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A GEOMETRIC AND ANALYTIC APPROACH TO SOME PROBLEMS ASSOCIATED WITH EMDEN EQUATIONS

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Introduction. The equations we shall consider here are the so-called classical Emden equations, namely

$$-\Delta u = \lambda e^u \qquad (\lambda > 0) \tag{1}$$

in a 3-dimensional domain and

$$-\Delta u = |u|^{q-1}u \qquad (q > 1) \tag{2}$$

in an N-dimensional domain, $N \geq 2$. Those equations play an important role in meteorology, physics, astrophysics and now Riemannian geometry and variational analysis, therefore they have been studied by many authors. The first works on radial solutions of (1)–(2) are due to Emden (1897), Fowler (1914–1931), Chandrasekhar (1939). An important fact for understanding limit phenomena is the existence of explicit radial singular solutions:

$$u_{\mathrm{s}}(x) = \mathrm{Ln}(1/|x|^2) + \mathrm{Ln}(2/\lambda) \tag{3}$$

for (1) and

$$u_{\mathrm{s}}(x) = \left(\left(\frac{2}{q-1}\right)\left(N - \frac{2q}{q-1}\right)\right)^{1/(q-1)} r^{-2/(q-1)} \tag{4}$$

for (2); in that case it is clear that we have to assume

$$q > N/(N-2), \tag{5}$$

which in fact contains the most interesting phenomena. We may look now for nonradial solutions of (1) as deformations of the radial ones of the following form:

$$u(r,\sigma) = \mathrm{Ln}(1/r^2) + \mathrm{Ln}(2/\lambda) + 2\omega(\sigma) \tag{6}$$

where (r,σ) are the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$; then ω must satisfy

$$\Delta_{S^2}\omega + e^{2\omega} - 1 = 0 \tag{7}$$

on S^2. Let G_2 be the set of all continuous, and therefore C^∞, solutions of (7); then it is classical that G_2 is the set of functions $\omega = \frac{1}{2}\operatorname{Ln}(\det|d\phi|)$ where ϕ is a conformal transformation of S^2. G_2 has a structure of a 3-dimensional noncompact manifold on which $\mathrm{SL}(2,\mathbb{C})$ acts transitively [6].

If we look for solutions of (2) of the form

$$u(r,\sigma) = r^{-2/(q-1)}\omega(\sigma) \tag{8}$$

then we see that

$$-\Delta_{S^{N-1}}\omega + l\omega - |\omega|^{q-1}\omega = 0 \tag{9}$$

where

$$l = \left(\frac{2}{q-1}\right)\left(N - \frac{2q}{q-1}\right). \tag{10}$$

Let $G_{N,q}$ be the set of $\omega \in C(S^{N-1})$ satisfying (9) and let $G^+_{N,q} = G_{N,q} \cap C^+(S^{N-1})$ be the set of nonnegative solutions of (9). If we assume that $q > N/(N-2)$, then $G^+_{N,q}$ contains at least the two constant elements 0 and $l^{1/(q-1)}$. A particular interesting case appears when $N \geq 4$ and $q = (N+1)/(N-3)$. If we write $d = N-1$, then $G^+_{d,(d+2)/(d-2)}$ is the set of nonnegative solutions of

$$\Delta_{S^d}\omega - \frac{d(d-2)}{4}\omega + \omega^{(d+2)/(d-2)} = 0 \tag{11}$$

on S^d. Let G_d be $G^+_{d,(d+2)/(d-2)}$. This set is completely known from the works of Obata [11], Aubin [1] or Uhlenbeck; it is a $(d+1)$-dimensional noncompact manifold on which the Möbius group of S^d acts transitively, and if we fix a point a on S^d and write ϱ for the geodesic distance, then G_d is the set of functions $\psi_{\mu,a}(\,\cdot\,)$ defined by

$$\psi_{\mu,a}(\sigma) = \left(\frac{d(d-2)}{4}\right)^{(d-2)/4}\left(\frac{\sqrt{\mu^2-1}}{\mu - \cos\varrho(a,\sigma)}\right)^{(d-2)/2}, \qquad \mu \in [1,\infty], \quad \sigma \in S^d. \tag{12}$$

The two questions we are interested in are:

Q1: When $G^+_{N,q}$ reduces to the constant functions?

