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SUPERPOSITION OF FUNCTIONS IN SOBOLEV SPACES OF FRACTIONAL ORDER. A SURVEY

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0. Introduction. The present paper is concerned with the study of the nonlinear operator

$$T_G : f \to G(f)\,, \tag{0.1}$$

where $G : \mathbb{R}^1 \to \mathbb{R}^1$ is a given function and f is taken from a generalized Sobolev space $H_p^s(\mathbb{R}^n)$ (cf. Section 1 for definitions). Operators of that type are called superposition or Nemytskiĭ operators, and play a crucial role in nonlinear analysis. Our aim here is to describe under what conditions one can establish an embedding of the form

$$T_G(H_p^{s_0}(\mathbb{R}^n)) \hookrightarrow H_p^{s_1}(\mathbb{R}^n)\,, \qquad s_1 \le s_0\,. \tag{0.2}$$

Since the paper of Dahlberg [7] it is known that one cannot expect $s_0 = s_1$ in general. The loss of smoothness under the superposition, even in the case $G \in C^\infty(\mathbb{R}^1)$, depends on the dimension n as well as on the smoothness and integrability properties of $f \in H_p^s(\mathbb{R}^n)$. This behaviour of T_G will be explained in what follows. Let us mention that all results are presented in the framework of the scale $H_p^s(\mathbb{R}^n)$. However, they remain true if one replaces $H_p^s(\mathbb{R}^n)$ by Slobodetskiĭ spaces $W_p^s(\mathbb{R}^n)$, the more general Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ or the Triebel–Lizorkin spaces $F_{p,q}^s(\mathbb{R}^n)$ (a generalization of $H_p^s(\mathbb{R}^n)$, cf. Triebel [30]). Let us refer also to the recent monograph by Appell and Zabreĭko [2], where such problems are investigated from a somewhat different point of view.

This survey summarizes recent results obtained by the Jena research group on function spaces around H. Triebel. It is based on a lecture given at the Stefan Banach International Center in Warsaw in November 1990.

1. Sobolev spaces of fractional order. The symbol $\mathbb{R}^n$ represents the Euclidean n-space, by $\mathbb{Z}$ we denote the set of all integers, and by $\mathbb{N}$ all natural numbers. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$, $S'(\mathbb{R}^n)$ the set of all tempered distributions on $\mathbb{R}^n$, F and F^{-1} the Fourier transform and its inverse on $S'(\mathbb{R}^n)$, respectively.

Definition. Let $1 < p < \infty$ and $s \geq 0$. The *Sobolev space* $H^s_p(\mathbb{R}^n)$ *of fractional order* s is the set of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f|H^s_p(\mathbb{R}^n)\| = \|F^{-1}(1+|\xi|^2)^{s/2}Ff|L_p(\mathbb{R}^n)\| < \infty\,. \tag{1.1}$$

Remark 1. We follow here the classical approach of Aronszajn–Smith [3] and Calderón [5]. Sometimes the spaces $H^s_p(\mathbb{R}^n)$ are also called Liouville spaces (in particular in the Russian literature) or Bessel-potential spaces.

Remark 2. A more explicit description of $H^s_p(\mathbb{R}^n)$ can be obtained with the help of differences. We put

$$(\Delta^1_h f)(x) = f(x+h) - f(x)\,, \qquad (\Delta^l_h f)(x) = \Delta^1_h(\Delta^{l-1}_h f)(x)\,, \qquad l = 2, 3, \ldots$$

Then we have with $l > s > 0$, $l \in \mathbb{N}$,

(1.2) $\quad f \in H^s_p(\mathbb{R}^n) \Leftrightarrow f \in L_p(\mathbb{R}^n)$ and

$$\|f|L_p(\mathbb{R}^n)\| + \bigg\|\bigg(\int\limits_0^1 r^{-2s}\Big(\int\limits_{\{h:|h|\le 1\}} |\Delta^l_{rh} f(\cdot)|dh\Big)^2 \frac{dr}{r}\bigg)^{1/2}\bigg|L_p(\mathbb{R}^n)\bigg\| < \infty\,.$$

Moreover, the expression in (1.2) yields an equivalent norm in $H^s_p(\mathbb{R}^n)$ (cf. Triebel [30]).

Basic properties. This scale generalizes the classical Sobolev spaces in a natural way:

(i) $H^s_p(\mathbb{R}^n)$ equipped with the norm in (1.1) is a Banach space,
(ii) $H^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$, $m = 1, 2, \ldots$,
(iii) $H^{s_0}_p(\mathbb{R}^n) \hookrightarrow H^{s_1}_p(\mathbb{R}^n) \hookrightarrow H^0_p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ if $s_0 \geq s_1 \geq 0$ ("$\hookrightarrow$" always means continuous embedding),
(iv) $f \in H^s_p(\mathbb{R}^n)$ implies $\partial f/\partial x_i \in H^{s-1}_p(\mathbb{R}^n)$, $i = 1, \ldots, n$, if $s \geq 1$,
(v) $H^s_p(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n) \Leftrightarrow H^s_p(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n) \Leftrightarrow s > n/p$ (cf. Fig 1).

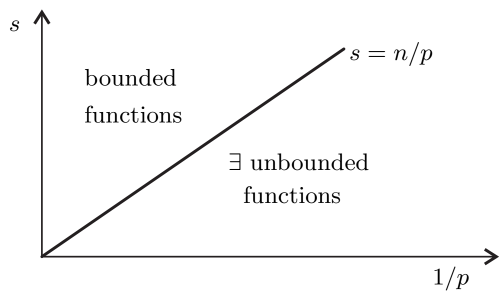

Fig. 1

For (i)–(v) we refer to [30].

