural mapping of X onto Y. Put $P(a) = \eta \circ F(a) \circ \eta^{-1}$. It is easy to see that P is an isomorphism of $(A, \cdot, ^{-1}, \le)$ into $(\mathrm{Rel}(Y), \circ, ^{-1}, \subset)$ and $P(1) = \Delta$. It follows that $P(a^*) = P(a) \cap P(1) = P(a) \cap \Delta = P(a)^*$. This completes the proof of Theorem 1.

P r o o f o f T h e o r e m s 2 a n d 3. Suppose that $(A, \cdot, ^{-1}, \wedge, 1)$ is a semilattice ordered involuted monoid and (4) holds. Let $\le$ be the canonical order relation of the semilattice $(A, \wedge)$. Put $a^* = a \wedge 1$. Obviously, $a^* \le 1$ and $a^* \le a$.

LEMMA 5. *The operations* $\cdot$, $^{-1}$, * *are monotonic.*

If $a \le b$, i.e. $a \wedge b = a$, then $a^{-1} \wedge b^{-1} = (a \wedge b)^{-1} = a^{-1}$, i.e. $a^{-1} \le b^{-1}$, and $a^* = a \wedge 1 \le b \wedge 1 = b^*$. Also if $a \le b$, i.e. $a \wedge b = a$, then $ac = (a \wedge b)c \le ac \wedge bc \le bc$ and $c(a \wedge b) \le ca \wedge cb \le cb$.

LEMMA 6. $xy \wedge z \le x(y \wedge x^{-1}z)$, $x \le xx^{-1}x$.

Indeed, $xy \wedge z = (y^{-1}x^{-1} \wedge z^{-1})^{-1} \le ((y^{-1} \wedge z^{-1}x)x^{-1})^{-1} = x(y \wedge xz)$, $x = x1 \wedge x \le x(1 \wedge x^{-1}x) \le xx^{-1}x$.

LEMMA 7. $(x^{-1})^* = x^*$, $x^*x^* = x^*$, $(x^*)^{-1} = x^*$.

Indeed, $x^* = x \wedge 1 = (x \wedge 1)1 \wedge 1 \le (x \wedge 1)(1 \wedge (x \wedge 1)^{-1}1) \le (x \wedge 1)^{-1} = x^{-1} \wedge 1 = (x^{-1})^*$ and $(x^{-1})^* \le ((x^{-1})^{-1})^* = x^*$. The second assertion follows from

$$x^*x^* = (x \wedge 1)(x \wedge 1) \le (x \wedge 1)1 \le x \wedge 1 = x^*$$

and

$$\begin{aligned} x^* = x \wedge 1 &= (x \wedge 1)1 \wedge 1 \le (x \wedge 1)(1 \wedge (x \wedge 1)^{-1}1) \\ &\le (x \wedge 1)(1 \wedge x^{-1}) = x^*(x^{-1})^* = x^*x^* . \end{aligned}$$

Finally, $(x^*)^{-1} = (x \wedge 1)^{-1} = x^{-1} \wedge 1^{-1} = x^{-1} \wedge 1 = (x^{-1})^* = x^*$.

LEMMA 8. $x^*y^* = x^* \wedge y^*$, $x^*y^* = y^*x^*$, $(x^*y^*)^* = x^*y^*$.

Since $x^*y^* \le x^*1 = x^*$ and $x^*y^* \le 1y^* = y^*$, we have $x^*y^* \le x^* \wedge y^*$. Conversely, $x^* \wedge y^* = x^*1 \wedge y^* \le x^*(1 \wedge (x^*)^{-1}y^*) \le x^*x^*y^* = x^*y^*$. Thus, $x^*y^* = x^* \wedge y^*$. It follows that $x^*y^* = x^* \wedge y^* = y^* \wedge x^* = y^*x^*$ and $(x^*y^*)^* = x^*y^* \wedge 1 = x^* \wedge y^* \wedge 1 = x^* \wedge y^* = x^*y^*$.

LEMMA 9. $(xyy^{-1}x^{-1})^* \le (xx^{-1})^*$, $(xx^{-1})^*x = x$.

Indeed, $(xyy^{-1}x^{-1})^* = xyy^{-1}x^{-1} \wedge 1 = (xyy^{-1}x^{-1} \wedge 1) \wedge 1 \le x(yy^{-1}x^{-1} \wedge x^{-1}1) \wedge 1 \le xx^{-1} \wedge 1 = (xx^{-1})^*$. For the second assertion, $(xx^{-1})^*x \le 1x = x$ and $x = 1x \wedge x \le (1 \wedge xx^{-1})x = (xx^{-1})^*x$.

LEMMA 10. $(xyy^{-1}x^{-1})^* = (x(yy^{-1})^*x^{-1})^*$.

Indeed,

$$\begin{aligned} (xyy^{-1}x^{-1})^* &= (x((yy^{-1})^*y)((yy^{-1})^*y)^{-1}x^{-1})^* \\ &= (x(yy^{-1})^*yy^{-1}((yy^{-1})^*)^{-1}x^{-1})^* \\ &\le (x(yy^{-1})^*((yy^{-1})^*)^{-1}x^{-1})^* \\ &= (x(yy^{-1})^*(yy^{-1})^*x^{-1})^* = (x(yy^{-1})^*x^{-1})^* \end{aligned}$$

and

$$(x(yy^{-1})^* x^{-1})^* \leq (xyy^{-1}x^{-1})^* .$$

According to Lemmas 5–10, $(A, \cdot, ^{-1}, ^*, 1, \leq)$ satisfies the conditions of Theorem 1. This immediately implies the conclusion of Theorem 2.

