PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 27
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1992

HYPOELLIPTIC SYSTEMS OF COMPLEX VECTOR FIELDS

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0. Introduction. Let $X_1, \ldots, X_{2n}$ be C^∞ real vector fields on Ω open in $\mathbb{R}^N$ and put

$$L_j = \tfrac{1}{2}(X_j + iX_{j+n}), \quad 1 \le j \le n, \; L = (L_1, \ldots, L_n).$$

The following hypotheses are assumed throughout:

(H1) (Hörmander condition) the brackets of length at most r of the X_j generate $T_x\Omega$, $\forall x \in \Omega$;
(H2) $d_{x\xi}\, X(x,\xi)$ are linearly independent;
(H3) $[L_j\,, L_k] = 0$, $\forall j, k \in [1, n]$.

Under (H1), it is well known that the system $\{X_j\}_{1\le j\le 2n}$ is $(1-1/r)$-*subelliptic*, i.e.

$$\|u\|_{1/r}^2 \le C\Big(\sum_{j=1}^{2n} \|X_j u\|^2 + \|u\|^2\Big), \quad u \in C_{\mathrm{c}}^\infty(\Omega),$$

cf. Bolley–Camus–Nourrigat [1]. In particular, for ω open $\subset \Omega$ we have

$$u \in \mathcal{D}'(\Omega), \quad X_j u \in C^\infty(\omega), \; 1 \le j \le 2n \quad \Rightarrow \quad u \in C^\infty(\omega).$$

Problem. Give geometric conditions to guarantee L to be *hypoelliptic*:

$$u \in \mathcal{D}'(\Omega), \quad L_j u \in C^\infty(\omega), \; 1 \le j \le n \quad \Rightarrow \quad u \in C^\infty(\omega).$$

Examples. 0) $N = 2n$, $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \Omega$, $X_j = \partial/\partial x_j$, $X_{n+j} = \partial/\partial y_j$, $L_j = \partial/\partial \bar z_j$. Then L is the Cauchy–Riemann operator, $r = 1$ and L is elliptic.

1) $N = 2n+1$, $(x_1, \ldots, x_n, y_1, \ldots, y_n, t) \in \Omega$, $X_j = \partial/\partial x_j - y_j\partial/\partial t$, $X_{n+j} = \partial/\partial y_j + x_j\partial/\partial t$, $r = 2$. Here, L is the induced Cauchy–Riemann operator on the hypersurface $M = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid \operatorname{Re} z_0 = \frac{1}{2}|z_1|^2 + \ldots + \frac{1}{2}|z_n|^2\}$; it is *not* hypoelliptic because when taking Fourier transforms in t, we get

$\widehat{L}_j = \overline{\partial}_{z_j} - \tau z_j/2 = e^{\tau|z|^2/2}\overline{\partial}_z e^{-\tau|z|^2/2}$ and for $\widehat{u}(z,\tau) = e^{\tau|z|^2/2}$ if $\tau < 0$, $= 0$ if $\tau > 0$, it is clear that $Lu = 0$ but $u \notin C^\infty$.

2) Same as above but M is defined by the equation $\operatorname{Re} z_0 = -\frac{1}{2}|z_1|^2 - \ldots - \frac{1}{2}|z_p|^2 + \frac{1}{2}|z_{p+1}|^2 + \ldots + \frac{1}{2}|z_n|^2$, $0 < p < n$. In this case, L satisfies

$$\sum_{j=1}^{2n} \|X_j u\|^2 \le C\Big(\sum_{j=1}^{n} \|L_j u\|^2 + \|u\|^2\Big), \qquad u \in C_{\mathrm{c}}^\infty(\Omega) \tag{1}$$

(cf. [3]) and is, therefore, hypoelliptic.

1. Maximal hypoellipticity

Remark. It is out of the question to characterize hypoellipticity for general systems like Nirenberg and Treves [8] did for $n = 1$. Even a characterization of *δ-subellipticity*:

$$\|u\|_{1-\delta}^2 \le C\Big(\sum_{j=1}^{n} \|L_j u\|^2 + \|u\|^2\Big), \qquad u \in C_{\mathrm{c}}^\infty(\Omega)\,,$$

seems beyond the scope of present techniques, except for $n = 1$ which is the Egorov–Hörmander theorem [4].

Definition. The system L is called *maximal hypoelliptic* in Ω if (1) holds. It is *maximal hypoelliptic at* $(x_0, \xi_0) \in T^*\Omega \setminus 0$ if $\sum_1^n \|\Psi X_j \cdot u\|^2 \le C\big(\sum_1^n \|L_j u\|^2 + \|u\|^2\big)$, $u \in C_{\mathrm{c}}^\infty(\Omega)$, where Ψ is an elliptic pseudodifferential operator at (x_0, ξ_0).

