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UNILATERAL PROBLEMS FOR ELLIPTIC SYSTEMS WITH GRADIENT CONSTRAINTS

T. N. ROZHKOVSKAYA

Institute of Mathematics, Russian Academy of Sciences, Siberian Branch
Universitetskiĭ Prosp. 4, 630090 Novosibirsk, Russia

We consider unilateral problems for elliptic operators $L = (L^1, \ldots, L^N)$ with diagonal principal part in a bounded domain $\Omega \subset \mathbb{R}^n$. Unilateral problems with gradient constraints have been studied in the scalar case ($N = 1$) for quasilinear elliptic and parabolic operators of general type. The main aim of this article is to extend the corresponding results to systems. The first step is to find an acceptable statement of a unilateral problem with gradient constraints. Comparison of formulations of classical problems and well-known unilateral problems leads to the following conclusion. In both cases we have to find a function $u(x)$ in Ω using information about the vectors $Lu(x) \in \mathbb{R}^N$, $x \in \Omega$. In classical problems we are given values and directions of the vectors $Lu(x)$, whereas in unilateral problems only directions at points $x \in \widetilde{\Omega}$, where $\widetilde{\Omega} \subset \Omega$ is not a priori known, are given. In the latter case the direction of $Lu(x)$ at $x \in \widetilde{\Omega}$ depends on the value of $u(x)$ (and possibly of $\nabla u(x)$).

In the scalar case ($N = 1$) the direction condition is formulated in terms of the sign of $Lu(x)$. Such an assumption appears in the obstacle problem and the Evans unilateral problem with gradient constraint. In the obstacle problem for a system the vectors $Lu(x) \in \mathbb{R}^N$ are directed along the outward normals to given convex sets $K(x) \subset \mathbb{R}^N$.

We give the statement of the unilateral problem with constraint on the solution and its gradient. The directions of the vectors $Lu(x) \in \mathbb{R}^N$ coincide with those of the solution $u(x) \in \mathbb{R}^N$ at the points $x \in \Omega$ where $u(x) \neq 0$ and the pair $(u(x), \nabla u(x))$ belongs to the boundary of a given convex set $K(x) \subset \mathbb{R}^N \times \mathbb{R}^{Nn}$. We give an equivalent statement of the problem in the form of a local quasivariational inequality with some additional conditions.

Using some ideas of the penalty method, we introduce a regularization as a boundary value problem with special penalty. The limit function is the solution of the problem considered. This solution satisfies the obstacle problem if we are given the convex constraint only on the solution (not on its gradient).

The paper consists of three sections. In the first some statements of unilateral problems and regularity results are discussed. The statement of the unilateral problem for a system with gradient constraint, and the existence theorems, are formulated in Section 2. A sketch of the proof is given in Section 3. We point out the difficulties which arose in the vector case and refer to earlier articles by the author for some technical details.

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1. Preliminaries

1.1. *Regularity of solutions of classical problems.* We recall the well-known results of [13]. Let u_0 be the solution of the Dirichlet problem

$$Lu_0 = 0 \quad \text{in } \Omega\,, \qquad u_0 = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ for the elliptic system $L = (L^1, \ldots, L^N)$, where

$$L^r v = a_{ij}^{rm}(x, v, \nabla v) v_{x_i x_j}^m + a^r(x, v, \nabla v) \quad (r = 1, \ldots, N)\,, \tag{1.2}$$

$v = (v^1, \ldots, v^N)$, $\nabla v = (v_{x_1}^1, \ldots, v_{x_n}^1, \ldots, v_{x_1}^N, \ldots, v_{x_n}^N)$. In the scalar case $(N = 1)$ the differential properties of the solution u_0 depend on the smoothness of the data and character of non-linearity. In the vector case $(N > 1)$ one has to take into account the type of the system. Thus, the regularity result for systems with diagonal principal part is quite similar to that in the scalar case if

$$a_{ij}^r(x, w, p) = a_{ij}(x, w, p) \quad (r = 1, \ldots, N)\,, \tag{1.3}$$

where $\nu_1|\xi|^2 \le a_{ij}\xi_i\xi_j$ $(\forall \xi \in \mathbb{R}^n)$ and the functions $a_{ij}(x, w, p)$, $a^r(x, w, p)$ are increasing in $p \in \mathbb{R}^{nN}$ so that

$$|a_{ij}(x, w, p)| \le \mu(|w|)|p|\,, \qquad |a^r(x, w, p)| \le \mu(|w|)|p|^{2-\delta}, \qquad \delta > 0\,,$$

where $\delta > 0$, $\mu(\tau)$ is a positive continuous function for $\tau > 0$. Namely, under the assumptions mentioned, the conditions $a_{ij}, a^r \in C^{l+\alpha-2}$, $\partial\Omega \in C^{l+\alpha}$ with $\alpha > 0$ imply $u_0 \in C^{l+\alpha}$. The analogous results hold for the second and third boundary value problems.

