0. Introduction. Let $X_1, \ldots, X_{2n}$ be $C^\infty$ real vector fields on $\Omega$ open in $\mathbb{R}^N$ and put
$$L_j = \frac{1}{2}(X_j + iX_{j+n}), \quad 1 \leq j \leq n, \quad L = (L_1, \ldots, L_n).$$

The following hypotheses are assumed throughout:

(H1) (Hörmander condition) the brackets of length at most $r$ of the $X_j$ generate $T_x\Omega$, $\forall x \in \Omega$;

(H2) $dx\xi X(x,\xi)$ are linearly independent;

(H3) $[L_j, L_k] = 0$, $\forall j, k \in [1, n]$.

Under (H1), it is well known that the system $\{X_j\}_{1 \leq j \leq 2n}$ is $(1 - 1/r)$-subelliptic, i.e.
$$\|u\|_{1,r}^2 \leq C \left( \sum_{j=1}^{2n} \|X_j u\|^2 + \|u\|^2 \right), \quad u \in C^\infty_c(\Omega),$$
cf. Bolley–Camus–Nourrigat [1]. In particular, for $\omega$ open $\subset \Omega$ we have
$$u \in D'(\Omega), \quad X_j u \in C^\infty(\omega), \quad 1 \leq j \leq 2n \Rightarrow u \in C^\infty(\omega).$$

Problem. Give geometric conditions to guarantee $L$ to be hypoelliptic:
$$u \in D'(\Omega), \quad L_j u \in C^\infty(\omega), \quad 1 \leq j \leq n \Rightarrow u \in C^\infty(\omega).$$

Examples. 0) $N = 2n$, $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \Omega$, $X_j = \partial/\partial x_j$, $X_{n+j} = \partial/\partial y_j$, $L_j = \partial/\partial \bar{z}_j$. Then $L$ is the Cauchy–Riemann operator, $r = 1$ and $L$ is elliptic.

1) $N = 2n + 1$, $(x_1, \ldots, x_n, y_1, \ldots, y_n, t) \in \Omega$, $X_j = \partial/\partial x_j - y_j \partial/\partial t$, $X_{n+j} = \partial/\partial \bar{z}_j + x_j \partial/\partial t$, $r = 2$. Here, $L$ is the induced Cauchy–Riemann operator on the hypersurface $M = \{(z_0, \ldots, z_n) \in C^{n+1} | \Re z_0 = \frac{1}{2}|z_1|^2 + \ldots + \frac{1}{2}|z_n|^2\}$; it is not hypoelliptic because when taking Fourier transforms in $t$, we get
\[ \hat{L}_j = \overline{\partial} z_j - \tau z_j / 2 = e^{\tau |\tau|^2 / 2} \overline{\partial} e^{-\tau |\tau|^2 / 2} \] and for \( \hat{u}(z, \tau) = e^{\tau |\tau|^2 / 2} \) if \( \tau < 0, = 0 \) if \( \tau > 0 \), it is clear that \( Lu = 0 \) but \( u \notin C^\infty \).

2) Same as above but \( M \) is defined by the equation \( \text{Re } z_0 = -\frac{1}{2} |z_1|^2 - \cdots - \frac{1}{2} |z_p|^2 + \frac{1}{2} |z_{p+1}|^2 + \cdots + \frac{1}{2} |z_n|^2, 0 < p < n \). In this case, \( L \) satisfies
\[
(1) \quad \sum_{j=1}^{2n} \|X_j u\|^2 \leq C \left( \sum_{j=1}^n \|L_j u\|^2 + \|u\|^2 \right), \quad u \in C_c^\infty(\Omega)
\]
(cf. [3]) and is, therefore, hypoelliptic.

1. Maximal hypoellipticity

Remark. It is out of the question to characterize hypoellipticity for general systems like Nirenberg and Treves [8] did for \( n = 1 \). Even a characterization of \( \delta \)-subellipticity:
\[
\|u\|_{1-\delta}^2 \leq C \left( \sum_{j=1}^n \|L_j u\|^2 + \|u\|^2 \right), \quad u \in C_c^\infty(\Omega),
\]
seems beyond the scope of present techniques, except for \( n = 1 \) which is the Egorov–Hörmander theorem [4].