Q2: What is the role of $G^+_{N,q}$, G_2, G_d for describing the asymptotics of the solutions of (1) and (2)?

Geometric aspects of Emden's equations. The main result of this section is the following [3]:

THEOREM 1. *Assume (M,g) is a compact Riemannian manifold without boundary of dimension $n \geq 2$, Δ_g is the Laplace–Beltrami operator on M, $q > 1$, $\lambda > 0$*

and u is a positive solution of

$$\Delta_g u + u^q - \lambda u = 0 \qquad \text{on } M. \tag{13}$$

Assume also that the spectrum $\sigma(R(x))$ of the Ricci tensor R of the metric g satisfies

$$\inf_{x\in M} \min \sigma(R(x)) \geq \frac{n-1}{n}(q-1)\lambda\,, \tag{14}$$

$$q \leq (n+2)/(n-2)\,. \tag{15}$$

Moreover, assume that one of the two inequalities (14), (15) *is strict if (M, g) is conformally diffeomorphic to (S^n, g_0). Then u is constant with value $\lambda^{1/(q-1)}$.*

Proof. The main tool for this proof is the Bochner–Lichnerowicz–Weitzenböck formula, which takes the following form when f is a 0-form:

$$\tfrac{1}{2}\Delta_g(|\nabla f|^2) = |\mathrm{Hess}\, f|^2 + \langle \nabla \Delta_g f, \nabla f\rangle + R(\nabla f, \nabla f) \tag{16}$$

where $\mathrm{Hess}\, f$ is the second covariant derivative of f and $\langle\,,\rangle$ the scalar product in TM associated with g.

For $\beta \in \mathbb{R} \setminus \{0\}$ we set $u = v^{-\beta}$; then v satisfies

$$\Delta_g v = (\beta+1)\frac{|\nabla v|^2}{v} + \frac{1}{\beta}(v^{1+\beta-\beta q} - \lambda v). \tag{17}$$

We take $f = v$ in (16), multiply by v^γ $(\gamma \in \mathbb{R})$ and integrate over M; we obtain

$$A + B + C + D = 0 \tag{18}$$

with

$$A = \int_M v^\gamma |\mathrm{Hess}\, v|^2 = \frac{1}{n}\int_M v^\gamma (\Delta_g v)^2 + J \tag{19}$$

where

$$J = \int_M v^\gamma \left(|\mathrm{Hess}\, v|^2 - \frac{1}{n}(\Delta_g v)^2\right) \geq 0 \tag{20}$$

from Schwarz' inequality, and

$$B = \int_M v^\gamma \langle \nabla \Delta_g v, \nabla v\rangle, \tag{21}$$

$$C = -\frac{1}{2}\int_M v^\gamma \Delta_g(|\nabla v|^2), \tag{22}$$

$$D = \int_M v^\gamma R(\nabla v, \nabla v). \tag{23}$$

After a lengthy computation we deduce from (17) that

$$J - a\int_M v^{\gamma-2}|\nabla v|^4 - b\int_M v^{\gamma-\beta(q-1)}|\nabla v|^2 + \int_M v^{\gamma}(R(\nabla v,\nabla v) + c|\nabla v|^2) = 0 \tag{24}$$

with

$$a = \frac{1}{2}\left(\gamma^2 + (3\beta+2)\gamma + 2\frac{n-1}{n}(\beta+1)^2\right), \tag{25}$$

$$b = \frac{1}{n}\left(q(n-1) + \frac{\gamma}{\beta}\frac{n+2}{2}\right), \tag{26}$$

$$c = \lambda\left(\frac{n-1}{n} + \frac{\gamma}{\beta}\frac{n+2}{2n}\right). \tag{27}$$

The proof of Theorem 1 then reduces to the search for $\beta \neq 0$ and γ such that

$$a \le 0, \qquad b \le 0, \qquad \inf_M \min \sigma(R(x)) \ge -c. \tag{28}$$

Writing $y = 1 + 1/\beta$, $\delta = -\gamma/\beta$ $(y \neq 1)$ implies that (28) becomes

$$2\frac{n-1}{n}y^2 - 2\delta y + \delta^2 - \delta \le 0, \tag{29}$$

$$2q\frac{n-1}{n+2} \le \delta, \tag{30}$$

$$\frac{2n}{n+2}\inf_M \min \sigma(R(x)) \ge \lambda\left(\delta - 2\frac{n-1}{n+2}\right). \tag{31}$$