LEMMA 11. *$x \wedge y(z \wedge tz) \leq y(y^{-1}xz^{-1} \wedge t)z$.*

Indeed,

$$y(z \wedge tz) \wedge x \leq y(z \wedge tz \wedge y^{-1}x)$$
$$\leq y((t \wedge zz^{-1})z \wedge y^{-1}x) \leq y(t \wedge zz^{-1} \wedge y^{-1}xz^{-1})z \leq y(t \wedge y^{-1}xz^{-1})z .$$

LEMMA 12. *If $x^{-1}x \leq 1$ then $x(y \wedge z) = xy \wedge xz$ and $(y \wedge z)x^{-1} = yx^{-1} \wedge zx^{-1}$.*

Indeed, $x(y \wedge z) \leq xy \wedge xz$ and $xy \wedge xz \leq x(y \wedge x^{-1}xz) \leq x(y \wedge 1z) = x(y \wedge z)$. The proof of the second assertion is similar.

LEMMA 13. *If $u^{-1}u \leq 1$, $v^{-1}v \leq 1$ and $x \wedge y \leq u^{-1}v$, then $x \wedge y = u^{-1}(uxv^{-1})^*(uyv^{-1})^*v$.*

Since $u^{-1}u \leq 1$ and $v^{-1}v \leq 1$, we have $u^{-1}(uxv^{-1})^*(uyv^{-1})^*v \leq u^{-1}(uxv^{-1})^*v \leq u^{-1}uxv^{-1}v \leq x$. Analogously, $u^{-1}(uxv^{-1})^*(uyv^{-1})^*v \leq y$. Thus, $u^{-1}(uxv^{-1})^*(uyv^{-1})^*v \leq x \wedge y$. Conversely, using Lemmas 11, 12, we obtain

$$x \wedge y = x \wedge y \wedge u^{-1}v$$
$$= x \wedge y \wedge u^{-1}(v \wedge 1v) = u^{-1}(u(x \wedge y)v^{-1} \wedge 1)v$$
$$= u^{-1}(uxv^{-1} \wedge uyv^{-1} \wedge 1)v = u^{-1}((uxv^{-1} \wedge 1) \wedge (uyv^{-1} \wedge 1))v$$
$$= u^{-1}((uxv^{-1})^* \wedge (uyv^{-1})^*)v = u^{-1}((uxv^{-1})^*(uyv^{-1})^*)v .$$

LEMMA 14. *Suppose that $\varrho, \pi, \alpha, \beta \in \mathrm{Rel}(X)$ and $\alpha^{-1} \circ \alpha \subset \Delta$, $\beta^{-1}\beta \subset \Delta$, $\varrho \cap \pi \subset \alpha^{-1} \circ \beta$. Then*

$$\varrho \cap \pi = \alpha^{-1} \circ (\alpha \circ \varrho \circ \beta^{-1})^* \circ (\alpha \circ \pi \circ \beta^{-1})^* \circ \beta .$$

It follows from $\alpha^{-1} \circ \alpha \subset \Delta$ and $\beta^{-1} \circ \beta \subset \Delta$ that α, β are functions. Therefore, we write $y = \alpha(x)$ and $y = \beta(x)$ instead of $(x, y) \in \alpha$ and $(x, y) \in \beta$. Since $\varrho \cap \pi \subset \alpha^{-1} \circ \beta$, for each pair $(x, y) \in \varrho \cap \pi$ there exists $z \in X$ such that $\alpha(z) = x$ and $\beta(z) = y$.

Suppose that $(x, y) \in \alpha^{-1} \circ (\alpha \circ \varrho \circ \beta^{-1})^* \circ (\alpha \circ \pi \circ \beta^{-1})^* \circ \beta$. Then $x = \alpha(z)$ and $y = \beta(z)$ for some z such that $(z, z) \in (\alpha \circ \varrho \circ \beta^{-1})^* \circ (\alpha \circ \pi \circ \beta^{-1})^*$. It follows that $(z, z) \in \alpha \circ \varrho \circ \beta^{-1}$ and $(z, z) \in \alpha \circ \pi \circ \beta^{-1}$, hence $(x, y) = (\alpha(z), \beta(z)) \in \varrho \cap \pi$. Conversely, let $(x, y) \in \varrho \cap \pi$. Since $x = \alpha(z)$ and $y = \beta(z)$ for some z, we have $(z, z) \in (\alpha \circ \varrho \circ \beta^{-1})^*$ and $(z, z) \in (\alpha \circ \pi \circ \beta^{-1})^*$, hence $(z, z) \in (\alpha \circ \varrho \circ \beta^{-1})^* \circ (\alpha \circ \pi \circ \beta^{-1})^*$ and $(x, y) = (\alpha(z), \beta(z)) \in \alpha^{-1} \circ (\alpha \circ \varrho \circ \beta^{-1})^* \circ (\alpha \circ \pi \circ \beta^{-1})^* \circ \beta$, which completes the proof of Lemma 14.

Theorem 2 and Lemmas 13, 14 immediately imply Theorem 3.

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