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BOUNDARY VALUE PROBLEM FOR FUCHSIAN PARTIAL DIFFERENTIAL EQUATIONS AND REFLECTION OF SINGULARITIES

TOSHINORI ÔAKU

Department of Mathematics, Yokohama City University
22-2 Seto, Kanazawa-ku, Yokohama, 236 Japan

Introduction. We study linear partial differential equations of Fuchsian type with respect to a hypersurface in $\mathbb{R}^n$. The notion of Fuchsian equations was introduced by Baouendi–Goulaouic [1] and is almost equivalent to the notion of equations with regular singularities in the weak sense defined by Kashiwara–Oshima [2]. We note here that these notions have been generalized to systems and to submanifolds of arbitrary codimension by Oshima [10] and Laurent–Monteiro Fernandes [3].

First, we present a method for defining boundary values of hyperfunction solutions to a Fuchsian partial differential equation with respect to a hypersurface. For single partial differential equations with regular singularities (in the weak sense), there have been two methods to define boundary values of hyperfunction solutions: one is a very general but rather abstract method of Kashiwara–Oshima [2], and the other is a very concrete but rather restrictive (the characteristic exponents must be constant) method by Oshima [9]. Our method is close to the latter, but not restrictive.

Our main purpose is to study the reflection of singularities of hyperfunction solutions to a Fuchsian linear partial differential equation under some boundary condition. One of the most typical examples is the equation

$$x_1\left(\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}\right)+a_1(x)\frac{\partial u}{\partial x_1}+a_2(x)\frac{\partial u}{\partial x_2}+b(x)u=0$$

in $\mathbb{R}^2 \ni x=(x_1,x_2)$, where a_1, a_2, b are real-analytic functions and u is a hyperfunction on $x_1>0$ satisfying a certain boundary condition on $x_1=0$. We give some sufficient conditions for the reflection of (analytic) singularities of u at the boundary $x_1=0$.

We can also treat, e.g., the following (rather special) equations:

$$\frac{\partial^2 u}{\partial x_1^2} - x_1^k \frac{\partial^2 u}{\partial x_2^2} = 0, \qquad x_1 \frac{\partial^2 u}{\partial x_1^2} - x_1^k \frac{\partial^2 u}{\partial x_2^2} + a_1(x_2) \frac{\partial u}{\partial x_1} = 0$$

with an integer $k \ge 0$ and a real-analytic function $a_1(x_2)$.

When the boundary is non-characteristic, such problems have been studied by Lax–Nirenberg [4] for the C^∞-singularity and by Schapira [11] for the analytic singularity. But it seems that there have been no results for degenerate equations including the above three examples.

1. Boundary values of hyperfunction solutions of Fuchsian partial differential equations

1.1. *Fuchsian partial differential equations and their formal elementary solutions following Tahara.* First let us recall the notion of Fuchsian operators with respect to a hypersurface in accordance with [1]. Let $x = (x_1, x_2, \ldots, x_n)$ be the coordinate of the n-dimensional real Euclidean space $M = \mathbb{R}^n$. We also write $x = (x_1, x')$ with $x' = (x_2, \ldots, x_n)$ and use the notation

$$D_j = D_{x_j} = \frac{\partial}{\partial x_j}, \qquad D = (D_1, D_2, \ldots, D_n) = (D_1, D'), \qquad D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\mathbb{N} = \{0, 1, 2, \ldots\}$. For the sake of simplicity we always assume that the hypersurface is $N = \{x = (x_1, x') \in \mathbb{R}^n \mid x_1 = 0\}$. (Since the definitions and properties we shall study are of local character, we can deform any real-analytic hypersurface to a hyperplane.) Let us consider a differential operator P (we always assume that a differential operator has real (or complex) analytic functions as its coefficients). P is said to be a *Fuchsian partial differential operator of weight* $m - k$ *with respect to* N (around $x^0 \in N$) if $0 \le k \le m$ and if, on a neighborhood of x^0, P is written in the form

$$P = P(x, D_x) = a(x)\Big(x_1^k D_1^m + \sum_{j=1}^{m} A_j(x, D') x_1^{\max(0, k-j)} D_1^{m-j}\Big), \tag{1}$$

where $A_j(x, D')$ is a linear partial differential operator of order at most j for $j = 1, \ldots, m$; $A_j(0, x', D')$ equals a function $a_j(x')$ for $j = 1, \ldots, k$; $a(x')$ is a real-analytic function with $a(x^0) \ne 0$. Then the *characteristic exponents* $0, 1, \ldots, m - k - 1, \lambda_j(x^0)$ $(j = 1, \ldots, k)$ of P at x^0 are defined as the roots of the *indicial equation*

$$\prod_{\nu=0}^{m-1} (\lambda - \nu) + \sum_{j=1}^{k} a_j(x^0) \prod_{\nu=0}^{m-j-1} (\lambda - \nu) = 0$$

in λ. In particular we call $\lambda_j(x^0)$ $(j = 1, \ldots, k)$ the *non-trivial characteristic exponents* of P.