Finally, we consider two distinguished families of functions. Let Ψ be a smooth cut-off function supported around zero and let $\alpha > 0$. Then we define

$$g_\alpha(x) = \Psi(x)|x|^\alpha \,, \tag{1.3}$$

$$f_\alpha(x) = \Psi(x)|x|^{-\alpha} \,. \tag{1.4}$$

It is known (cf. Stein [26], Triebel [30]) that

$$g_\alpha \in H^s_p(\mathbb{R}^n) \ \Leftrightarrow \ s < n/p + \alpha \,, \tag{1.5}$$

$$f_\alpha \in H^s_p(\mathbb{R}^n) \ \Leftrightarrow \ s < n/p - \alpha \,. \tag{1.6}$$

In particular, the family g_α shows great similarity between measuring smoothness in the H^s_p-scale and in the C^s-scale (Hölder spaces).

If there is no danger of confusion we shall omit $\mathbb{R}^n$ in notations.

2. Boundedness of superposition operators. Our programme is to discuss the following three principal cases for the outer function G:

(i) $G(t) = t^m$, $m = 2, 3, \ldots$,
(ii) $G(t) = |t|^\mu$, $\mu > 1$,
(iii) $G(t) \in C^\infty(\mathbb{R}^1)$.

To do this we follow the way in which the pertinent results were proved. As we shall see the most striking feature will be the different behaviour of T_G for bounded and unbounded functions. In this survey much attention is paid to describe the embedding (1.2) with proper inequalities.

2.1. *Powers of f.* First we investigate powers f^m of f. It is a nonlinear problem, of course, but we can deal with it as a linear one, considering the family of operators

$$T_{[g_1,\ldots,g_{m-1}]}(f) = (g_1 \circ \ldots \circ g_{m-1}) \circ f \,, \qquad f \in H^s_p \,,$$

where $g_1, \ldots, g_{m-1} \in H^s_p$ are fixed functions. Nowadays this problem is well understood. It is the problem of pointwise multipliers with respect to H^s_p.

THEOREM 1 ([23]). *Let $m = 2, 3, \ldots$*

(i) *Let $s > n/p$. Then there exists a constant c such that*

$$\|f^m | H^s_p\| \le c \|f | H^s_p\|^m \quad \textit{for all } f \in H^s_p \,. \tag{2.1}$$

(ii) *Let $0 < s < n/p$. Let*

$$s_m = s - (m-1)(n/p - s) > 0 \,. \tag{2.2}$$

Then there exists a constant c such that

$$\|f^m | H^{s_m}_p\| \le c \|f | H^s_p\|^m \quad \textit{for all } f \in H^s_p \,. \tag{2.3}$$

Remark 3. Whereas for bounded functions ($s > n/p$) the result shows a good correspondence to that in the case of Hölder spaces C^s, the second part of Theorem 1 requires some further comments. Since $(f_\alpha)^m$ is of the same type as

f_α, but with a local singularity of order $m\alpha$, we can apply (1.6) to both functions. This yields $f_\alpha^m \in H_p^r \Leftrightarrow r < n/p - \alpha m$. For $f_\alpha \in H_p^s$, $\alpha \uparrow (n/p - s)$ we get

$$r \le n/p - m(n/p - s) = s - (m-1)(n/p - s) = s_m\,.$$

This shows that each multiplication leads to a loss of smoothness of order $n/p - s$. Also the condition (2.2) can be interpreted with the help of the family f_α. The inequality $s_m > 0$ simply ensures $f_\alpha^m \in L_p = H_p^0$.

Remark 4. The statement (i) is a simple consequence of the fact that H_p^s, $s > n/p$, forms a multiplication algebra, a famous result of Strichartz [27]. The second statement in Theorem 1 was proved by Yamazaki [32] with the help of the paramultiplication principle. For a more detailed description (also in case $s = n/p$) and further references we refer to the survey [23].

2.2. *The real powers $|f|^\mu$, $\mu > 1$.* A new phenomenon appears when investigating $G(t) = |t|^\mu$, $\mu > 1$, as the outer function. The finite smoothness of $|t|^\mu$ leads to a restriction on the smoothness of the superposition $G(f)$.

THEOREM 2 ([20], [24]). *Let $\mu > 1$.*

(i) *Let $n/p < s < \mu$. Then there exists a constant c such that*

$$\| \, |f|^\mu | H_p^s | | \le c \|f | H_p^s\|^\mu \quad \textit{for all } f \in H_p^s\,. \tag{2.4}$$

(ii) *Let $0 < s < n/p$. Let*

$$0 < s_\mu = s - (\mu - 1)(n/p - s) < \mu\,. \tag{2.5}$$

Then there exists a constant c such that

$$\| \, |f|^\mu | H_p^\mu \| \le c \|f | H_p^s\|^\mu \quad \textit{for all } f \in H_p^s\,. \tag{2.6}$$

Remark 5. For (ii) we can argue as in Theorem 1: again using the family f_α one derives that (2.5) ensures that $T_\mu^* : f \to |f|^\mu$ maps H_p^s into L_p.

Remark 6. Part (i) is a consequence of a more general result proved by Runst [20]. A proof of (ii) may we found in Sickel [24]. Partial results may also be found in Triebel [31] and Edmunds–Triebel [9].

A remark on the proof and a first generalization. In both cases the proof is based on the use of the Taylor expansion of $G(t) = |t|^\mu$, $\mu > 1$. The estimate of the Taylor polynomial reduces to an application of Theorem 1. To obtain an estimate of the remainder one has to investigate the integral means

$$(I_k^\mu f)(x) = \int\limits_{|z| \le 2^{-k}} |f(x+z) - f(x)|^\mu \, dz\,, \qquad k \in \mathbb{Z}\,. \tag{2.7}$$

In Runst [20] and Sickel [24] different estimates for these means were derived by using maximal-function techniques (Fefferman–Stein–Peetre maximal inequality, Hardy–Littlewood maximal inequality).