1.2. *Regularity of solutions of variational inequalities.* Let L be of divergence form, i.e., $L = B$, where

$$B^r v = \frac{d}{dx_i} b_i^r(x, v, \nabla v) + a^r(x, v, \nabla v) \quad (r = 1, \ldots, N)\,. \tag{1.4}$$

Under some assumptions on $b_j^r(x, w, p)$, $a^r(x, w, p)$ we can define an operator

$A : \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N) \to W^{-1}_m(\Omega;\mathbb{R}^N)$ by the formula

$$\langle A\xi,\eta\rangle = \sum_{r=1}^{N} \int_{\Omega} [b_i^r(x,\xi,\xi_x)\eta^r_{x_i} + a^r(x,\xi,\xi_x)\eta^r]\,dx\,. \tag{1.5}$$

For $\xi,\eta \in C_0^\infty(\Omega;\mathbb{R}^N)$ we have

$$\langle A\xi,\eta\rangle = -\int_{\Omega} B\xi\cdot\eta\,dx\,.$$

The solution u_0 of the Dirichlet problem (1.1) satisfies the variational equality

(1.6) *find* $u_0 \in \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)$ *such that*

$$\langle Au_0, v\rangle = 0 \quad \textit{for all } v \in \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)\,,$$

which is a special case of the variational inequality

(1.7) *find* $u \in K$ *such that*

$$\langle Au, v-u\rangle \ge 0 \quad \textit{for all } v \in K\,,$$

where K *is a convex closed set in* $\overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)$.

If the operator A is coercive and pseudomonotone, the classical problem (1.6) has a solution. The latter is considered as a weak solution of the boundary value problem (1.1). It follows from general theorems of convex analysis that the variational inequality (1.7) also has a solution under the same conditions on A and for a convex closed set $K \in \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)$.

However, the regularity questions for variational inequalities differ from those for equations. Indeed, let us take data (i.e., $a^r_{ij}, a^r, \partial\Omega$) so that $u_0 \in C_0^\infty(\overline{\Omega};\mathbb{R}^N)$. Take a convex closed set $\mathbf{K} \subset \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)$ and let u be a solution of (1.7) with the same data. For some $\mathbf{K}$ the differential properties of u can be better than those of an arbitrary function in $\overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)$, but there exists a function space W such that $C_0^\infty(\overline{\Omega};\mathbb{R}^N) \subset W \subset \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N)$, and $u \notin W$. Note that we have such a situation in variational inequalities with pointwise constraints. For example, in the obstacle problem, i.e., the variational inequality (1.7) with the set

$$\mathbf{K} = \{v \in \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N) : v(x) \in K(x) \text{ a.e. in } \Omega\,,\ K(x) \text{ is a convex closed set in } \mathbb{R}^N\}\,, \tag{1.8}$$

and in the problem with gradient constraint, i.e. (1.7) with

$$\mathbf{K} = \{v \in \overset{\circ}{W}{}^1_m(\Omega;\mathbb{R}^N) : \nabla v(x) \in K(x) \text{ a.e. in } \Omega\,,\ K(x) \text{ is a convex closed set in } \mathbb{R}^{Nn}\}\,, \tag{1.9}$$

we have $u \notin C^2(\Omega;\mathbb{R}^N)$ (see the counterexample in [15, Ch. 2, Sec. 8.7]).

The first result on the regularity of solutions of variational inequalities with sets $\mathbf{K}$ of type (1.8) or (1.9) was obtained by H. Lewy and G. Stampacchia in 1969 [14]. Up to now, the scalar case has been studied in detail. The most complete results are established for obstacle problems. Here we mention some regularity results and refer to the books [7, 12] and the survey [25] for precise formulations and bibliography.

In the scalar case ($N = 1$) we have three basic forms of constraints.

(a) *The obstacle problem* (1.7), (1.8). The limit regularity ($u \in W^2_\infty$) was proved in the case of quasilinear operators. For a sketch of proof see [25].

(b) *The thin obstacle problem*, i.e., variational inequality of type (1.7), (1.8) but with constraint $u(x) \in K(x)$ for $x \in S$, where S is a given $(n-1)$-dimensional surface in $\overline{\Omega}$, for example $S = \partial\Omega$. The best regularity result (Hölder continuity of the first derivatives) was proved for linear operators in [5, 11, 24] by different methods.

(c) *The problem with gradient constraint*. In the case $K(x) = \{p \in \mathbb{R}^n : |p| \leq 1\}$, $Lv = \Delta v + \mu$, $\mu = \text{const}$, the variational inequality (1.7), (1.9) is the well-known elastic-plastic torsion problem. The limit regularity ($u \in W^2_\infty$) was apparently proved only in this special case [5]. W^2_q-regularity was established for divergence-type quasilinear operators and some class of strictly convex bounded sets $K(x) \subset \mathbb{R}^n$ [27, 17, 10]; see also [7].