First we briefly review the theory of elementary solutions in formal differential operators following H. Tahara [12]. We work in the complex Euclidean space $X =$

$\mathbb{C}^n$ and put $Y = \{z = (z_1, z') \in X \mid z_1 = 0\}$. Let $\widetilde{X} = \mathbb{C}^n \times \mathbb{C}^{n-1} \ni (z, w')$ with $z = (z_1, z') = (z_1, z_2, \dots, z_n)$ and $w' = (w_2, \dots, w_n)$. Let $\widetilde{Y}$ be the hypersurface of $\widetilde{X}$ defined by $z_1 = 0$. Then we denote by $\mathcal{O}_{\widetilde{X}}$ the sheaf of holomorphic functions on $\widetilde{X}$, and by $\mathcal{O}_{\widetilde{X}}|_{\widetilde{Y}}$ the sheaf theoretic restriction of $\mathcal{O}_{\widetilde{X}}$ to $\widetilde{Y}$. For $\varepsilon > 0$, we put

$$W_{\varepsilon,j} := \{(0, z', w') \in \widetilde{Y} \mid |z'| < \varepsilon,\ |w'| < \varepsilon,\ z_k \neq w_k \text{ if } 2 \le k \le n \text{ and } k \neq j\}\,,$$
$$W_\varepsilon := \bigcap_{j=2}^{n} W_{\varepsilon,j}\,.$$

Then the set of the formal differential operators at $0 \in Y$ is defined as

$$\widetilde{\mathcal{D}}_0 = \varinjlim \Big(\mathcal{O}_{\widetilde{X}}|_{\widetilde{Y}}(W_\varepsilon) \Big/ \sum_{j=2}^{n} \mathcal{O}_{\widetilde{X}}|_{\widetilde{Y}}(W_{\varepsilon,j})\Big)\,,$$

where the inductive limit is taken with respect to ε tending to 0. Hence a holomorphic function $K(z, w')$ defined on a neighborhood in $\widetilde{X}$ of

$$\{(0, z', w') \mid |z'| < \varepsilon,\ |w'| < \varepsilon,\ z_k \neq w_k\ (2 \le k \le n)\}$$

for some $\varepsilon > 0$ defines a formal differential operator $\mathcal{K} = [K(z, w')] \in \widetilde{\mathcal{D}}_0$. Here $K(z, w')$ is called a *kernel function* of $\mathcal{K}$.

For any point $p = (0, z'^0)$ of Y, we define $\widetilde{\mathcal{D}}_p$ by replacing the inequalities $|z'| < \varepsilon$ and $|w'| < \varepsilon$ in the definition of $W_{\varepsilon,j}$ by $|z' - z'^0| < \varepsilon$ and $|w' - z'^0| < \varepsilon$ respectively. Thus the sheaf $\widetilde{\mathcal{D}}$ on Y of rings of formal differential operators is defined so that its stalk at p coincides with $\widetilde{\mathcal{D}}_p$.

The sheaf $\widetilde{\mathcal{D}}$ operates on $\mathcal{O}_X|_Y$ naturally. For example, if $K(z, w')$ is an element of $\mathcal{O}_{\widetilde{X}}|_{\widetilde{Y}}(W_\varepsilon)$ and $f(z)$ is holomorphic on a neighborhood of $\{(0, z') \in Y \mid |z'| < \varepsilon\}$ in X, then we have

$$([K(z, w')]f)(z) = \int_{\gamma(z)} K(z, w') f(z_1, w')\, dw' \in \mathcal{O}_X|_Y(\{(0, z') \mid |z'| < \varepsilon\})\,,$$

where $\gamma(z)$ is an $(n-1)$-cycle defined by $\{w' \in \mathbb{C}^{n-1} \mid |w_k - z_k| = \delta\ (2 \le k \le n)\}$ with a sufficiently small $\delta > 0$.

For a linear partial differential operator $P = P(z, D_z)$ with holomorphic coefficients, the notion of Fuchsian operator is defined similarly with N replaced by Y. In order to define the boundary values of hyperfunction solutions of Fuchsian partial differential equations, we use the following deep results obtained by H. Tahara [12].

Proposition 1 (Tahara [12, Corollary 1.2.13 and Theorem 1.2.14]). *Let $P = P(z, D_z)$ be a Fuchsian partial differential operator of weight $m-k$ with respect to Y defined on a neighborhood of $0 \in X$. Assume that its characteristic exponents $\lambda_1(0), \dots, \lambda_k(0)$ at 0 satisfy $\lambda_i(0) \notin \mathbb{Z}$ $(i = 1, \dots, k)$ and $\lambda_i(0) - \lambda_j(0) \notin \mathbb{Z}$ if $i \neq j$. Then there are unique elements $[K_i(z, w')]$ $(i = 0, 1, \dots, m-k-1)$ and*