However, only the following qualitative properties of $G(t) = |t|^\mu$ are used:

$$G : \mathbb{R}^1 \to \mathbb{R}^1\,, \tag{2.8}$$

$$|G^{(l)}(t)| \le c_l |t|^{\mu-l}, \qquad l = 0, \ldots, N,\ N \in \mathbb{N}\,, \tag{2.9}$$

$$\sup_{t_0 \ne t_1} \frac{|G^{(N)}(t_1) - G^{(N)}(t_0)|}{|t_1 - t_0|^\tau} \le c < \infty, \qquad \tau + N = \mu,\ 0 < \tau \le 1\,. \tag{2.10}$$

A simple reformulation of the conditions (2.8)–(2.10) is given by

$$G \text{ is } N \text{ times continuously differentiable,} \tag{2.11}$$

$$G^{(l)}(0) = 0, \qquad l = 0, \ldots, N, \tag{2.12}$$

$$G^{(N)} \in \operatorname{Lip} \tau\,, \tag{2.13}$$

where the Lipschitz space $\operatorname{Lip}\tau$ is characterized by (2.10). To make a composition $G(f)$ meaningful, we restrict ourselves to real-valued functions f.

Definition. Let $1 < p < \infty$ and $s \ge 0$. By $\widetilde{H}^s_p$ we denote the subspace of H^s_p consisting of all real-valued functions $f \in H^s_p$, equipped with the norm (1.1).

Theorem 3 ([20], [24], [25]). *Let G be a function such that* (2.11)–(2.13) *are satisfied for some $\mu > 1$. Then Theorem* 2 *remains true if we replace $|f|^\mu$ by $G(f)$ and H^s_p by $\widetilde{H}^s_p$.*

2.3. *The case $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$.* As usual, $C^m(\mathbb{R}^1)$ denotes the set of functions f such that

(i) $f, \ldots, f^{(m)}$ are uniformly continuous,
(ii) $\|f|C^m(\mathbb{R}^1)\| = \max_{0\le l\le m} \sup_{t\in\mathbb{R}^1} |f^{(l)}(t)| < \infty$.

We put

$$C^\infty(\mathbb{R}^1) = \bigcap_{m=1}^\infty C^m(\mathbb{R}^1)\,.$$

To overcome the restriction (2.12) in Theorem 3 one uses the splitting

$$G(t) = \bigg(G(t) - \sum_{j=0}^N \frac{G^{(j)}(0)}{j!} t^j\bigg) + \sum_{j=0}^N \frac{G^{(j)}(0)}{j!} t^j = H_N(t) + P_N(t)\,.$$

Then $P_N(f)$ is estimated by Theorem 1, and $H_N(f)$ by Theorem 3. If $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$, then N and τ are at our disposal. If $n/p < s$ we choose $\mu > \max(1,s)$, $\mu \downarrow \max(1,s)$. If $s < n/p$ the situation is more complicated. Both μ and s_μ are upper bounds for the smoothness of $G(f)$. Since s_μ decreases if μ increases the optimal choice is $s_\mu = \mu$. We have

$$\mu = s_\mu = \frac{n}{p} - \mu\bigg(\frac{n}{p} - s\bigg) \Leftrightarrow \mu\bigg(\frac{n}{p} - s + 1\bigg) = \frac{n}{p} \Leftrightarrow \mu = \frac{n/p}{n/p - s + 1}\,.$$

From this point of view the following result is not surprising.

THEOREM 4. *Let* $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$.

(i) *Let* $s > n/p$. *Then there exists a constant* c *such that*

$$\|G(f)|H_p^s\| \le c(\|f|H_p^s\| + \|f|H_p^s\|^{\max(1,s)}) \quad \text{for all } f \in \widetilde{H}_p^s\,. \tag{2.14}$$

(ii) *Let* $1 < s < n/p$. *Let*

$$\varrho(s, n/p) = \varrho = \frac{n/p}{n/p - s + 1}\,. \tag{2.15}$$

Then there exists a constant c *such that*

$$\|G(f)|H_p^\varrho\| \le c(\|f|H_p^s\| + \|f|H_p^s\|^\varrho) \quad \text{for all } f \in \widetilde{H}_p^s\,. \tag{2.16}$$

(iii) *Let* $0 \le s \le 1$. *Then there exists a constant* c *such that*

$$\|G(f)|H_p^s\| \le c\|f|H_p^s\| \quad \text{for all } f \in \widetilde{H}_p^s\,. \tag{2.17}$$

Some comments. (i) The case $s > n/p$. With regard to this case there are numerous references. The first is Mizohata [15], who had discovered $T_G(\widetilde{H}_2^s) \hookrightarrow H_2^s$, $s > n/2$, in 1965. Fifteen years later Meyer [14] established $T_G(\widetilde{H}_p^s) \hookrightarrow H_p^s$ by using the elegant method of paradifferential operators. Inspired by Meyer's work there exist further extensions to the classes $B_{p,q}^s$ and $F_{p,q}^s$ (Runst [19]), to anisotropic spaces (Yamazaki [32]), and to weighted spaces (Marschall [13]). Runst [20] applied maximal function techniques to this problem. However, the simple structure of (2.14), including the exponents, seems to be new. Note that at least for the Sobolev spaces H_p^m $(= W_p^m)$ these exponents are optimal. We refer to Sickel [25].

(ii) The case $0 \le s \le 1$. Because our function G is smooth one can apply the chain rule. Now, (2.17) is a simple consequence for $s = 1$. If $0 < s < 1$ then (2.17) follows from (1.2). In case $s = 0$ inequality (2.17) is again obvious.

(iii) The case $1 < s < n/p$. First, note that the restriction on s implies $1 < \varrho < s$, so we have some loss of smoothness. The reason becomes clear by the following example. Again we use the family f_α defined in (1.4). We have

$$\frac{\partial^m}{\partial x_1^m} G(f(x)) \sim G^m(f(x))\left(\frac{\partial f}{\partial x_1}\right)^m + \text{ lower order terms} \sim (|x|^{-\alpha-1})^m \qquad \text{as } |x| \to 0\,,$$

at least if $G^{(m)}(t) \nrightarrow 0$ as $t \to \infty$. Compare this with

$$\frac{\partial^m f}{\partial x_1^m}(x) \sim |x|^{-\alpha-m} \quad \text{as } |x| \to 0\,.$$

Hence, superpositions with even very smooth functions G create stronger singularities in the derivatives of order $m \ge 2$. Inequality (2.16) is proved in Sickel [24].