From (14) we see that there exists δ such that (30) and (31) hold. If (14) is strict we can choose δ such that one of the two inequalities (30), (31) is strict. If (14) is strict then $\delta = 2q(n-1)/(n+2)$. In order to find y satisfying (29) we must have

$$\varDelta' = \frac{\delta}{n}(2n - 2 - (n-2)\delta) \ge 0\,. \tag{32}$$

If $n = 2$ we always have $\varDelta' > 0$.

If $n \ge 3$, then $\varDelta' > 0 \Leftrightarrow \delta < 2(n-1)/(n-2)$ and we are confronted with three cases:

(i) $q < (n+2)/(n-2) \Leftrightarrow 2(n-1)q/(n+2) < 2(n-1)/(n+2)$. Then we can find δ such that $2(n-1)q/(n+2) < \delta < 2(n-1)/(n+2)$, and we also find y.

(ii) $q = (n+2)/(n-2)$ and $(M,g) = (S^n, kg_0)$ for some $k > 0$, $k \in C^\infty(M)$. We take $\delta = 2(n-1)/(n-2)$, $y = (n-1)^2/(n(n-2)) \neq 1$ and (14) is strict.

(iii) $q = (n+2)/(n-2)$ and $(M,g) \neq (S^n, kg_0)$. We take δ and y as in (ii). Even if (14) is not strict we have

$$\int_M v^\gamma \left(|\operatorname{Hess} v|^2 - \frac{1}{n}(\Delta_g v)^2 \right) > 0 \tag{33}$$

unless v is constant, which ends the proof.

Remark 1. When $(M,g) = (S^{N-1}, g_0)$, our result reads

$$\lambda(q-1) \le N-1, \qquad q \le (N+1)/(N-3) \tag{34}$$

with one inequality being strict.

An interesting consequence of Theorem 1 is that it gives improved estimates for the value of

$$S_{\lambda,q} = \inf\{Q_{\lambda,q}(u) : u \in W^{1,2}(M) \setminus \{0\}\} \tag{35}$$

where

$$Q_{\lambda,q}(u) = \int_M (|\nabla u|^2 + \lambda u^2) \Big/ \Big(\int_M |u|^{q+1} \Big)^{2/(q+1)}. \tag{36}$$

The following result is a consequence of Theorem 1 of [3]:

COROLLARY 1. *Assume* (M,g) *is a compact Riemannian manifold without boundary of dimension* $n \ge 2$, *and*

$$\Lambda = \inf_{x \in M} \min \sigma(R(x)) > 0. \tag{37}$$

Assume $1 < q \le (n+2)/(n-2)$.

(i) *If* $0 \le \lambda \le \Lambda$ *then*

$$S_{\lambda,q} = \lambda(\operatorname{Vol} M)^{(q-1)/(q+1)}, \tag{38}$$

(ii) *if* $\lambda > \Lambda$ *then*

$$\Lambda(\operatorname{Vol} M)^{(q-1)/(q+1)} < S_{\lambda,q} \le \lambda(\operatorname{Vol} M)^{(a-1)/(a+1)}. \tag{39}$$

Remark 2. Similar results to Theorem 1 and Corollary 1 hold if ∂M is not empty but convex and (15) holds with homogeneous Neumann boundary conditions [10].

OPEN PROBLEMS. Under what conditions on (M,g), $\lambda > 0$, q and $\alpha \ge 0$ is any positive solution of

$$\operatorname{div}_g(|\nabla\omega|^{p-2}\nabla\omega) + \omega^q - \lambda\omega^{p-1} = 0, \tag{40}$$

$1 < p < q+1 \le np/(n-p)$ (if $n > p$) or

$$\operatorname{div}_g((\alpha^2\omega^2 + |\nabla\omega|^2)^{(p-2)/2}\nabla\omega) + \omega^q - \lambda(\alpha^2\omega^2 + |\nabla\omega|^2)^{(p-2)/2}\omega = 0 \tag{41}$$

a constant?