$[L_j(z,w')]$ $(j=1,\dots,k)$ *of* $\widetilde{\mathcal{D}}_0$ *(formal elementary solutions) such that*

$$\begin{array}{rl} P(z,D_z)\,[K_i(z,w')] = 0 & (i=0,\dots,m-k-1),\\ P(z,D_z)\,[L_j(z,w')] = 0 & (j=1,\dots,k),\\ [D_{z_1}^l K_i(0,z',w')] = \delta_{i,l}\delta(z'-w') & (i,l=0,\dots,m-k-1),\\ [L_j(0,z',w')] = \delta(z'-w') & (j=0,\dots,m-k-1), \end{array}$$

where $\delta_{i,l}$ *denotes Kronecker's* δ, *and*

$$\delta(z'-w') = \left[\frac{1}{(-2\pi\sqrt{-1})^{n-1}}\frac{1}{(z_2-w_2)\dots(z_n-w_n)}\right].$$

PROPOSITION 2 (Tahara [12, Theorems 1.3.8 and 1.3.9]). *Let* P *be a Fuchsian partial differential operator of the form* (1) *satisfying the same condition as in Proposition* 1. *Let* $K_i(z,w')$ *and* $L_j(z,w')$ *be kernel functions of the formal elementary solutions of* P *constructed in Proposition* 1. *Put*

$$A := \begin{pmatrix} 0 & z_1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 0 & z_1 & & & & \\ & & & 0 & 1 & & & \\ & & & & 1 & 1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & k-1 & 1 \\ -z_1A_m & \dots & \dots & -z_1A_{k+1} & -A_k & \dots & -A_2 & -A_1+k \end{pmatrix},$$

$$U(z,w') := (\vec{K}_0(z,w')\ \dots\ \vec{K}_{m-k-1}(z,w')\ \vec{L}_1(z,w')\ \dots\ \vec{L}_k(z,w'))$$

with

$$\vec{K}_i(z,w') := \begin{pmatrix} K_i(z,w') \\ D_{z_1}K_i(z,w') \\ \vdots \\ D_{z_1}^{m-k-1}K_i(z,w') \\ z_1D_{z_1}^{m-k}K_i(z,w') \\ \vdots \\ z_1^kD_{z_1}^{m-1}K_i(z,w') \end{pmatrix} \qquad (i=0,\dots,m-k-1),$$

$$\vec{L}_j(z,w') := \begin{pmatrix} z_1^{m-k-1}L_j(z,w') \\ z_1^{m-k-2}\Lambda_1(\lambda_j(w'))L_j(z,w') \\ \vdots \\ \Lambda_{m-k-1}(\lambda_j(w'))L_j(z,w') \\ \Lambda_{m-k}(\lambda_j(w'))L_j(z,w') \\ \vdots \\ \Lambda_{m-1}(\lambda_j(w'))L_j(z,w') \end{pmatrix} \qquad (j=1,\dots,k)$$

where $\lambda_1(z'), \dots, \lambda_k(z')$ are the non-trivial characteristic exponents of P, and

$$\begin{aligned}&\Lambda_i(\lambda_j(w'))\\ &\quad := (z_1 D_{z_1} + \lambda_j(w'))(z_1 D_{z_1} + \lambda_j(w') - 1) \dots (z_1 D_{z_1} + \lambda_j(w') - i + 1)\,.\end{aligned}$$

Then the differential equation $Pu = 0$ is equivalent to the system of differential equations $(z_1 D_{z_1} - A)\vec{u} = 0$ by the correspondence

$$\vec{u} = \begin{pmatrix} u \\ D_{z_1} u \\ \vdots \\ D_{z_1}^{m-k-1} u \\ z_1 D_{z_1}^{m-k} u \\ \vdots \\ z_1^k D_{z_1}^{m-1} u \end{pmatrix}$$

and $\mathcal{U} := [U(z,w')]$ is an invertible matrix with elements in $\widetilde{\mathcal{D}}_0$ satisfying

$$\begin{aligned}&\mathcal{U}^{-1}(z_1 D_{z_1} - A(z, D_z))\mathcal{U}\\ &\quad = z_1 D_{z_1} - \begin{pmatrix} 0 &&&&& \\ & \ddots &&&& \\ && 0 &&& \\ &&& \lambda_1(z') - m + k + 1 && \\ &&&& \ddots & \\ &&&&& \lambda_k(z') - m + k + 1 \end{pmatrix}.\end{aligned}$$