2.4. *The counterexample of Dahlberg.* As mentioned in the introduction more than ten years ago Dahlberg [7] proved: If $G \in C^2(\mathbb{R}^1)$ such that $G(f) \in W_p^m$ for all $f \in \widetilde{W}_p^m$, where $1+1/p < m < n/p$, then G is a linear function. The example he used is a function of the type

$$f(x) = \sum_{j=1}^{\infty} j^{\beta} u(j^{\alpha}(x - z^j)) , \tag{2.18}$$

where $u \in C_0^{\infty}$, $u(x) = u(x_1, \ldots, x_n) = x_1$ if $|x| \le 1$, $u(x) = 0$ if $|x| \ge 2$, $\{z^j\}_{j=1}^{\infty}$ is an appropriate sequence in $\mathbb{R}^n$ and α, β are positive real numbers.

By using the same example the degeneracy result was extended to $\widetilde{B}_{p,q}^s$ and $\widetilde{F}_{p,q}^s$ by Bourdaud [4] and Runst [20]. The problem of measuring this loss of smoothness was first treated in Sickel [24]. Again we applied the construction (2.18).

THEOREM 5 ([24]). *Let $1<s<n/p$. Let $\tau>0$. Let G be τ-periodic, sufficiently smooth, and non-trivial. Then for all $\varepsilon > 0$ there exists $f_\varepsilon \in \widetilde{H}_p^s$ (with arbitrarily small support) such that $G(f_\varepsilon) \notin H_p^{\varrho+\varepsilon}$.*

Remark 7. Theorem 5 proves that Theorem 4(ii) is sharp in the sense that the exponent ϱ cannot be improved in general.

We make a simple observation concerning the loss of smoothness. Let n and p be fixed such that $n/p > 1$. We define

$$d(s) = s - \varrho(s, n/p) .$$

One easily checks $\lim_{s\downarrow 1} d(s) = \lim_{s\uparrow n/p} d(s) = 0$. Furthermore, $\varrho < s$ if $1 < s < n/p$ and $d(s)$ is concave there. Hence, $d(s)$ has a maximum on $(1, n/p)$. It is taken at the point

$$s_0 = n/p - \sqrt{n/p} + 1 , \tag{2.19}$$

and

$$d(s_0) = (\sqrt{n/p} - 1)^2 \tag{2.20}$$

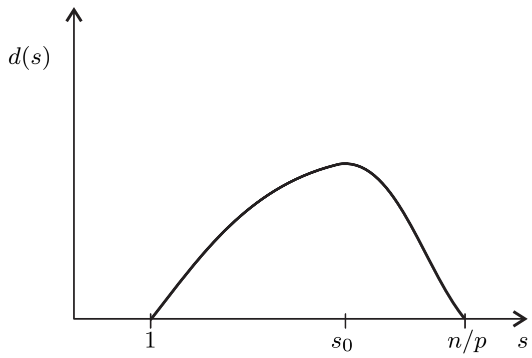

Fig. 2

(cf. Fig. 2). Consequently, $d(s)$ can become arbitrarily large if $n/p \to \infty$. To make the behaviour of T_G more clear, we draw a further figure.

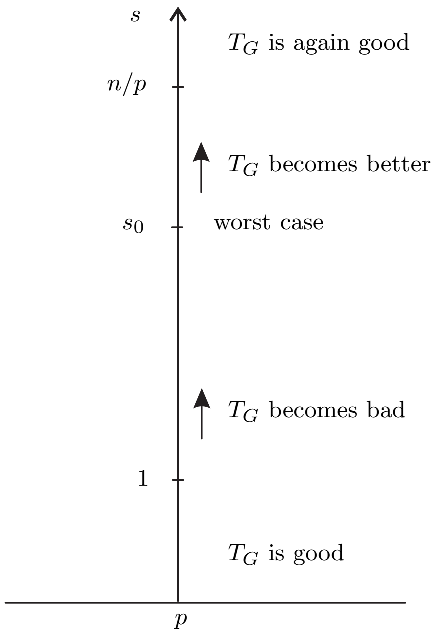

Fig. 3

Here "T_G is good" means T_G maps a space H^s_p into itself and "T_G becomes bad" means $d(s)$ is increasing. On the other hand, "T_G becomes better" is used for $d(s)$ decreasing.

Figure 3 shows that the behaviour of nonlinear operators can be completely different from that of linear ones. Since T_G is good on H^s_p, $s > n/p$, and on H^s_p, $s \le 1$, by interpolation one would also expect a good behaviour with respect to $[H^{s_0}_p, L_p]_\theta = H^s_p$, $0 < \theta < 1$, $s = (1-\theta)s_0$ (cf. Triebel [29]). But this is false by Fig. 3.

Remark 8. Note that τ-periodicity of G in Theorem 5 is not necessary. One needs the existence of a sequence of disjoint intervals $\{I_j\}_{j=1}^\infty$ with

$$\inf_j |I_j| \ge A > 0\,, \tag{2.21}$$

$$I_j \subset \{t : |G^{(m+1)}(t)| \ge B > 0\}\,, \tag{2.22}$$

where $m+1 = [\varrho + 1]$ (integer part) for some $A, B > 0$.

Since a function like $(1+t^2)^{-\alpha}$, $\alpha > 0$, cannot satisfy (2.21), (2.22), the following degeneracy result is also of interest.

Theorem 6 ([24]). *Let $1 + 1/p < s < n/p$. Let*

$$\varrho^*\left(s, \frac{n}{p}\right) = \varrho^* = \frac{\frac{n}{p} + \frac{1}{p}\left(\frac{n}{p} - s\right)}{\frac{n}{p} - s + 1}\,. \tag{2.23}$$

Put $m = [\varrho^*]$ (integer part). Let G be sufficiently smooth and let $G^{(m+1)}$ be non-trivial. Then for any $\varepsilon > 0$ there exists $f_\varepsilon \in \widetilde{H}^s_p$ (with arbitrarily small support) such that $G(f) \notin H^{\varrho^*+\varepsilon}_p$.