Let $\Sigma = \{(x,\xi) \in T^*\Omega \backslash 0 \mid X_j(x,\xi) = 0\,,\ 1 \le j \le 2n\}$ be the characteristic set of $X_1, \ldots, X_{2n}$ and for $(x,\xi) \in \Sigma$ define the Levi matrix $\mathcal{L}(x,\xi)$ by $\mathcal{L}_{jk}(x,\xi) = i\{L_j, \overline{L}_k\}(x,\xi)\,,\ 1 \le j,k \le n$, where $\{\ ,\ \}$ is the Poisson bracket. Denote by $\lambda_1(x,\xi), \ldots, \lambda_n(x,\xi) \in \mathbb{R}$ the eigenvalues of $\mathcal{L}(x,\xi)$.

Theorem (Nourrigat [9]). *The system L is maximal hypoelliptic at (x_0, ξ_0) if and only if there exists a neighbourhood U of (x_0, ξ_0) and $C > 0$ such that*

$$\max(|\lambda_1|, \ldots, |\lambda_n|) \le C \max(0, \lambda_1, \ldots, \lambda_n) \quad \textit{in } U. \tag{D(0)}$$

Comments. 1) Sufficiency of D(0) was proved in [6] when $L = \overline{\partial}_{\mathrm{b}}$.

2) In the non-degenerate case, i.e., $\lambda_1 \ne 0, \ldots, \lambda_n \ne 0$ everywhere, D(0) is equivalent to the condition Y(0) of Folland–Kohn [2].

3) The theorem covers the subellipticity result of Egorov–Hörmander [4] provided we add $\sum |X_j(x,\xi)|$ on the right hand side of D(0).

4) The proof relies on a general theorem by Helffer–Nourrigat [3] which reduces maximal hypoellipticity to injectivity in $\mathcal{S}(\mathbb{R}^{2n})$ of operators with polynomial coefficients.

2. Case of $(0,q)$-forms for $\overline{\partial}_{\mathrm{b}}$. From now on, we consider the induced Cauchy–Riemann operator on a real hypersurface of $\mathbb{C}^n$; therefore, we may suppose that locally

$$L_j = \frac{\partial}{\partial \bar z_j} + i\frac{\partial f/\partial \bar z_j}{1 + i\,\partial f/\partial t}\partial t$$

with $f : \Omega \to \mathbb{R}$ of class C^∞, Ω open $\subset \mathbb{C}^n \times \mathbb{R}$. Then L gives the $\overline{\partial}_{\mathrm{b}}$ complex

$$0 \longrightarrow \Lambda^{0,0}C^\infty(\Omega) \xrightarrow{L^0} \Lambda^{0,1}C^\infty(\Omega) \xrightarrow{L^1} \dots \xrightarrow{L^{n-1}} \Lambda^{0,n}C^\infty(\Omega) \longrightarrow 0\,,$$

where $\Lambda^{0,q}C^\infty(\Omega) = \{\sum_{|J|=q} u_J\, d\bar z_J \mid u_J \in C^\infty(\Omega)\}$ and

$$L^q\Big(\sum_{|J|=q} u_J\, d\bar z_J\Big) = \sum_{|J|=q,\,1\le j\le n} L_j u_J\, d\bar z_j \wedge d\bar z_J. \tag{2}$$

To $\overline{\partial}_{\mathrm{b}}$ we associate its formal adjoint complex $\overline{\partial}_{\mathrm{b}}^*$:

$$0 \longrightarrow \Lambda^{0,n}C^\infty(\Omega) \xrightarrow{L^{n-1*}} \Lambda^{0,n-1}C^\infty(\Omega) \longrightarrow \dots \xrightarrow{L^{0*}} \Lambda^{0,0}C^\infty(\Omega) \longrightarrow 0\,.$$

Remark. The following general result by Kohn [5] on L^2 complexes:

$$L^{q-1}L^{q-1*} + L^{q*}L^q \text{ subelliptic} \Rightarrow \ker L^q = \operatorname{im} L^{q-1}$$

shows that regularity implies solvability.

Definition. The operator $\overline{\partial}_{\mathrm{b}}$ is *maximal hypoelliptic* on $(0,q)$-forms if

$$\|\mathrm{Re}\, L^q u\|^2 + \|\mathrm{Im}\, L^q u\|^2 \le C(\|L^q u\|^2 + \|L^{q-1*}u\|^2 + \|u\|^2)\,, \qquad u \in \Lambda^{0,q}C^\infty_{\mathrm{c}}(\Omega)\,,$$

where $\mathrm{Re}\, L^q$, $\mathrm{Im}\, L^q$ are defined as in (2) with X_j, X_{n+j} on the right hand side.