Example (the Euler–Poisson–Darboux equation). Put $n = 2$ and

$$P = z_1(D_{z_1}^2 - D_{z_2}^2) + aD_{z_1} + bD_{z_2}$$

with constants $a, b \in \mathbb{C}$. Then P is a Fuchsian operator of weight 1 and the non-trivial characteristic exponent is $1 - a$. Assume $a \notin \mathbb{Z}$. Then the formal elementary solutions of P are given by

$$\begin{aligned}K_0(z_1, z_2, w_2) &= \frac{-1}{2\pi\sqrt{-1}} \frac{1}{z_1 + z_2 - w_2} F\left(1, \frac{a+b}{2}, a; \frac{2z_1}{z_1 + z_2 - w_2}\right),\\ L_1(z_1, z_2, w_2) &= \frac{-1}{2\pi\sqrt{-1}} \frac{1}{z_1 + z_2 - w_2} F\left(1, 1 - \frac{a-b}{2}, 2 - a; \frac{2z_1}{z_1 + z_2 - w_2}\right),\end{aligned}$$

where F denotes the Gauss hypergeometric function. We remark that in this case F can be explicitly given by

$$F(1, \beta, \gamma; z) = (\gamma - 1) z^{1-\gamma} (1 - z)^{\gamma - \beta - 1} \int_0^z \tau^{\gamma - 2} (1 - \tau)^{\beta - \gamma} \, d\tau$$

for $|z| < 1$ if $\mathrm{Re}(\gamma) > 1$.

1.2. *Boundary values of hyperfunction solutions to Fuchsian partial differential equations.* We use the notation of 1.1. Put $M_+ = \{x \in M \mid x_1 > 0\}$. Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on M. We define the sheaf $\mathcal{B}_{N|M_+}$ on N so that its stalk at $p = (0, x'^0) \in N$ is

$$(\mathcal{B}_{N|M_+})_p = \varinjlim \mathcal{B}_M(\{x \in M_+ \mid |x-p| < \varepsilon\}),$$

where the inductive limit is taken with respect to $\varepsilon > 0$ tending to 0.

In [7], we constructed a sheaf $\widetilde{\mathcal{B}}_{N|M_+}$ on N containing $\mathcal{B}_{N|M_+}$ as a subsheaf such that the sheaf $\widetilde{\mathcal{D}}$ of formal differential operators operates on $\widetilde{\mathcal{B}}_{N|M_+}$.

Let P be a Fuchsian partial differential operator of weight $m-k$ with respect to N defined on a neighborhood of $x^0 \in N$. Assume that its characteristic exponents $\lambda_1(x^0), \ldots, \lambda_k(x^0)$ at x^0 satisfy $\lambda_i(x^0) \notin \mathbb{Z}$ and $\lambda_i(x^0) - \lambda_j(x^0) \notin \mathbb{Z}$ if $i \neq j$. Let $\mathcal{K}_i = [K_i(z, w')]$ and $\mathcal{L}_j = [L_j(z, w')]$ be the formal elementary solutions of P. Then by Proposition 2, any element $u(x)$ of $\widetilde{\mathcal{B}}_{N|M_+}$ satisfying $Pu = 0$ is written in the form

$$u(x) = \sum_{i=0}^{m-k-1} \mathcal{K}_i u_i(x') + \sum_{j=1}^{k} \mathcal{L}_j(x_1^{\lambda_j(x')} v_j(x'))$$

as an identity in $\widetilde{\mathcal{B}}_{N|M_+}$ with unique hyperfunctions

$$u_0(x'), \ldots, u_{m-k-1}(x'), \quad v_1(x'), \ldots, v_k(x')$$

on a neighborhood of x_0 in N. In particular, for a hyperfunction solution $u(x) \in \mathcal{B}_{N|M_+}$ of $Pu = 0$, the regular and the singular boundary values

$$\gamma_{+\,\mathrm{reg}}(u) = \begin{pmatrix} u_0(x') \\ \vdots \\ u_{m-k-1}(x') \end{pmatrix} \in (\mathcal{B}_N)^{m-k}, \quad \gamma_{+\,\mathrm{sing}}(u) = \begin{pmatrix} v_1(x') \\ \vdots \\ v_k(x') \end{pmatrix} \in (\mathcal{B}_N)^k$$

are defined as above. We call $\gamma_{+\,\mathrm{reg}}(u)$ and $\gamma_{+\,\mathrm{sing}}(u)$ the boundary values of $u(x)$.

2. Main results on reflection of singularities. We use the same notation as in the preceding section and define the purely imaginary cosphere bundles of M and N by

$$\sqrt{-1}S^*M = M \times \sqrt{-1}S^{n-1} = \{(x, \sqrt{-1}\langle \xi, dx \rangle \infty) \mid x \in M,\ \xi \in \mathbb{R}^n \setminus \{0\}\},$$
$$\sqrt{-1}S^*N = N \times \sqrt{-1}S^{n-2} = \{(0, x', \sqrt{-1}\langle \xi', dx' \rangle \infty) \mid x', \xi' \in \mathbb{R}^{n-1} \setminus \{0\}\},$$

where $\langle \xi, dx \rangle = \sum_{j=1}^n \xi_j dx_j$, and the symbol ∞ denotes taking cosets in $S^{n-1} = \mathbb{R}^n/\mathbb{R}_+$ with $\mathbb{R}_+$ being the set of positive real numbers.