Remark 9. A short calculation gives $1+1/p < \varrho^* < s$, $\varrho < \varrho^*$ if $1+1/p < s < n/p$, so for any non-trivial G we have some loss of smoothness after superposition.

Remark 10. Positive results for the number ϱ^*, i.e. improvements on Theorem 4(ii) under additional assumptions on G are not known to the author.

Remark 11. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded C^∞-domain. Let

$$H^s_p(\Omega) = \{f \in L_p(\Omega) : \exists g \in H^s_p(\mathbb{R}^n) \text{ such that } g|_\Omega = f\}\,, \tag{2.24}$$

$$\|f|H^s_p(\Omega)\| = \inf_{g|_\Omega = f} \|g|H^s_p(\mathbb{R}^n)\|\,. \tag{2.25}$$

Theorems 5 and 6 are also applicable in this situation, since we can make the support of f_ε as small as we want.

2.5. *Boundedness of superposition operators in Sobolev spaces of fractional order $s \le 1 + 1/p$.* Theorems 5 and 6 make it plausible that under additional conditions on G the operator T_G maps H^s_p into H^s_p if $s \le 1 + 1/p$.

THEOREM 7 ([25]). *Let $1 < p < 2$. Let $0 \le t < s \le 2/p$. Let G be a function with*

(i) $G(0) = 0$,
(ii) $G'' \in L_1(\mathbb{R}^1)$.

Then there exists a constant c such that

$$\|G(f)|H^t_p\| \le c\|f|H^s_p\| \quad \text{for all } f \in \widetilde{H}^s_p\,. \tag{2.26}$$

Remark 12. Theorem 7 is a consequence of the following result of Bourdaut [4]: If G is a function with properties (i) and (ii), then there exists a constant c such that

$$\|G(f)|W^2_1\| \le c\|f|\widetilde{W}^2_1\| \quad \text{for all } f \in \widetilde{W}^2_1\,. \tag{2.27}$$

In Sickel [25] a further extension of (2.27) is obtained with the help of interpolation of nonlinear operators (cf. Peetre [18], Maligranda [11]).

2.6. *An overview.* Our aim is to explain in three figures the different behaviour of T_G for $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$. For simplicity we assume $G \not\equiv 0$.

(i) *The case $n = 1$.* In that case we have a very simple and nice behaviour shown in Fig. 4 (cf. Theorem 4). Here A stands for any space H^s_p, where the couple $(s, 1/p)$ is taken from the shaded region.

(ii) *The case $n = 2$.* As a consequence of Theorems 4 and 7 we obtain the situation as in Fig. 5.

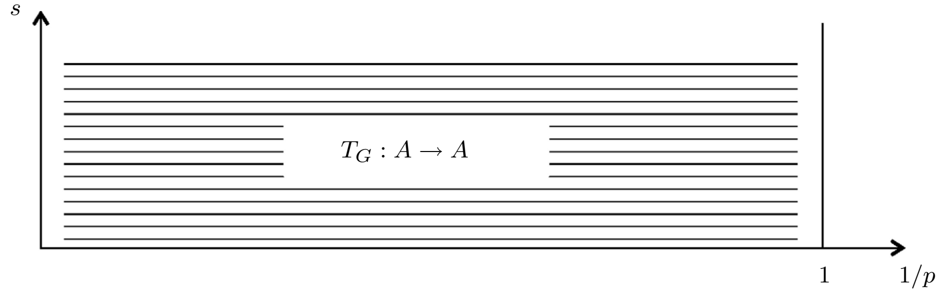

Fig. 4

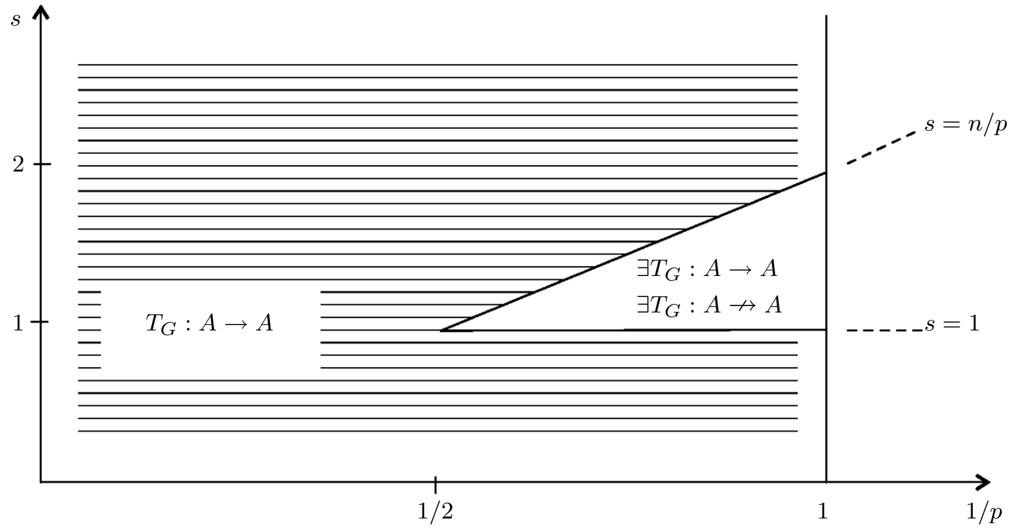

Fig. 5

In the non-shaded region $1 < p < 2$, $1 < s < 2/p$ the symbol "$\exists T_G : A \not\to A$" is used for the fact that there exists some G (cf. Theorem 4 and Remark 8) such that T_G does not map A into A, while "$\exists T_G : A \to A$" means that there exists some G (cf. Theorem 7) such that T_G maps A into A.

(iii) *The general case* $n \geq 3$. Now we have to use Theorems 4–7 (see Fig. 6).