Further, the question arises whether the regularity results obtained for the case $N=1$ can be extended to elliptic systems. The answer is affirmative if we consider the obstacle problem for some systems with diagonal principal part. W^2_q-regularity for quasilinear systems was proved by S. Hildebrandt and K.-O. Widman [8]. A. A. Arkhipova and N. N. Ural'tseva have considered the thin obstacle problem [4]. The limit regularity in the obstacle problems for linear diagonal systems with special constraints was also proved; see formulations of results and references in [25, 4, 1].

The natural regularity result in obstacle problems for a strongly elliptic system is W^2_2-regularity. The corresponding theorems for some modifications of the obstacle problems are given in [25]; see also [2].

The regularity question for the variational inequality for systems with gradient constraint remains open.

1.3. *Unilateral problems. The scalar case.* The Dirichlet problem is considered both in a weak formulation (1.6) and in a strong form of a boundary value problem (1.1). These statements are formally equivalent for a divergence-type operator L.

By analogy, the variational inequality (1.7) can be treated as a weak formulation of a boundary value problem depending on the convex set $\mathbf{K}$. We refer to these boundary problems and their variants as "unilateral problems" and use the term "variational inequality" only for (1.7). We emphasize that these terms are often identified. This does not matter if we restrict ourselves to the obstacle problems with divergence-type operators, since in this case the statements are equivalent. However, we have to distinguish between the statements even in the case $N = 1$, $Lv = \Delta v + f(x)$ if we deal with gradient constraints.

We begin with the scalar obstacle problem. Let $N = 1$ and u be a solution of (1.7), (1.8). Any convex closed set $K(x) \subseteq \mathbb{R}$ can be written in the form $K(x) = [\varphi_1(x), \varphi_2(x)]$, where $\varphi_1(x) \le \varphi_2(x)$ (the case $\varphi_1 = -\infty$ or $\varphi_2 = +\infty$ is admitted). Since $v = \psi w + (1-\psi)u \in \mathbf{K}$ for $\psi \in C_0^\infty(\Omega)$, $0 \le \psi \le 1$, and $w \in \mathbf{K}$, the variational inequality (1.7), (1.8) is formally equivalent to the pointwise inequality

$$Lu(x) \cdot (w(x) - u(x)) \le 0 \qquad (\forall w \in \mathbf{K}) \tag{1.10}$$

with the boundary condition $u = 0$ on $\partial\Omega$. It follows from (1.10) that a smooth solution u of (1.7), (1.8) satisfies the boundary value problem

$$\begin{gathered} \varphi_1(x) \le u(x) \le \varphi_2(x) \quad \text{in } \Omega\,, \\ Lu(x) \begin{cases} \ge 0 & \text{if } u(x) = \varphi_2(x),\ \varphi_1(x) < \varphi_2(x), \\ = 0 & \text{if } \varphi_1(x) < u(x) < \varphi_2(x), \\ \le 0 & \text{if } u(x) = \varphi_1(x),\ \varphi_1(x) < \varphi_2(x), \end{cases} \\ \text{the sign of } Lu(x) \text{ is arbitrary if } \varphi_1(x) = \varphi_2(x), \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.11}$$

Note that the sign of $Lu(x)$ is not defined for $x \in \Omega$ such that the corresponding set $K(x)$ is degenerate, i.e., $\operatorname{int} K(x) = \emptyset$ or $K(x) = \{\varphi_1(x)\} = \{\varphi_2(x)\}$. For every convex set $K(x) \subset \mathbb{R}$ there exists a convex function $G(x,\cdot)$ such that $G(x,\cdot) \in C^2(\mathbb{R})$ and

$$\begin{aligned} K(x) &= \{w \in \mathbb{R} : G(x,w) \le 0\} = [\varphi_1(x), \varphi_2(x)]\,, \\ \operatorname{int} K(x) &= \{w \in \mathbb{R} : G(x,w) < 0\} = (\varphi_1(x), \varphi_2(x))\,. \end{aligned}$$

Moreover, $\operatorname{int} K(x) = \emptyset$ if and only if the value $\varphi_1(x) = \varphi_2(x)$ is the minimum point of $G(x,\cdot)$. Therefore, in this case $D_w G(x, \varphi_1(x)) = 0$. Hence we can rewrite (1.11) in terms of $G(x,w)$ as follows:

$$\begin{gathered} G(x,u(x)) \le 0 \quad \text{in } \Omega\,, \\ Lu(x) \upuparrows \nabla_w G(x,u(x)) \quad \text{if } G(x,u(x)) = 0\,,\ D_w G(x,u(x)) \ne 0\,, \\ Lu(x) = 0 \quad \text{if } G(x,u(x)) < 0\,, \\ \text{the sign of } Lu(x) \text{ is not defined if } G(x,u(x)) = 0 \text{ and } D_w G(x,u(x)) = 0, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.12}$$

Here $a \upuparrows b$ means that either $a = 0$ or $a \ne 0$ has the same sign as b. If $0 \in K(x)$ $(\forall x \in \overline{\Omega})$, then (1.12) reduces to the following boundary value problem:

$$\begin{gathered} u(x) \in K(x)\,, \quad Lu(x) \cdot u(x) \ge 0 \quad \text{in } \Omega\,, \\ Lu(x) = 0 \quad \text{if } u(x) \in \operatorname{int} K(x)\,, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.13}$$