Definition. The system \( L \) is called maximal hypoelliptic in \( \Omega \) if (1) holds. It is maximal hypoelliptic at \( (x_0, \xi_0) \in T^* \Omega \setminus \emptyset \) if \( \sum_{j=1}^n \|\Psi X_j, u\|^2 \leq C \left( \sum_{j=1}^n \|L_j u\|^2 + \|u\|^2 \right) \), \( u \in C_c^\infty(\Omega) \), where \( \Psi \) is an elliptic pseudodifferential operator at \( (x_0, \xi_0) \).

Let \( \Sigma = \{ (x, \xi) : (x, \xi) \in T^* \Omega \setminus \emptyset \mid X_j(x, \xi) = 0, \ 1 \leq j \leq 2n \} \) be the characteristic set of \( X_1, \ldots, X_{2n} \) and for \( (x, \xi) \in \Sigma \) define the Levi matrix \( \mathcal{L}(x, \xi) \) by \( L_{jk}(x, \xi) = \lambda_j(x, \xi), 1 \leq j, k \leq n \), where \( \{ \lambda_j \} \) is the Poisson bracket. Denote by \( \lambda_1(x, \xi), \ldots, \lambda_n(x, \xi) \in \mathbb{R} \) the eigenvalues of \( \mathcal{L}(x, \xi) \).

Theorem (Nourrigat [9]). The system \( L \) is maximal hypoelliptic at \( (x_0, \xi_0) \) if and only if there exists a neighbourhood U of \( (x_0, \xi_0) \) and \( C > 0 \) such that
\[
D(0) \quad \max(|\lambda_1|, \ldots, |\lambda_n|) \leq C \max(0, \lambda_1, \ldots, \lambda_n) \quad \text{in } U.
\]

Comments. 1) Sufficiency of \( D(0) \) was proved in [6] when \( L = \overline{\partial} \).

2) In the non-degenerate case, i.e., \( \lambda_1 \neq 0, \ldots, \lambda_n \neq 0 \) everywhere, \( D(0) \) is equivalent to the condition \( Y(0) \) of Folland–Kohn [2].

3) The theorem covers the subellipticity result of Egorov–Hörmander [4] provided we add \( \sum |X_j(x, \xi)| \) on the right hand side of \( D(0) \).

4) The proof relies on a general theorem by Helffer–Nourrigat [3] which reduces maximal hypoellipticity to injectivity in \( S(\mathbb{R}^{2n}) \) of operators with polynomial coefficients.

2. Case of \( (0, q) \)-forms for \( \overline{\partial} \). From now on, we consider the induced Cauchy–Riemann operator on a real hypersurface of \( \mathbb{C}^n \); therefore, we may
suppose that locally
\[ L_j = \frac{\partial}{\partial z_j} + i \frac{\partial f / \partial \bar{z}_j}{1 + \partial f / \partial t} \partial t \]
with \( f : \Omega \rightarrow \mathbb{R} \) of class \( C^\infty \), \( \Omega \) open \( \subset \mathbb{C}^n \times \mathbb{R} \). Then \( L \) gives the \( \mathcal{D}_b \) complex
\[ 0 \rightarrow A^{0,0}C^\infty(\Omega) \xrightarrow{L^0} A^{0,1}C^\infty(\Omega) \xrightarrow{L^1} \ldots \xrightarrow{L^{n-1}} A^{0,n}C^\infty(\Omega) \rightarrow 0, \]
where \( A^{0,q}C^\infty(\Omega) = \{ \sum_{|j|=q} u_j d\bar{z}_j \mid u_j \in C^\infty(\Omega) \} \) and
\[ (2) \quad L^q \left( \sum_{|j|=q} u_j d\bar{z}_j \right) = \sum_{|j|=q, 1 \leq j \leq n} L_j u_j d\bar{z}_j \land d\bar{z}_j. \]

To \( \mathcal{D}_b \) we associate its formal adjoint complex \( \mathcal{D}_b^* \):
\[ 0 \rightarrow A^{0,0}C^\infty(\Omega) \xrightarrow{L_0^*} A^{0,1}C^\infty(\Omega) \xrightarrow{L_1^*} \ldots \xrightarrow{L_{n-1}^*} A^{0,n}C^\infty(\Omega) \rightarrow 0. \]

Remark. Only the non-degenerate case and some weakly pseudoconvex cases, e.g. \( M = \{ z \in \mathbb{C}^{n+1} \mid \text{Re} \, z_0 = f(z_1, \ldots, z_n) \} \supset 0 \) and the Levi matrix degenerates only at the origin. Then \( \mathcal{D}_b \) is maximal hypoelliptic on \( (0,q) \)-forms if and only if \( \text{Re}(\mathcal{D}_b) \) holds, i.e. the index of \( \mathcal{L} \) is different from \( q \) and \( n - q \) (cf. Folland–Kohn [2], and [3]).