In the region $1 < p < n$, $\max(1, 2/p) < s < 1 + 1/p$ it is an open problem whether there exists some $G \in C^\infty(\mathbb{R}^1)$ such that $T_G : A \to A$ holds. Note that for $1/(n-1) < 1/p < 1$, $1 + 1/p < s < n/p$ we have $T_G : A \not\to A$ for any $G \not\equiv 0$.

2.7. *Some further results on boundedness of superposition operators*

2.7.1. *Moser-type inequalities.* It is known that by restriction to bounded functions one can improve several of the results collected in 2.1–2.3. A first example is the embedding

$$(W_p^m \cap L_\infty) \circ (W_p^m \cap L_\infty) \hookrightarrow W_p^m, \qquad m = 1, 2, \ldots, \tag{2.28}$$

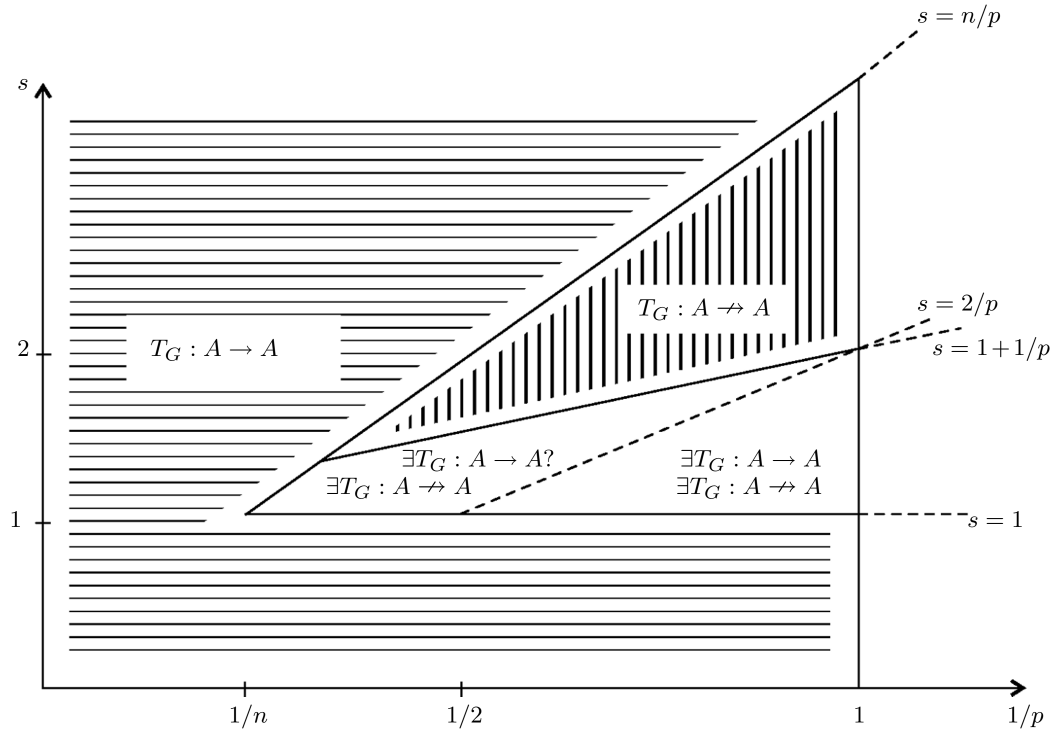

Fig. 6

which holds true without the restriction $m > n/p$ (cf. Nirenberg [17]). Later on Moser [16] dealt with some extensions. Recall that the results in 2.1–2.3 are based on assertions on pointwise multipliers. So, (2.28) gives some hope of improving Theorems 1–4.

THEOREM 8 ([20], [24], [25]). *Let $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$.*

(i) *Let $m = 2, 3, \ldots$ Then there exists a constant c such that*

$$\|f^m|H^s_p\| \le c\|f|H^s_p\|\|f|L_\infty\|^{m-1} \quad \text{for all } f \in H^s_p \cap L_\infty\,. \tag{2.29}$$

(ii) *Let $\mu > 1$ and $s < \mu$. Then there exist a constant c such that*

$$\|\,|f|^\mu|H^s_p\| \le c\|f|H^s_p\|\|f|L_\infty\|^{\mu-1} \quad \text{for all } f \in H^s_p \cap L_\infty\,. \tag{2.30}$$

(iii) *There exists a constant c such that*

$$\|G(f)|H^s_p\| \le c(\|f|H^s_p\| + \|f|H^s_p\|\|f|L_\infty\|^{\max(0,s-1)}) \quad \text{for all } f \in \widetilde{H}^s_p \cap L_\infty\,. \tag{2.31}$$

Remark 13. Further contributions to this subject can be found in Peetre [18] and Adams–Frazier [1]. The first deals with $B^s_{p,q} \cap L_\infty$ (Besov spaces), whereas the second is concerned with the action of T_G on $H^s_p \cap \mathrm{BMO}$.

2.7.2. *An improvement of the integrability properties.* In Sickel [24, 25] we considered the possibility that one can improve the results of the preceding subsections concerning integrability properties. We ask now for an embedding

$$T_G(H^s_{p_0}) \hookrightarrow H^s_{p_1}, \quad p_0 \ge p_1 . \tag{2.32}$$

It is not our aim to treat (2.32) in its full generality. We only mention the following two interesting lemmata.

LEMMA 1. *Let* $0 < s < n/p$.