Note that the signs of $Lu(x)$ and of the solution $u(x)$ have to coincide at x such that $u(x) \in \partial K(x)$. Finally, if $K(x) = (-\infty, \varphi_2(x)]$ and $0 \in K(x)$ $(\forall x \in \overline{\Omega})$, i.e., $\varphi_2 \geq 0$, we obtain from (1.11) the boundary value problem in the form

$$\min\{Lu(x), -G(x, u(x))\} = 0 \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,. \tag{1.14}$$

Considering (1.11)–(1.14) with a non-divergence-type operator L we lose the connection of the boundary value problems with variational inequalities and in this case the weak solvability does not follow from general results of convex analysis. The obstacle problem with non-divergence-type operators was considered in [3, 23], where the solvability in W_p^2 was proved. Note that the existence theorem and the limit regularity ($u \in W_\infty^2$) also follow from the results of [21, 22].

It is easy to write out the corresponding boundary value problems (unilateral problems) for variational inequalities with thin obstacles, boundary obstacles etc. Indeed, in this case the functions $v = \psi u + (1-\psi)w$ are admissible in (1.7) for any $\psi \in C_0^\infty(\Omega)$, $0 \leq \psi \leq 1$, and $w \in \mathbf{K}$.

Now we analyze the variational inequality (1.7), (1.9) with gradient constraint. One can show that a solution $u(x)$ of the variational inequality (1.7), (1.9) formally satisfies the conditions

$$\begin{gathered} (u(x), \nabla u(x)) \in K(x) \quad \text{in } \Omega\,, \\ Lu(x) = 0 \quad \text{if } (u(x), \nabla u(x)) \in \operatorname{int} K(x)\,, \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.15}$$

However, we cannot obtain any information about the sign of $Lu(x)$ directly from (1.7), (1.9). It is obvious that the problem (1.15) is not well-posed. L. C. Evans considered the problem (1.15) with an additional condition on the sign of $Lu(x)$ in Ω [6]. Namely, he proved the solvability of the problem (cf. (1.14))

$$\begin{gathered} \min\{Lu(x), -F(x, u(x), \nabla u(x))\} = 0 \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.16}$$

in the case

$$\begin{gathered} Lv = a_{ij}(x) v_{x_i x_j} + a_i(x) v_{x_i} + a(x) v + f(x)\,, \qquad f \geq 0\,,\ a \leq 0\,, \\ F(x, w, p) = |p|^2 - g(x)\,, \qquad g \geq 0\,, \end{gathered}$$

i.e., the sets $K(x) = \{p \in \mathbb{R}^n : |p|^2 \leq g(x)\}$ are balls in $\mathbb{R}^n$ centered at the origin with radius $g(x)$. The assumption $f \geq 0$ in Ω provides the inequality $u \geq 0$ in Ω. Hence (1.16) implies that $Lu(x) \uparrow\uparrow u(x)$ at $x \in \Omega$ such that $F(x, u(x), \nabla u(x)) = |\nabla u(x)|^2 - g(x) = 0$. Note that 0 is a minimal value of u. Hence $\nabla u(x) = 0$ at $x \in \Omega$ such that $u(x) = 0$. Consequently, $F(x, u(x), \nabla u(x)) < 0$ and $Lu(x) = 0$ on the set of zeros of $u(x)$.

The results of L. C. Evans were generalized in [9, 16, 26, 18–22].

1.4. *Unilateral problems. The vector case.* We consider the obstacle problem (1.7), (1.8) in the case $N > 1$. Let $0 \in K(x)$ $(\forall x \in \Omega)$ and $\operatorname{int} K(x) = \emptyset$ if and only if $K(x) = \{0\}$; moreover, $\partial K(x) \in C^1$ for $x \in \Omega$ such that $\operatorname{int} K(x) \neq \emptyset$. It follows that the solution u of (1.7), (1.8) satisfies the pointwise inequality

$$Lu \cdot (v-u) \le 0 \qquad (\forall v \in C_0^\infty(\Omega),\ v(x) \in K(x)). \tag{1.17}$$

For vectors $a \in \mathbb{R}^N$, $b \in \mathbb{R}^N$ we shall write $a \upuparrows b$ if there exists $\lambda \ge 0$ such that $a = \lambda b$. In particular, $a \upuparrows 0$ means $a = 0$. It is easy to see that (1.7), (1.8) is formally equivalent to the next problem:

$$\begin{gathered} u(x) \in K(x) \quad \text{in } \Omega, \\ Lu(x) = 0 \quad \text{if } u(x) \in \operatorname{int} K(x),\ \operatorname{int} K(x) \neq \emptyset, \\ Lu(x) \upuparrows \mathbf{n}(x, u(x)) \quad \text{if } u(x) \in \partial K(x),\ \operatorname{int} K(x) \neq \emptyset, \\ \text{the direction of } Lu(x) \text{ is arbitrary if } \operatorname{int} K(x) = \emptyset, \text{ i.e., } K(x) = \{0\}, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.18}$$

where $\mathbf{n}(x, w)$ denotes the unit outward normal vector to $\partial K(x)$ at $w \in \partial K(x)$. Let $G(x, w)$ be a convex function such that

$$\begin{gathered} G(x,0) \le 0, \qquad G(x,\cdot) \in C^1, \\ K(x) = \{w \in \mathbb{R}^N : G(x,w) \le 0\}, \qquad \operatorname{int} K(x) = \{w \in \mathbb{R}^N : G(x,w) < 0\}, \end{gathered}$$