Theorem. Suppose \( M = \{ z \in \mathbb{C}^{n+1} \mid \text{Re} \, z_0 = f(z_1, \ldots, z_n) \} \supset 0 \) and the Levi matrix degenerates only at the origin. Then \( \mathcal{D}_b \) is maximal hypoelliptic on \( (0,q) \)-forms if and only if \( \text{Y}(\mathcal{D}_b) \) holds on \( M \setminus \{ 0 \} \).

The necessity is due to Helffer–Nourrigat [3]. For sufficiency we know (cf. [7]) that the sublevel sets of any localized polynomial of \( f \) at 0 have the same homology as those of a quadratic form of index \( \neq q \) and \( \neq n - q \). It is then possible to solve in \( \mathcal{S}(\mathbb{R}^{2n}) \) the equation \( \mathcal{L} v = u \), \( v \in A^{0,q-1}S(\mathbb{R}^{2n}), \) where \( \mathcal{L} \) is the operator associated to any localized polynomial of \( f \), provided \( L^q_0 u = 0 \). If, moreover, \( L^{q-1*}u = 0 \), then \( u = 0 \). This proves the injectivity of \( (\mathcal{L}^{q-1*}, \mathcal{L}^q) \) in \( \mathcal{S}(\mathbb{R}^{2n}) \). Maximal hypoellipticity follows from [3].

Remark. Our proof certainly generalizes.

3. Hypersurfaces with Levi form having an isolated singularity

Remark. Only the non-degenerate case and some weakly pseudoconvex cases, e.g. \( M = \{ z \in \mathbb{C}^{n+1} \mid \text{Re} \, z_0 = (\|z_1\|^2 + \ldots + \|z_n\|^2)^k \} \), were previously known examples of hypersurfaces satisfying the hypothesis of our theorem.
In order to get more examples we first restrict ourselves to homogeneous real polynomials with non-vanishing hessian; more precisely, let
\[ \mathcal{H}_{p,q}^{(m)} = \{ P \in \mathbb{R}[x_1, \ldots, x_n] \mid P \text{ homogeneous of degree } m \text{ and } x \neq 0 \Rightarrow (P''(x)) \text{ has } p \text{ negative and } q \text{ positive eigenvalues} \}. \]

The following results are proved in [7].

**Proposition.** If \( P \in \mathcal{H}_{p,q}^{(m)} \) and \( n = p + q \geq 3 \) then the map \( P' : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0 \) is a diffeomorphism.

**Proposition.** For \( k \geq 1 \) and \( n = p + q \geq 2 \) we have
a) \( \mathcal{H}_{p,q}^{(2k+1)} \neq \emptyset \iff p = q = 1 \);
b) \( \mathcal{H}_{p,q}^{(2)} \neq \emptyset, \forall p, \forall q; \)
c) \( \mathcal{H}_{p,q}^{(4)} \neq \emptyset \iff p = q = 1 \) or \( p = 0 \) or \( q = 0 \);
d) \( \mathcal{H}_{p,q}^{(2k)} \neq \emptyset, \forall k \geq 3, \forall p, \forall q. \)

**Example.** \( M = \{ z \in \mathbb{C}^{n+1} \mid \text{Re } z_0 = -(|z_1|^2 + \ldots + |z_p|^2)^{k+1} + \varepsilon (|z_1|^2 + \ldots + |z_p|^2)(|z_{p+1}|^2 + \ldots + |z_n|^2)^k + (|z_{p+1}|^2 + \ldots + |z_n|^2)^{k+1} \} \) is strictly \( p \)-pseudoconcave and \( q \)-pseudoconvex away from the origin if \( k \geq 2 \) and \( \varepsilon < 2/k^2 \).

**References**