Analytic aspects of Emden's equations. A natural way to study the structure of the set of solutions of (1) or (2) is to study their asymptotics, the isolated singularities for example: we assume that u is a solution of (1) (resp. (2)) in $B_1(0)\setminus\{0\}=\{x\in\mathbb{R}^N: 0<|x|<1\}$ and we write it as

$$u(r,\sigma)=\mathrm{Ln}(1/r^2)+\mathrm{Ln}(2/\lambda)+2v(t,\sigma) \tag{42}$$

where $t=\mathrm{Ln}(1/r)$ (resp.

$$u(r,\sigma)=r^{-2/(q-1)}v(t,\sigma) \tag{43}$$

and $t\in(0,\infty)$). In the case of equation (1) the function v defined by (42) satisfies

$$v_{tt}-v_t+\Delta_{S^2}v+e^{2v}-1=0 \tag{44}$$

on $(0,\infty)\times S^2$, and in the case of (2) the function v defined by (43) satisfies

$$v_{tt}-\left(N-2\frac{q-1}{q+1}\right)v_t+\Delta_{S^{N-1}}v-lv+v|v|^{q-1}=0 \tag{45}$$

on $(0,\infty)\times S^2$. An important fact is that in (45) the coefficient of v_t vanishes if and only if $q=(N+2)/(N-2)$. A general form for (44), (45) is

$$\phi_{tt}+a\phi_t+\Delta_g\phi+f(\phi)=0. \tag{46}$$

THEOREM 2. *Assume (M,g) is a compact manifold without boundary, Δ_g is the Laplace–Beltrami operator on M, f is a $C^{1,\gamma}$ real function for some $\gamma\in(0,1)$ and $a\neq0$. If ϕ is any solution of (46) uniformly bounded on $(0,\infty)\times M$, then there exists a compact and connected subset ξ of the set E of C^2 solutions of*

$$\Delta_g\omega+f(\omega)=0 \tag{47}$$

on M such that $\phi(t,\cdot)$ converges to ξ as t tends to ∞ in the $C^2(M)$-topology.

Proof. From the boundedness of ϕ and the regularity theory of elliptic equations there exists $k_1>0$ such that

$$\left|\frac{\partial^\alpha}{\partial t^\alpha}\nabla^\beta\phi\right|\le k_1 \tag{48}$$

on $[1,\infty)\times M$, for any $\alpha+|\beta|\le3$. Therefore the ω-limit set ξ of the positive trajectory of ϕ defined by

$$\xi=\bigcap_{t>0}\overline{\bigcup_{\tau\ge t}\phi(\tau,\cdot)}^{C^2} \tag{49}$$

is a nonempty compact connected subset of $C^2(M)$.

Multiplying (46) by ϕ_t and integrating on M yields

$$a\int_1^T\int_M\phi_t^2=\left[\int_M\left(\frac12|\nabla\phi|^2-\frac12\phi_t^2-F(\phi)\right)\right]_{t=1}^{t=T} \tag{50}$$

for any $T > 1$, where $F(r) = \int_0^r f(s)\,ds$. From the boundedness of ϕ and $a \neq 0$ we see that

$$\int_1^\infty \int_M \phi_t^2 < \infty. \tag{51}$$

From this estimate and (46) we deduce that

$$\int_1^\infty \int_M \phi_{tt}^2 < \infty, \tag{52}$$

which implies

$$\lim_{t\to\infty} (\|\phi_t(t,\,\cdot\,)\|_{L^2(M)} + \|\phi_{tt}(t,\,\cdot\,)\|_{L^2(M)}) = 0 \tag{53}$$

by using the uniform continuity of ϕ_t and ϕ_{tt}. Therefore

$$\xi \subset E = \{\omega \in C^2(M) : \Delta_g \omega + f(\omega) = 0\}, \tag{54}$$

which ends the proof.