We consider a linear partial differential operator P with analytic coefficients of the form

$$P(x, D) = x_1 P_m(x, D) + Q(x, D)$$

defined on a neighborhood (in M) of a point x^0 of N. We assume

(A-1) $P_m(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)D^\alpha$ is a linear partial differential operator of order m and N is non-characteristic for P at x^0 (i.e. $a_{(m,0,\dots,0)}(x^0) \neq 0$).

(A-2) $Q(x,D) = \sum_{|\alpha|\le m-1} b_\alpha(x)D^\alpha$ is a linear partial differential operator of order $m-1$.

Then P is a Fuchsian linear partial differential operator of weight $m-1$ with respect to N in the sense of Baouendi–Goulaouic [1]. The non-trivial characteristic exponent of P at x^0 is

$$\lambda(x^0) = m - 1 - \frac{b_{(m-1,0,\dots,0)}(x^0)}{a_{(m,0,\dots,0)}(x^0)}\,.$$

We assume

(A-3) $\lambda(x^0) \notin \mathbb{Z}\,.$

Let $u(x)$ be a hyperfunction on $U \cap M_+$ satisfying $Pu(x) = 0$ with a sufficiently small open neighborhood U of x^0 in M. Then the boundary values of $u(x)$

$$\gamma_{+\,\mathrm{reg}}(u) \in \mathcal{B}_N(U\cap N)^{m-1}, \qquad \gamma_{+\,\mathrm{sing}}(u) \in \mathcal{B}_N(U\cap N)$$

have been defined in the preceding section. Put

$$p_m(x,\xi) = \sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha\,, \qquad q_{m-1}(x,\xi) = \sum_{|\alpha|=m-1} b_\alpha(x)\xi^\alpha$$

and let $x^* = (x^0, \sqrt{-1}\langle \xi'^0, dx'\rangle\infty)$ be a point of $\sqrt{-1}S^*N$. Assume

(A-4) P_m is microlocally strictly hyperbolic at x^*; i.e., the roots of the equation $p_m(x,\zeta,\xi') = 0$ in ζ are all real and distinct if $x \in M$ with $|x - x^0| < \varepsilon$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\xi' - \xi'^0| < \varepsilon$ for some $\varepsilon > 0$.

Then there are m bicharacteristics $b_j(x^*)$ $(j = 1,\dots,m)$ of P_m issuing from

$$x^{*(j)} = (x^0, \sqrt{-1}(\xi_1^{(j)}dx_1 + \langle \xi'^0, dx'\rangle)\infty) \in \sqrt{-1}S^*M|_N\,,$$

where $\zeta = \xi_1^{(j)}$ $(j = 1,\dots,m)$ are the roots of $p_m(x^0,\zeta,\xi'^0) = 0$. (We may assume $\xi_1^{(0)} < \xi_1^{(1)} < \dots < \xi_1^{(m)}$.) We denote by $b_j^+(x^*)$ the part of $b_j(x^*)$ where $x_1 > 0$. We define a function e on $\sqrt{-1}S^*M$ by

$$e((x, \sqrt{-1}\langle \xi, dx\rangle\infty)) = -\frac{q_{m-1}(x,\xi)}{(\partial/\partial\xi_1)p_m(x,\xi)}\,.$$

Example. Put

$$P = x_1(D_1^2 + \dots + D_k^2 - D_{k+1}^2 - \dots - D_n^2) + \sum_{j=1}^n a_j(x)D_j + b(x)$$

with $1 \le k \le n-1$ and real-analytic functions $a_j(x)$ and $b(x)$ defined on a neighborhood of $x^0 \in N$ such that $a_1(x^0) \notin \mathbb{Z}$. Then P satisfies (A-1)–(A-4) for $x^* = (x^0, \sqrt{-1}\langle \xi'^0, dx\rangle\infty) \in \sqrt{-1}S^*N$ if $\xi'^0 = (\xi_2^0,\dots,\xi_n^0)$ satisfies

$$(\xi_2^0)^2 + \dots + (\xi_k^0)^2 < (\xi_{k+1}^0)^2 + \dots + (\xi_n^0)^2\,.$$

For a hyperfunction u defined on an open subset of M, we denote by $\mathrm{SS}(u)$ the singularity spectrum (= analytic wave front set) of u, which is a subset of $\sqrt{-1}S^*M$.

THEOREM 1. *Let $u(x)$ be a hyperfunction on $U \cap M_+$ with an open neighborhood U of x^0 in M. Let k be an arbitrary integer with $1 \le k \le m$ and assume* (*in addition to* (A-1)–(A-4))

- $Pu(x) = 0$,
- $\gamma_{+\,\mathrm{reg}}(u)$ *is micro-analytic at* x^*,
- $\mathrm{SS}(u) \cap b_k^+(x^*) = \emptyset$,
- $e(x^{*(k)}) \notin \{-1, -2, -3, \ldots\}$.

Then $\mathrm{SS}(u) \cap b_j^+(x^*) = \emptyset$ *for any j and* $\gamma_{+\,\mathrm{sing}}(u)$ *is micro-analytic at* x^*.