Example. When the Levi matrix $\mathcal{L}$ is non-degenerate, then $\overline{\partial}_{\mathrm{b}}$ is maximal hypoelliptic on $(0,q)$-forms if and only if the condition $\mathrm{Y}(q)$ holds, i.e. the index of $\mathcal{L}$ is different from q and $n-q$ (cf. Folland–Kohn [2], and [3]).

Theorem. *Suppose $M = \{z \in \mathbb{C}^{n+1} \mid \mathrm{Re}\, z_0 = f(z_1,\dots,z_n)\} \ni 0$ and the Levi matrix degenerates only at the origin. Then $\overline{\partial}_{\mathrm{b}}$ is maximal hypoelliptic on $(0,q)$-forms if and only if $\mathrm{Y}(q)$ holds on $M \setminus \{0\}$.*

The necessity is due to Helffer–Nourrigat [3]. For sufficiency we know (cf. [7]) that the sublevel sets of any localized polynomial of f at 0 have the same homology as those of a quadratic form of index $\ne q$ and $\ne n-q$. It is then possible to solve in $\mathcal{S}(\mathbb{R}^{2n})$ the equation $\widetilde{L}^{q-1}v = u$, $v \in \Lambda^{0,q-1}\mathcal{S}(\mathbb{R}^{2n})$, where $\widetilde{L}$ is the operator associated to any localized polynomial of f, provided $\widetilde{L}^q u = 0$. If, moreover, $\widetilde{L}^{q-1*}u = 0$, then $u = 0$. This proves the injectivity of $(\widetilde{L}^{q-1*}, \widetilde{L}^q)$ in $\mathcal{S}(\mathbb{R}^{2n})$. Maximal hypoellipticity follows from [3].

Remark. Our proof certainly generalizes.

3. Hypersurfaces with Levi form having an isolated singularity

Remark. Only the non-degenerate case and some weakly pseudoconvex cases, e.g. $M = \{z \in \mathbb{C}^{n+1} \mid \mathrm{Re}\, z_0 = (\|z_1\|^2 + \dots + \|z_n\|^2)^k\}$, were previously known examples of hypersurfaces satisfying the hypothesis of our theorem.

In order to get more examples we first restrict ourselves to homogeneous real polynomials with non-vanishing hessian; more precisely, let

$$\mathcal{H}_{p,q}^{(m)} = \{P \in \mathbb{R}[x_1, \ldots, x_n] \mid P \text{ homogeneous of degree } m \text{ and}$$
$$x \neq 0 \ \Rightarrow\ (P''(x)) \text{ has } p \text{ negative and } q \text{ positive eigenvalues}\}\,.$$

The following results are proved in [7].

Proposition. *If* $P \in \mathcal{H}_{p,q}^{(m)}$ *and* $n = p+q \geq 3$ *then the map* $P' : \mathbb{R}^n \backslash 0 \to \mathbb{R}^n \backslash 0$ *is a diffeomorphism.*

Proposition. *For* $k \geq 1$ *and* $n = p+q \geq 2$ *we have*

a) $\mathcal{H}_{p,q}^{(2k+1)} \neq \emptyset \ \Leftrightarrow\ p = q = 1$;

b) $\mathcal{H}_{p,q}^{(2)} \neq \emptyset$, $\forall p, \forall q$;

c) $\mathcal{H}_{p,q}^{(4)} \neq \emptyset \ \Leftrightarrow\ p = q = 1$ or $p = 0$ or $q = 0$;

d) $\mathcal{H}_{p,q}^{(2k)} \neq \emptyset$, $\forall k \geq 3$, $\forall p, \forall q$.

Example. $M = \{z \in \mathbb{C}^{n+1} \mid \operatorname{Re} z_0 = -(|z_1|^2 + \ldots + |z_p|^2)^{k+1} + \varepsilon(|z_1|^2 + \ldots + |z_p|^2)^k(|z_{p+1}|^2 + \ldots + |z_n|^2) - \varepsilon(|z_1|^2 + \ldots + |z_p|^2)(|z_{p+1}|^2 + \ldots + |z_n|^2)^k + (|z_{p+1}|^2 + \ldots + |z_n|^2)^{k+1}\}$ is strictly p-pseudoconcave and q-pseudoconvex away from the origin if $k \geq 2$ and $\varepsilon < 2/k^2$.

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