The main problem is now to prove that ξ reduces to a single element. Thanks to Simon's results [13]–[15] we know two cases where this holds:

THEOREM 3. *Assume the hypotheses of Theorem* 2 *to hold and let* f *be a real-analytic function in some neighborhood of the closure of the range of* ϕ. *Then there exists* $\omega \in E$ *such that* $\phi(t,\,\cdot\,)$ *converges to* ω *in the* $C^k(M)$*-topology*, *for any* $k \in \mathbb{N}$.

The proof [3], [14], [15] is an application of a very deep and difficult result of [13].

THEOREM 4. *Assume the hypotheses of Theorem* 2 *and that there exists* $\omega \in \xi$ *such that* E *is hyperbolic near* ω, *that is*, $E \cap \mathcal{O}$ *is a* d*-dimensional manifold for some neighborhood* $\mathcal{O}$ *of* ω *and*

$$\dim \operatorname{Ker}(\Delta_g + f'(\omega)I) = d. \tag{55}$$

Then $\phi(t,\,\cdot\,)$ *converges to* ω *in the* $C^2(M)$*-topology.*

Remark 3. The boundedness assumption upon ϕ has the following meaning for equations (1) and (2):

$$u(x) - \operatorname{Ln}(1/|x|^2) \in L^\infty_{\mathrm{loc}}(B_1(0)) \tag{56}$$

for (1), and

$$|x|^{2/(q-1)} u(x) \in L^\infty_{\mathrm{loc}}(B_1(0)) \tag{57}$$

for (2). In the case of (1) no estimate of type (56) is known. In the case of (2) estimate (57) is only known when $u \geq 0$ and $q < (N+2)/(N-2)$ ([3], [9]).

Remark 4. The hyperbolicity assumption in Theorem 4 is not easy to check in general. In the particular case of the set G_d ($d \geq 2$) this hyperbolicity property can be checked thanks to the transitivity of the conformal group action.

We now give some applications of Theorems 3–4 [3].

THEOREM 5. *Assume u is a solution of* (1) *in* $B_1(0) \setminus \{0\}$ *such that*

$$|x|^2 e^{u(x)} \leq k \tag{58}$$

for some constant k and any $0 < |x| \leq 1/2$. *Then either*

(i) *u is a smooth solution of* (1) *in* $B_1(0)$, *or*

(ii) *there exists* $\gamma < 0$ *such that*

$$-\Delta u = \lambda e^u + 4\pi\gamma\delta_0 \tag{59}$$

in $\mathcal{D}'(B_1(0))$ *and*

$$\lim_{x \to 0} |x| u(x) = \gamma, \tag{60}$$

or

(iii) *there exists* $\omega \in G_2$ *such that*

$$\lim_{r \to 0} \left(u(r,\,\cdot\,) - \operatorname{Ln} \frac{1}{r^2} \right) = \operatorname{Ln} \frac{2}{\lambda} + 2\omega(\,\cdot\,) \tag{61}$$

in the $C^k(S^2)$*-topology, for any* $k \in \mathbb{N}$.

For the exterior problem we have

THEOREM 6. *Assume u is a solution of* (1) *in* $\mathbb{R}^N \setminus B_1(0)$ *such that* (58) *holds for* $|x| \geq 2$ *and some constant k. Then there exists* $\omega \in G_2$ *such that*

$$\lim_{r \to \infty} \left(u(r,\,\cdot\,) - \operatorname{Ln} \frac{1}{r^2} \right) = \operatorname{Ln} \frac{2}{\lambda} + 2\omega(\,\cdot\,) \tag{62}$$

in the $C^k(S^2)$*-topology, for any* $k \in \mathbb{N}$.