(i) *Let* $m = 2, 3, \ldots$ *and let*

$$1 < r < \infty, \quad \frac{p}{m} \le r \le \frac{n}{s + m(n/p - s)} . \tag{2.33}$$

Then there exists a constant c such that

$$\|f^m | H^s_r\| \le c \|f | H^s_p\|^m \quad \text{for all } f \in H^s_p . \tag{2.34}$$

(ii) *Let* $\max(1, s) < \mu$ *and let*

$$1 < r < \infty, \quad \frac{p}{\mu} \le r \le \frac{n}{s + \mu(n/p - s)} . \tag{2.35}$$

Then there exists a constant c such that

$$\| |f|^\mu | H^s_r\| \le c \|f | H^s_p\|^\mu \quad \text{for all } f \in H^s_p . \tag{2.36}$$

LEMMA 2. *Let Ω be a bounded C^∞-domain. Let $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$. Let $0 < s < n/p$ and*

$$1 < r < \frac{n}{s + \max(1, s)(n/p - s)} . \tag{2.37}$$

Then there exists a constant c such that

$$\|G(f) | H^s_r(\Omega)\| \le c(\|f | H^s_p(\Omega)\| + \|f | H^s_p(\Omega)\|^{\max(1,s)}) \quad \text{for all } f \in \widetilde{H}^s_p(\Omega) . \tag{2.38}$$

R e m a r k 14. From the embedding relations for H^s_p-spaces we know that

$$H^s_r \hookrightarrow H^{s_\mu}_p, \quad r = \frac{n}{s + \mu(n/p - s)} > 1 , \tag{2.39}$$

and $H^s_{r+\varepsilon} \hookrightarrow H^{s_\mu + \delta}_p$, $\varepsilon > 0$, $\delta = \delta(\varepsilon) > 0$ (cf. Triebel [30]). Thus, the number r cannot be improved since s_μ is best possible (cf. Theorem 2, Remarks 3 and 5).

R e m a r k 15. Of course, (2.33), (2.35), and (2.37) also imply further restrictions on s. For instance, from (2.33), (2.35) we find

$$s > \frac{n}{p} - \frac{1}{\mu - 1}\left(n - \frac{n}{p}\right) \tag{2.40}$$

to guarantee $n/(s+\mu(n/p-s)) > 1$. Using a similar condition to (2.40), Cazenave and Weissler [6] proved a corresponding statement for homogeneous Besov spaces.

2.7.3. *Minimal smoothness conditions on G.* It is more or less clear that $G \in C^\infty(\mathbb{R}^1)$ is far from optimal. A more detailed examination of our approach yields that $G \in C^r(\mathbb{R}^1)$ with $r > s$ (in the case of (2.14), (2.27)) or $r > \varrho$ (in the case of (2.15)) is always sufficient. Moreover, the constants which appear in these inequalities have the form $c = c'\|G|C^r(\mathbb{R}^1)\|$, c' independent of G. However, this is not optimal either. For Sobolev spaces W^1_p, it is known (cf. Marcus and Mizel [12]) that T_G maps W^1_p into W^1_p if and only if

(i) G is locally Lipschitz continuous if either $p > n$, or $n = 1$ and $p \geq 1$,
(ii) G is uniformly Lipschitz continuous if $p < n$.

Also the result of Bourdaud [4] mentioned in Remark 12 cannot be improved, at least if $W^2_1 \not\hookrightarrow L_\infty$. A more general result in this direction is the following.

THEOREM 9 ([25]). *To have an embedding*

$$T_G(H^s_p) \hookrightarrow H^s_p \tag{2.41}$$

it is necessary that $G \in H^{s,\mathrm{loc}}_p(\mathbb{R}^1)$.

Remark 16. In this connection let us refer to Szigeti [28] who stated that

$$T_G(W^m_p([a,b])) \hookrightarrow W^m_p([a,b]), \qquad -\infty < a < b < \infty\,, \tag{2.42}$$

if $G \in W^m_p(\mathbb{R}^1)$ and $m \geq 2$. Moreover, he investigated the example $f(x) = |x|^{\alpha-1/p}\psi(x)$, $x \in \mathbb{R}^1$, $\alpha > 1/p$, and $G(t) = |t|^{\beta-1/p}\psi(t)$, $t \in \mathbb{R}^1$, $\beta > 1/p$ (cf. (1.3), (1.5)). The superposition results in

$$G(f(x)) \sim |x|^{(\alpha-1/p)(\beta-1/p)} \qquad \text{near zero, which gives}$$
$$G(f(x)) \in H^r_p(\mathbb{R}^1), \qquad r < (\alpha-1/p)(\beta-1/p) + 1/p\,.$$

Because of $f \in H^s_p(\mathbb{R}^1)$, $s < \alpha$, it is necessary to have

$$(\alpha - 1/p)(\beta - 1/p) + 1/p > \alpha$$

to guarantee the embedding (2.42). This means $\beta - 1/p > 1$. Hence, in that case $G \in H^s_p(\mathbb{R}^1)$, $s > 1 + 1/p$, is necessary to have (2.42).

In the literature some attention is also paid to the mappings $f \to |f|$ or equivalently to $f \to \max(0, f) = f_+$, $f \to \min(0, f) = f_-$. As a supplement to Theorem 7 and to the above-mentioned result of Marcus and Mizel [12] we have proved the following in Runst–Sickel [22]:

THEOREM 10. *Let $\varepsilon > 0$. Let $1 < p < 2$. Let $0 \leq s < 2/p$. Then there exists a constant c_ε such that*

$$\|\,|f|\,|H^s_p\| \leq c_\varepsilon \|f|H^{s+\varepsilon}_p\| \quad \textit{for all } f \in H^{s+\varepsilon}_p\,. \tag{2.43}$$

Remark 17. The proof in [22] is based on the fact that the translates and dilates of the hut function N (see Fig. 7) form a dense set in H^s_p, $1 < p < \infty$, $0 \leq s < 1 + 1/p$. Furthermore, one can use the formula $|\sum_j \alpha_j N(t-j)| = \sum_j |\alpha_j| N(t-j)$, $t \in \mathbb{R}^1$.

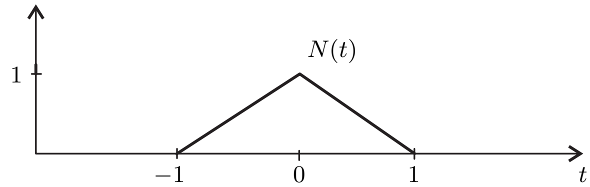

Fig. 7

2.7.4. $\mathbb{R}^m \to \mathbb{R}^1$ *functions* G. Using similar ideas to the case of $\mathbb{R}^1 \to \mathbb{R}^1$ functions one obtains the following generalizations of Theorems 4 and 8.