$\operatorname{int} K(x) = \emptyset$ if and only if $\nabla_w G(x,0) = 0$. To construct $G(x,w)$ one can use the distance function. For example, in the case of balls $K(x) = \{w \in \mathbb{R}^N : |w|^2 \le g(x)\}$ we can take $G(x,w) = |w|^2 - g(x)$. If $g(x_0) > 0$ for $x_0 \in \overline{\Omega}$ and $|w|^2 = g(x_0)$, then the vector $\nabla_w G(x,w)$ is directed along the outward normal to $\partial K(x)$, i.e., $\mathbf{n}(x,w) = \lambda \nabla_w G(x,w) = 2\lambda w$, where $\lambda > 0$. Thus, the unilateral problem (1.18) can be written as follows:

$$\begin{gathered} G(x,u(x)) \le 0 \quad \text{in } \Omega, \\ Lu(x) = 0 \quad \text{if } G(x,u(x)) < 0, \\ Lu(x) \upuparrows \nabla_w G(x,u(x)) \quad \text{if } G(x,u(x)) = 0,\ \nabla_w G(x,u(x)) \neq 0, \\ \text{the direction of } Lu(x) \text{ is arbitrary if } \nabla_w G(x,u(x)) = 0, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.19}$$

2. The main result. In this section we formulate a problem with gradient constraint as a local quasivariational inequality under some additional conditions. We show that its regular solution satisfies a unilateral problem and give formulations of the existence theorems. A sketch of the proof is given in Section 3. First we write out assumptions on data.

2.1. *The operator A.* Let us introduce

(A1) $$\langle Au, v\rangle = \sum_{r=1}^{N} \int_\Omega (a_{ij}(x) u^r_{x_i} v^r_{x_j} + (a_{ij}(x))_{x_j} u^r_{x_i} v^r - a^r(x,u,\nabla u) v^r)\, dx,$$

where $a_{ij} \in C^2(\overline{\Omega})$, $a^r \in C^2(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{Nn})$, Ω is a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega \in C^3$. Suppose that the ellipticity condition is satisfied:

$$\text{(A2)} \qquad \nu_1|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \mu_1|\xi|^2 \quad (\forall \xi \in \mathbb{R}^n)\,,$$

and for $x \in \overline{\Omega}$, $|w| > M$, $p \in \mathbb{R}^{Nn}$

$$\text{(A3)} \qquad a(x,w,p)\cdot w = \sum_{r=1}^{N} a^r(x,w,p)w^r < 0\,,$$

while for $x \in \overline{\Omega}$, $|w| \le M$, $|p| > R_1$

$$\text{(A4)} \qquad |a^r(x,w,p)| \le \mu_2(|w|)(1+|p|^{1+\alpha_1})\,,$$

where ν_1, μ_1, M, R_1 denote positive constants, the function $\mu_2(\tau), \tau > 0$, is strictly positive, and $0 < \alpha_1 < 1$, $1/(1-\alpha_1) < n$. Then (A1) defines an operator $A : \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^N) \to W_2^{-1}(\Omega;\mathbb{R}^N)$. Moreover, assume that for $|p| > R_1$

$$\text{(A5)} \qquad \left|\frac{\partial a^r(x,w,p)}{\partial x_k}\right| + \left|\frac{\partial a^r(x,w,p)}{\partial w^m}\right| \le \mu_3(|w|)(1+|p|^{1+\alpha_1})\,,$$

$$\text{(A6)} \qquad \left|\frac{\partial a^r(x,w,p)}{\partial p_k^l}\right| \le \mu_4(|w|)(1+|p|^{\alpha_1})\,,$$

where $1 \le k \le n$ and $1 \le l,\ m \le N$, the functions $\mu_3(\tau), \mu_4(\tau)$ are strictly positive and continuous for $\tau > 0$.

2.2. *The operator L.* Define

$$\text{(L1)} \qquad Lv = a_{ij}(x)v^r_{x_ix_j} + a^r(x,w,p)\,.$$

It is easy to see that the integral identity

$$\text{(L2)} \qquad \langle Au, v\rangle = -\int_\Omega Lu\cdot v\,dx$$

holds for functions $u \in W_p^2(\Omega;\mathbb{R}^N)$, $v \in \overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^N)$.