Remark 1. The last assumption in Theorem 1 is necessary as is seen in the following example: Put

$$P = x_1(D_1^2 - D_2^2) + aD_1 + (2-a)D_2$$

in $\mathbb{R}^2$ with $a \in \mathbb{C} \setminus \mathbb{Z}$ and put $x^* = (0, \sqrt{-1}dx_2\infty)$. Then we have $\lambda = 1 - a$ and

$$x^{*(1)} = (0, \sqrt{-1}(dx_1 + dx_2)\infty), \qquad x^{*(2)} = (0, \sqrt{-1}(-dx_1 + dx_2)\infty),$$
$$b_1^+(x^*) = \{(x_1, x_2, \sqrt{-1}(dx_1 + dx_2)\infty) \mid x_1 > 0, x_2 = -x_1\},$$
$$b_2^+(x^*) = \{(x_1, x_2, \sqrt{-1}(-dx_1 + dx_2)\infty) \mid x_1 > 0, x_2 = x_1\},$$
$$e(x^{*(1)}) = -1, \qquad e(x^{*(2)}) = 1 - a.$$

Define a hyperfunction u on M_+ by

$$u(x_1, x_2) = x_1^{1-a}(x_2 + \sqrt{-1}\,0 - x_1)^{a-1}\,.$$

Then u satisfies $Pu = 0$ and

$$\gamma_{+\,\mathrm{reg}}(u) = 0, \qquad \gamma_{+\,\mathrm{sing}}(u) = (x_2 + \sqrt{-1}\,0)^{a-1}\,, \qquad \mathrm{SS}(u) = b_2^+(x^*)\,.$$

THEOREM 2. *Let $u(x)$ be a hyperfunction on $U \cap M_+$ with an open neighborhood U of x^0 in M. Let k be an arbitrary integer with $1 \le k \le m$ and assume* (*in addition to* (A-1)–(A-4))

- $Pu(x) = 0$,
- $\gamma_{+\,\mathrm{sing}}(u)$ *is micro-analytic at* x^*,
- $\mathrm{SS}(u) \cap b_j^+(x^*) = \emptyset$ *for any* $j \ne k$,
- $e(x^{*(k)}) \notin \{0, 1, 2, \ldots\}$.

Then $\mathrm{SS}(u) \cap b_k^+(x^*) = \emptyset$ *and* $\gamma_{+\,\mathrm{reg}}(u)$ *is micro-analytic at* x^*.

Remark 2. The last assumption in Theorem 2 is necessary as is seen in the following example: Put

$$P = x_1(D_1^2 - D_2^2) + aD_1 + aD_2$$

in $\mathbb{R}^2$ with $a \in \mathbb{C} \setminus \mathbb{Z}$ and put $x^* = (0, \sqrt{-1}\, dx_2 \infty)$. Then we have $\lambda = 1 - a$ and

$$x^{*(1)} = (0, \sqrt{-1}(dx_1 + dx_2)\infty), \qquad x^{*(2)} = (0, \sqrt{-1}(-dx_1 + dx_2)\infty),$$

$$b_1^+(x^*) = \{(x_1, x_2, \sqrt{-1}(dx_1 + dx_2)\infty) \mid x_1 > 0, x_2 = -x_1\},$$

$$b_2^+(x^*) = \{(x_1, x_2, \sqrt{-1}(-dx_1 + dx_2)\infty \mid x_1 > 0, x_2 = x_1\},$$

$$e(x^{*(1)}) = -a, \qquad e(x^{*(2)}) = 0.$$

Define a hyperfunction u on M_+ by

$$u(x_1, x_2) = (x_2 + \sqrt{-1}\, 0 - x_1)^{1-a}.$$

Then u satisfies $Pu = 0$ and

$$\gamma_{+\,\mathrm{reg}}(u) = (x_2 + \sqrt{-1}\, 0)^{1-a}, \qquad \gamma_{+\,\mathrm{sing}}(u) = 0, \qquad \mathrm{SS}(u) = b_2^+(x^*).$$

As an application of these theorems, we give results for rather special equations. The first is concerning a non-characteristic boundary value problem with two bicharacteristics issuing from the same point:

Theorem 3. *Let P be a linear partial differential operator of the form*

$$P = D_1^2 - x_1^k A(x', D'),$$

with $k \geq 0$ an integer, defined on a neighborhood of $x^0 \in N$, where A is a second order linear partial differential operator free of x_1 and D_1 such that $\sigma_2(A)(x', \xi')$ is real-valued for real x', ξ' and that $\sigma_2(A)(x^0, \xi'^0) > 0$. Let b_1^+, b_2^+ be the two bicharacteristics in $\sqrt{-1}S^*M|_{M_+}$ of P issuing from $x^* = (x^0, \sqrt{-1}\langle \xi'^0, dx' \rangle \infty)$. Let $u(x)$ be a hyperfunction on $U \cap M_+$ with an open neighborhood U of x^0 such that

- $Pu(x) = 0$,
- $u|_{x_1 \to +0}$ *or* $D_1 u|_{x_1 \to +0}$ *is micro-analytic at* x^*,
- $\mathrm{SS}(u) \cap b_1^+ = \emptyset$.