THEOREM 11 ([25]). *Let* $G : \mathbb{R}^m \to \mathbb{R}^1$, $G(0, \ldots, 0) = 0$ *and* $G \in C^\infty(\mathbb{R}^n)$.

(i) *Let* $s \ge 0$. *Then there exists a constant* c *such that*

$$\|G(f_1, \ldots, f_m)|H_p^s\| \le c \max_{i=1,\ldots,m} (\|f_i|H_p^s\| + \|f_i|H_p^s\| \|f_i|L_\infty\|^{\max(0,s-1)}) \tag{2.44}$$

for all $(f_1, \ldots, f_m) \in (\widetilde{H}_p^s \cap L_\infty)^m$.

(ii) *Let* $1 < s < n/p$. *Let* ϱ *be defined as in* (2.15). *Then there exists a constant* c *such that*

$$\|G(f_1, \ldots, f_m)|H_p^\varrho\| \le c \max_{i=1,\ldots,m} (\|f_i|H_p^s\| + \|f_i|H_p^s\|^\varrho) \tag{2.45}$$

for all $(f_1, \ldots, f_m) \in (\widetilde{H}_p^s)^m$.

(iii) *Let* $0 \le s \le 1$. *Then there exists a constant* c *such that*

$$\|G(f_1, \ldots, f_m)|H_p^s\| \le c \max_{i=1,\ldots,m} (\|f_i|H_p^s\|) \tag{2.46}$$

for all $(f_1, \ldots, f_m) \in (\widetilde{H}_p^s)^m$.

3. Continuity and differentiability of T_G.

In most applications continuity and smoothness properties of T_G are also of interest.

3.1. *Continuity of* T_G. The following simple trick yields the continuity of T_G as a consequence of its boundedness. We apply the interpolation inequality

$$\|G(f) - G(g)|H_p^s\| \le \|G(f) - G(g)|H_p^{s_0}\|^{1-\theta} \|G(f) - G(g)|L_p\|^\theta , \tag{3.1}$$

where $0 < \theta < 1$, $s = (1-\theta)s_0$ (cf. Triebel [29, 30]). Then the L_p continuity of T_G in connection with its $H_p^{s_0}$ boundedness yield the continuity of T_G as a mapping from $H_p^{s_0}$ into H_p^s. By choosing θ sufficiently small, the defect $s_0 - s$ can be made arbitrarily small.

A little more elegant is the following application of Theorem 11, which works for bounded functions. We use the identity

$$\begin{aligned} G(f) - G(g) &= \frac{G(f) - G(g)}{f - g}(f - g) - G'(0)(f - g) + G'(0)(f - g) \\ &= H(f, g)(f - g) + G'(0)(f - g) \end{aligned}$$

and the fact that $H_p^s \cap L_\infty$ is a multiplication algebra. This leads to the following theorem (cf. Franke–Runst [10], Drabek–Runst [8], Sickel [25]).

THEOREM 12. *Let $G \in C^\infty(\mathbb{R}^1)$, $G(0) = 0$. Then T_G is locally Lipschitz continuous as a mapping of $\widetilde{H}_p^s \cap L_\infty$ into itself. Moreover*,

$$\begin{aligned}(3.2)\quad &\|G(f) - G(g)|H_p^s\| \\ &\le c(\|f-g|H_p^s\| + \|f-g|L_\infty\| \max(\|f|H_p^s\| + \|f|H_p^s\|\|f|L_\infty\|^{\max(0,s-1)}, \\ &\qquad \|g|H_p^s\| + \|g|H_p^s\|\|g|L_\infty\|^{\max(0,s-1)})\end{aligned}$$

for all $f, g \in \widetilde{H}_p^s \cap L_\infty$.

3.2. *Differentiability*. Sometimes also differentiability properties of T_G are of interest. Here we only present the following result.

THEOREM 13 ([25]). *Let G be an infinitely differentiable function on $\mathbb{R}^1$. Let Ω be a bounded C^∞-domain. Let $s > n/p$. Then the operator T_G is infinitely differentiable as a mapping from $H_p^s(\Omega)$ into $H_p^s(\Omega)$. We have*

$$(3.3)\qquad (T_G(f))^{(j)}[g_1, \ldots, g_j] = G^{(j)}(f) g_1 \circ \ldots \circ g_j, \qquad j = 1, 2, \ldots,$$

$f \in H_p^s(\Omega)$, $g_1, \ldots, g_j \in H_p^s(\Omega)$. Moreover,

$$(3.4)\qquad \left\| G(f+g) - \sum_{j=0}^{N} \frac{G^{(j)}(f)}{j!} g^j \middle| H_p^s(\Omega) \right\| \le c\|g|H_p^s(\Omega)\|^{N+1}(1 + \|g|H_p^s(\Omega)\|)$$

for all $f, g \in H_p^s$ and all $N = 1, 2, \ldots$

A final remark. In Runst [19–21], Runst–Sickel [22], Triebel [31] and Sickel [23–25] boundedness and continuity of superposition operators are investigated in the scales $B_{p,q}^s$ and $F_{p,q}^s$. On the one hand, this is a natural extension of the case treated above because of $F_{p,2}^s = H_p^s$; on the other hand, $F_{p,q}^s$ and $B_{p,q}^s$ are meaningful also for $p \le 1$. Beside some technical difficulties, also the problem itself then becomes complicated. For instance, in case $n = 1$ or $n = 2$, $G \in C^\infty(\mathbb{R}^1)$ we obtain similar figures as in the general case $n \ge 3$ (cf. Figs. 4–6), since the critical triangle starts at $(s, 1/p) = (1, 1/n)$ (cf. Fig. 6). Also our considerations in 2.7.2 make it meaningful to deal with $p \le 1$.

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