2.3. *The function $G : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}$.* Define

$$\text{(G1)} \qquad G(x,w) = B(x)w\cdot w = \sum_{l,m=1}^{N} b^{lm}(x)w^lw^m\,,$$

where $B(x) = (b^{lm}(x))$ is a symmetric positive-definite matrix,

$$\text{(G2)} \qquad B(x)\zeta\cdot\zeta \ge \nu_2|\zeta|^2 \quad (\forall \zeta \in \mathbb{R}^N)\,,$$

where $\nu_2 = \text{const} > 0$, with elements $b^{lm} \in C^2(\overline{\Omega})$. Note that $G(x,w)$ is nonnegative and strictly convex with respect to $w \in \mathbb{R}^N$; moreover,

$$\text{(G3)} \qquad G(x,w) = 0 \Leftrightarrow \nabla_w G(x,w) = 0 \Leftrightarrow B(x) = 0 \Leftrightarrow w = 0\,.$$

2.4. *The function $F : \overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to \mathbb{R}$.* Set

$$\text{(F1)} \qquad D = \{(x,w,p) \in \overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN} : F(x,w,p) > 0\}\,,$$

$$\text{(F2)} \qquad K(x) = \{(w,p) \in \mathbb{R}^N\times\mathbb{R}^{Nn} : F(x,w,p) \le 0\}\,.$$

Suppose that

$$F \in C^{1+\alpha_2}(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{Nn}) \cap C^2(D)\,, \tag{F3}$$

where $0 < \alpha_2 < 1$, and let $F(x, w, p)$ be convex with respect to $(w, p) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$. Then the set $K(x)$ is convex. Moreover, by the definition of a convex function we have for $x \in \Omega$, $w \in \mathbb{R}^N$, $p \in \mathbb{R}^{Nn}$

$$\sum_{m=1}^{N} \frac{\partial F(x,w,p)}{\partial w^m} w^m + \sum_{l=1}^{N}\sum_{k=1}^{n} \frac{\partial F(x,w,p)}{\partial p_k^l} p_k^l \ge F(x,w,p) - F(x,0,0)\,. \tag{F4}$$

We also suppose that $F(x, w, p)$ is strictly convex on D with respect to $p \in \mathbb{R}^{Nn}$, i.e.,

$$\sum_{m,l=1}^{N} \frac{\partial^2 F}{\partial u^m \partial u^l} \zeta^m \zeta^l + 2\sum_{m,l=1}^{N}\sum_{k=1}^{n} \frac{\partial^2 F}{\partial u^m \partial p_k^l} \zeta^m \eta_k^l + \sum_{m,l=1}^{N}\sum_{k,s=1}^{n} \frac{\partial^2 F}{\partial p_k^l \partial p_s^m} \eta_k^l \eta_s^m \tag{F5}$$
$$\ge \nu_2|\zeta|^2 + \nu_3(|w|)[1+|p|^2]^{(\alpha_2-1)/2}|\eta|^2 \qquad (\forall \zeta \in \mathbb{R}^N,\ \eta \in \mathbb{R}^{nN})\,,$$

where $(x, w, p) \in D$, $\zeta = (\zeta^1, \ldots, \zeta^N) \in \mathbb{R}^N$, $\nu_2 = \text{const} \ge 0$ and $\nu_3(\tau)$ is a strictly positive continuous function for $\tau > 0$. Furthermore,

$$F(x,0,0) < 0 \qquad (\forall x \in \overline{\Omega}) \tag{F6}$$

and there exists a function $R_2(\tau)$ such that for all $x \in \overline{\Omega}$

$$K(x) \subset \{(w,p) \in \mathbb{R}^N \times \mathbb{R}^{nN} : |p| < R_2(|w|)\}\,, \tag{F7}$$

and for $x \in \overline{\Omega}$, $u \in \mathbb{R}^N$, $|p| \ge R_2(|w|)$

$$F(x,w,p) \ge \nu_4|p|^{1+\alpha_2}\,. \tag{F8}$$

Moreover, for $(x, w, p) \in D$ the following estimates hold:

$$F(x,w,p) + \left|\frac{\partial F(x,w,p)}{\partial x_i}\right| + \left|\frac{\partial^2 F(x,w,p)}{\partial x_i \partial x_j}\right| \le \mu_5(|w|)[1+|p|^{1+\alpha_2}], \tag{F9}$$

$$\left|\frac{\partial^2 F(x,w,p)}{\partial x_i \partial p_k^l}\right| + \left|\frac{F(x,w,p)}{\partial p_k^l}\right| \le \mu_6(|w|)[1+|p|^{\alpha_2}], \tag{F10}$$

$$\left|\frac{\partial F(x,w,p)}{\partial w^m}\right| + \left|\frac{\partial^2 F(x,w,p)}{\partial x_i \partial w^m}\right| \le \mu_7(|w|)[1+|p|^{1+\alpha_2-\delta}]\,, \tag{F11}$$

where $1 \le i, j, k \le n$, $1 \le l, m \le N$, $0 < \alpha_2 < 1$, $\delta > 0$, the functions $\mu_5(\tau)$, $\mu_6(\tau)$, $\mu_7(\tau)$ are continuous and positive for $\tau \ge 0$.