Then $\mathrm{SS}(u) \cap b_2^+ = \emptyset$, and both $u|_{x_1 \to +0}$ and $D_1 u|_{x_1 \to +0}$ are micro-analytic at x^*.

The last result is for a Fuchsian partial differential equation with two bicharacteristics issuing from the same point ($k \geq 2$) or both tangent to the boundary ($k = 0$).

Theorem 4. *Let P be a linear partial differential operator of the form*

$$P = x_1 D_1^2 - x_1^k A(x', D') + a_1(x') D_1,$$

with an integer $k \geq 2$ or $k = 0$ defined on a neighborhood of $x^0 \in N$, where A is a second order linear partial differential operator satisfying the same conditions as in Theorem 3, and $a_1(x')$ is a real-analytic function defined on a neighborhood of x^0 with $a_1(x^0) \notin \mathbb{Z}$ and $2(1 - a_1(x^0))/(k+1) \notin \mathbb{Z}$. Let b_1^+, b_2^+ be the two bicharacteristics in $\sqrt{-1}S^*M|_{M_+}$ of P issuing from $(x^0, \sqrt{-1}\langle \xi'^0, dx' \rangle \infty)$ (when $k \geq 2$) or issuing from $(x^0, \sqrt{-1}\, dx_1 \infty)$ (when $k = 0$). Let $u(x)$ be a hyperfunction on $U \cap M_+$ with U an open neighborhood of x^0 such that

- $Pu(x) = 0$,
- $\gamma_{+\,\mathrm{reg}}(u)$ *or* $\gamma_{+\,\mathrm{sing}}(u)$ *is micro-analytic at* x^*,
- $\mathrm{SS}(u) \cap b_1^+ = \emptyset$.

Then $\mathrm{SS}(u) \cap b_2^+ = \emptyset$, *and both* $\gamma_{+\,\mathrm{reg}}(u)$ *and* $\gamma_{+\,\mathrm{sing}}(u)$ *are micro-analytic at* x^*.

3. Proof of the main results

Proof of Theorem 1. There exists a hyperfunction $\widetilde{u}$ on U such that $\widetilde{u} = u$ in $U \cap M_+$, $\widetilde{u} = 0$ in $U \cap M_-$ and that

$$P\widetilde{u}(x) = \sum_{\mu=0}^{m-2} \delta^{(\mu)}(x_1) \sum_{\nu=0}^{m-2-\mu} B_{\mu,\nu}(x', D')\gamma_{+\,\mathrm{reg}}(u)_\nu\,,$$

where $\gamma_{+\,\mathrm{reg}}(u)_\nu$ denotes the $(\nu-1)$th component of $\gamma_{+\,\mathrm{reg}}(u)$, and $B_{\mu,\nu}(x', D')$ is a linear partial differential operator (cf. [12, Proposition 2.3.10]). Hence $P\widetilde{u}(x)$ is micro-analytic on a neighborhood of

$$\{(x^0, \sqrt{-1}(\xi_1 dx_1 + \langle \xi', dx'\rangle)\infty) \in \sqrt{-1}S^*M \mid \xi_1 \in \mathbb{R},\ \xi' = \xi'^0\}\,.$$

The assumption (A-4) implies that P_m is of real simple characteristics near $x^{*(j)}$'s. Since $e(x^{*(k)})$ is not a negative integer, we can apply the theorem on branching of singularities by Ôaku [6, Theorem 3.5] and get $x^{*(k)} \not\in \mathrm{SS}(\widetilde{u})$. (Note that $\mathrm{SS}(\widetilde{u}) \subset \{x_1 \geq 0\}$.) Hence we have

$$\mathrm{SS}(\widetilde{u}) \cap \{(x^0, \sqrt{-1}(\xi_1 dx_1 + \langle \xi'^0, dx'\rangle)\infty) \in \sqrt{-1}S^*M \mid \xi_1^{(k-1)} < \xi_1 < \xi_1^{(k+1)}\} = \emptyset\,,$$

where we put $\xi_1^{(0)} = -\infty$, $\xi_1^{(m+1)} = \infty$. Since P_m is micro-hyperbolic in the direction $dx_1 + d\xi_1$, we get

$$x^{*(j)} \not\in \mathrm{SS}(\widetilde{u}) \tag{2}$$

for $j = k-1$ and $k+1$. Hence we get (2) for any j by induction. This implies $\mathrm{SS}(u) \cap b_j^+(x^*) = \emptyset$ for any j. The theorem on the propagation of micro-analyticity up to the boundary by Ôaku [8, Theorem 2.2] implies that $\gamma_{+\,\mathrm{sing}}(u)$ is micro-analytic at x^*. This completes the proof of Theorem 1.