THEOREM 7. *Assume* $1 < q < (N+2)/(N-2)$, λ *is such that*

$$\lambda < l = \frac{2}{q-1} \left(N - \frac{2q}{q-1} \right) \tag{63}$$

and u is a nonnegative solution of

$$-\Delta u = u^q + \frac{\lambda}{|x|^2} u \tag{64}$$

in $B_1(0)$. *Then either*

(i) *there exists* $\omega \in C^\infty(S^{N-1})$ *such that*

$$-\Delta_{S^{N-1}} \omega + (l - \lambda)\omega - \omega^q = 0 \tag{65}$$

on S^{N-1} *and*

$$\lim_{r \to 0} r^{2/(q-1)} u(r,\,\cdot\,) = \omega(\,\cdot\,) \tag{66}$$

in the $C^k(S^{N-1})$*-topology, or*

(ii) *there exists* $\gamma \geq 0$ *such that*

$$\lim_{x\to 0} u(x)/\mu(x) = \gamma \tag{67}$$

where

$$\mu(x) = |x|^{(2-N+\sqrt{(N-2)^2-4\lambda})/2}. \tag{68}$$

Soliton solutions. In the particular case of (2) with $q = (N+2)/(N-2)$ the method developed in Theorems 2–4 does not work. If we study the more general conformally invariant equation

$$-\Delta u = |u|^{4/(N-2)}u + \frac{\lambda}{|x|^2}u \tag{69}$$

in an N-dimensional domain we first notice that the quantity

$$A(r) = r^{-1}\int_{|x|=r}\Big(|Du|^2|x|^2 - 2(Du\cdot x)^2 - (N-2)u(Du\cdot x) - \frac{\lambda}{|x|^2}u^2 - \frac{N-2}{N}|u|^{2N/(N-2)}\Big)\,dS \tag{70}$$

is independent of r (this corresponds to the fact that the coefficient of v_t in (45) is zero). It is interesting to notice the existence of soliton solutions of (69), that is, solutions of the form

$$u(r,\sigma) = r^{-(N-2)/2}\omega(e^{-(\mathrm{Ln}\, r)A}(\sigma)) \tag{71}$$

where A is a skew-symmetric matrix. If L_A is the Lie derivative associated with the vector field $\sigma \mapsto A(\sigma)$, then ω satisfies

$$\Delta_{S^{N-1}}\omega + L_A L_A\omega + |\omega|^{4/(N-2)}\omega - \bigg(\Big(\frac{N-2}{2}\Big)^2 - \lambda\bigg)\omega = 0 \tag{72}$$

on S^{N-1}, which is the Euler–Lagrange equation of the functional

$$J_A(\omega) = \frac{1}{2}\int_{S^{N-1}}\bigg(|\nabla\omega|^2 + (L_A\omega)^2 + \bigg(\Big(\frac{N-2}{2}\Big)^2 - \lambda\bigg)\omega^2 - \frac{N-2}{N}|u|^{2N/(N-2)}\bigg)\,d\sigma \tag{73}$$

defined on $W^{1,2}(S^{N-1})$.

A natural problem is to study the existence of nontrivial or non-A-invariant solutions of (72). By using the method of Theorem 1 we can prove the following result:

THEOREM 8. *Assume* $\lambda \geq (2-N)/4$. *Then any positive solution of* (72) *is a constant.*

Remark 5. In the particular case where $\lambda = 0$, Theorem 8 is a consequence of the very deep results contained in [4].

By using the Lyusternik–Schnirelmann theory [12] we can prove the following [3]:

THEOREM 9. *There exist infinitely many different solutions of* (72), *all corresponding to different critical values of* J_A.

Our conjecture concerning the role of the solutions of (72) is the following:

Assume u *is a nonnegative solution of* (69) *in* $B_1(0) \setminus \{0\}$. *Then either*

(i) *there exists a skew-symmetric matrix* A *and a nonconstant positive solution of* (72) *on* S^{N-1} *such that*

$$\lim_{r\to 0}(r^{(N-2)/2}u(r,\sigma) - \omega(e^{-(\operatorname{Ln} r)A}(\sigma)) = 0, \tag{74}$$

or

(ii) *there exists a nonnegative solution* ψ *of*

$$\psi_{rr} + \psi^{(N+2)/(N-2)} - \left(\frac{N-2}{2}\right)^2 - \lambda\psi = 0 \tag{75}$$

such that

$$\lim_{r\to 0} r^{(N-2)/2}(u(r,\sigma) - \psi(r)) = 0. \tag{76}$$

A similar conjecture can be made for the solutions of

$$\Delta u = |u|^{4/(N-2)}u - \frac{\lambda}{|x|^2}u \tag{77}$$

in a punctured ball (notice that all the solutions are uniformly bounded by $C|x|^{(2-N)/2}$ near 0).

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