2.5. *Formulation of the problem.* For $v \in C(\Omega; \mathbb{R}^N)$ write

$$\Omega_v = \{x \in \Omega : v(x) \ne 0\}\,, \tag{2.1}$$

$$M(x, v(x)) = \{w \in \mathbb{R}^N : G(x,w) \le G(x, v(x))\} \tag{2.2}$$

and for $v \in C^1(\Omega_v; \mathbb{R}^N)$ set

$$\Omega_{vF} = \{x \in \Omega_v : F(x, v(x), \nabla v(x)) < 0\}\,. \tag{2.3}$$

Note that the sets Ω_v, Ω_{vF} are open in Ω and the $M(x, v(x))$ are convex closed sets for all $x \in \Omega$. It is obvious that $v(x) \in \partial M(x, v(x))$ $(\forall x \in \Omega)$.

PROBLEM (quasivariational inequality). *Find a function*

$$u \in \mathring{W}_2^1(\Omega; \mathbb{R}^N) \cap C(\Omega; \mathbb{R}^N) \cap C^1(\Omega_u; \mathbb{R}^N) = V \tag{QV1}$$

such that

$$\langle Au, (v-u)\eta\rangle \geq 0 \tag{QV2}$$

for all $\eta \in C_0^\infty(\Omega)$ *with* $\operatorname{supp}\eta \subset \Omega_u$, $\eta \geq 0$, *and all* $v \in \mathring{W}_2^1(\Omega; \mathbb{R}^N)$ *with* $v(x) \in M(x, u(x))$ *a.e. in* Ω,

$$F(x, u(x), \nabla u(x)) \leq 0 \quad \text{in } \Omega_u\,, \tag{QV3}$$

$$Au = 0 \quad \text{in } \Omega_{uF} \text{ (as distributions)}. \tag{QV4}$$

PROBLEM (unilateral problem). *Find a function*

$$u \in C(\overline{\Omega}; \mathbb{R}^N) \cap W^2_{p,\mathrm{loc}}(\Omega_u; \mathbb{R}^N) \tag{U1}$$

such that

$$F(x, u(x), \nabla u(x)) \leq 0 \quad \text{in } \Omega_u\,, \tag{U2}$$

$$Lu(x) = \lambda(x)\nabla_w G(x, u(x)) = \lambda(x)B(x)u(x) \quad \text{a.e. in } \Omega_u\,, \tag{U3}$$

where $\lambda(x) \geq 0$ *is an a priori unknown function and*

$$\lambda(x) = 0 \quad \text{for } x \in \Omega_{uF}\,, \tag{U4}$$

$$u = 0 \quad \text{on } \partial\Omega\,. \tag{U5}$$

R e m a r k 2.1. The problem (U1)–(U5) is a free boundary problem. The domain Ω is divided into three sets:

- $\Omega_1 : u = 0$;
- $\Omega_2 : u \neq 0$, $F(x, u, \nabla u) < 0$, $Lu = 0$,
- $\Omega_3 : u \neq 0$, $F(x, u, \nabla u) = 0$, either $Lu = 0$ or $Lu(x)$ is directed along the outward normal to $M(x, u(x))$ at $u(x) \in \partial M(x, u(x))$.

Note that the set $M(x, u(x))$ is degenerate for $x \in \Omega_1$, i.e., for x such that $M(x, u(x)) = \{0\} \Leftrightarrow \operatorname{int} M(x, u(x)) = \emptyset$.

R e m a r k 2.2. The scalar variant of (U1)–(U5) can be written as follows:

$$\begin{gathered} F(x, u(x), \nabla u(x)) \leq 0 \quad \text{in } \Omega_u\,, \\ Lu(x) \cdot u(x) \geq 0 \quad \text{a.e. in } \Omega\,, \\ Lu(x) = 0 \quad \text{in } \Omega_{uF}\,, \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered}$$

i.e., the sign of $Lu(x) \neq 0$ is the same as that of the solution $u(x)$.

2.6. PROPOSITION. *If* $u \in C(\overline{\Omega}; \mathbb{R}^N) \cap W^2_{p,\mathrm{loc}}(\Omega_u; \mathbb{R}^N)$ *satisfies* (QV1)–(QV4), *then* u *is a solution of* (U1)–(U5).

P r o o f. (U2) is a consequence of (QV3); and (U1), (U5) are valid because of the regularity assumption. To verify (U3) we take $w \in \mathring{W}_2^1(\Omega;\mathbb{R}^N)$ such that $w(x) \in M(x,u(x))$ a.e. in Ω and $\xi \in C_0^\infty(\Omega)$ with $0 \le \xi \le 1$. Then $v = \xi(x)u + (1-\xi(x))w$ is admissible in (QV2). Hence we can deduce from (QV2) the pointwise inequality

$$Lu(x)\cdot(w(x)-u(x)) \le 0 \quad \text{a.e. in } \Omega_u\,.$$

It means that $Lu(x)$ either vanishes or else its direction coincides with that of the outward normal $\mathbf{n}(x,u(x))$ to $\partial M(x,w)$ at $w = u(x)$. Since $\mathbf{n}(x,u(x)) = \lambda\nabla_w G(x,u(x))$, $\lambda > 0$, we obtain (U3). Equality (QV4) yields (U4).