Proof of Theorem 2. By (A-4) and [5, Theorem 2] there exists a hyperfunction $\widetilde{u}(x)$ on a neighborhood $V \subset U$ of x^0 with x_1 as a real-analytic parameter such that $P\widetilde{u} = 0$ and that $\gamma_{+\,\mathrm{reg}}(\widetilde{u}-u)_\nu = D_1^\nu \widetilde{u}|_{x_1=0} - \gamma_{+\,\mathrm{reg}}(u)_\nu$ is micro-analytic at x^* for any $\nu = 0, \ldots, m-2$. Moreover, $\gamma_{+\,\mathrm{sing}}(\widetilde{u} - u) = -\gamma_{+\,\mathrm{sing}}(u)$ is micro-analytic at x^* by the assumption. Then by the micro-local uniqueness theorem for Fuchsian boundary value problems [8, Theorem 2.1] we have

$$\mathrm{SS}(u - \widetilde{u}) \cap \{(x, \sqrt{-1}\langle \xi, dx\rangle\infty) \mid |x| < \varepsilon,\ x_1 > 0,\ |\xi' - \xi'^0| < \varepsilon\} = \emptyset\,,$$

namely, $\mathrm{SS}(\widetilde{u}) = \mathrm{SS}(u)$ holds on

$$\{(x, \sqrt{-1}\langle \xi, dx\rangle\infty) \mid |x| < \varepsilon,\ x_1 > 0,\ |\xi' - \xi'^0| < \varepsilon\}$$

for some $\varepsilon > 0$. Since $\widetilde{u}$ has x_1 as a real-analytic parameter it follows that
$\mathrm{SS}(\widetilde{u}) \cap \{(x^0, \sqrt{-1}(\xi_1 dx_1 + \langle \xi'^0, dx' \rangle)\infty) \in \sqrt{-1}S^*M \mid \xi_1 < \xi_1^{(1)} \text{ or } \xi_1 > \xi_1^{(m)}\} = \emptyset$.
Since P is micro-hyperbolic in the direction $dx_1 + d\xi_1$, (2) holds for $j = 1$ and $j = m$ if $j \neq k$. Hence we get

$$\mathrm{SS}(\widetilde{u}) \cap \{(x^0, \sqrt{-1}(\xi_1 dx_1 + \langle \xi'^0, dx' \rangle)\infty) \in \sqrt{-1}S^*M \mid \xi_1 \neq \xi_1^{(k)}\} = \emptyset\,.$$

Again by [6, Theorem 3.5] and the condition that $e(x^{*(k)}) \notin \{0, 1, 2, \ldots\}$ we get (2) for $j = k$ and hence $\mathrm{SS}(u) \cap b_k^+(x^*) = \emptyset$. This completes the proof of Theorem 2.

Proof of Theorem 3. We use a local coordinate transformation $x_1 = t^\alpha$ with $\alpha = 2/(k+2)$. We define a hyperfunction $v(t, x')$ by $v(t, x') = u(t^\alpha, x')$ for $t > 0$. Then $v(t, x')$ satisfies $Qv(t, x') = 0$ with

$$Q = tD_t^2 + (1 - \alpha)D_t - \alpha^2 tA(x', D')\,.$$

This Q satisfies (A-1)–(A-4). In fact, for Q we have

$$\lambda = \alpha \notin \mathbb{Z}, \qquad e(x^{*(j)}) = \alpha - 1 \notin \mathbb{Z} \quad \text{for } j = 1, 2\,.$$

Moreover, we can show

$$u(+0, x') = \gamma_{+\,\mathrm{reg}}(v)(x'), \qquad D_1 u(+0, x') = \gamma_{+\,\mathrm{sing}}(v)(x')\,.$$

Hence Theorem 3 follows from Theorems 1 and 2.

Proof of Theorem 4. We use a local coordinate transformation $x_1 = t^\alpha$ with $\alpha = 2/(k+1)$. We define a hyperfunction $v(t, x')$ by $v(t, x') = u(t^\alpha, x')$ for $t > 0$. Then $v(t, x')$ satisfies $Qv(t, x') = 0$ with

$$Q = tD_t^2 + (1 + \alpha - \alpha a_1(x'))D_t - \alpha^2 tA(x', D')\,.$$

This Q satisfies (A-1)–(A-4). In fact, for Q we have

$$\lambda = \alpha(a_1(x^0) - 1) \notin \mathbb{Z}\,,$$
$$e(x^{*(j)}) = (1 + \alpha - \alpha a_1(x'))/2 \notin \mathbb{Z} \quad \text{for } j = 1, 2\,.$$

Moreover, we can show

$$\gamma_{+\,\mathrm{reg}}(u)(x') = \gamma_{+\,\mathrm{reg}}(v)(x')\,, \qquad \gamma_{+\,\mathrm{sing}}(u)(x') = \gamma_{+\,\mathrm{sing}}(v)(x')\,.$$

Hence Theorem 4 follows from Theorems 1 and 2.

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