2.7. *The penalty problem.* Consider the boundary value problem

$$Lu^\varepsilon = \beta_\varepsilon(F(x,u^\varepsilon,\nabla u^\varepsilon))B(x)u^\varepsilon \quad \text{in } \Omega\,, \tag{PP1}$$

$$u^\varepsilon = 0 \quad \text{on } \partial\Omega\,, \tag{PP2}$$

where $\varepsilon > 0$, $\beta_\varepsilon(\tau) = (1/\varepsilon)\beta_0(\tau)$, $\tau \in \mathbb{R}$, and $\beta_0 \in C^2(\mathbb{R})$ satisfies

$$\beta_0(\tau) = 0 \quad \text{for } \tau \le 0\,, \qquad \beta_0(\tau) > 0 \quad \text{for } \tau > 0\,,$$
$$\beta_0'(\tau) > 0\,, \quad \beta_0''(\tau) \ge 0 \quad \text{for } \tau > 0\,,$$
$$\beta_0(\tau) = \tau^{1+\alpha_0} \quad \text{for } \tau \ge 1\,.$$

The constant α_0, $0 < \alpha_0 < 1$, is such that

$$\max\left\{0, \frac{\alpha_1-\alpha_2}{1+\alpha_2}\right\} < \alpha_0 < \frac{1-\alpha_2}{1+\alpha_2}\,. \tag{PP3}$$

By (F9) we have for $(x,w,p) \in D \cap \{(x,w,p) : |w| \le M\}$

$$0 \le \beta_\varepsilon(F(x,w,p)) = \frac{1}{\varepsilon}[F(x,w,p)]^{1+\alpha_0} \le \frac{1}{\varepsilon}\mu_5(M)(1+|p|^\gamma)\,,$$

where $\gamma = (1+\alpha_0)(1+\alpha_2)$. In view of (PP3) we have $\gamma < 2$. Moreover, by (G2),

$$\beta_\varepsilon(F(x,w,p))B(x)w\cdot w \ge 0\,.$$

From Theorem 5.2 of [13, Ch. 8] we conclude that there exists a solution $u^\varepsilon \in C^3(\Omega;\mathbb{R}^N) \cap C^1(\overline{\Omega};\mathbb{R}^N)$ of the penalty problem (PP1), (PP2).

2.8. *Formulation of the theorems.* We assume the conditions from 2.1–2.4 to be satisfied.

THEOREM 2.1. *Suppose that*

$$\nabla_w G\cdot\nabla_w F + \sum_{l,m=1}^{N}\sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}b^{lm}p_k^m \ge \nu_5|p|^{1+\alpha_2} - \mu_7(|w|)p^{1+\alpha_2-\gamma_2}\,, \tag{2.4}$$

where $\nu_5 = \text{const} > 0$ *and* $\mu_7(\tau)$ *is a positive continuous function for* $\tau \ge 0$. *Then there exist a sequence* u^ε *and a function* $u \in V \cap W^2_{p,\mathrm{loc}}(\Omega_u;\mathbb{R}^N)$ *such that* $u^\varepsilon \to u$ *in* $W_2^{-1}(\Omega;\mathbb{R}^N)$ *and* u *satisfies* (QV1)–QV4).

THEOREM 2.2. *Let* u *be the solution of* (QV1)–(QV4) *from Theorem* 2.1. *If* $B(x)$ *is the unit matrix then* $u \in C^{0,1}(\overline{\Omega};\mathbb{R}^N)$.

COROLLARY. *Under the assumption of Theorem* 2.2 *the solution* u *of* (QV1)–(QV4) *from Theorem* 2.1 *is a solution of the unilateral problem* (U1)–(U5).

If constraint is given only on the solution, then the regularizations introduced above yield the solution of the obstacle problem. We show this in a simple case and refer to the article of S. Hildebrandt and K.-O. Widman [8] for regularity results in the general case. Let

$$\text{(O1)} \qquad Lv = \Delta v^l + a^{lm}(x)w^m + a^l(x) \quad (l = 1, \dots, N),$$

$$\text{(O2)} \qquad G(x, w) = |w|^2 ,$$

$$\text{(O3)} \qquad F(x, w, p) = |w|^2 - f(x) ,$$

where $f(x) \geq f_0 > 0$. In this case $\nabla_w F = \nabla_w G = w$ and $B(x)$ is the unit matrix.

THEOREM 2.3. *Under the conditions* (O1)–(O3) *there exist a subsequence* u^ε *and a function* $u \in V$ *so that* $u^\varepsilon \to u$ *in* $W_2^{-1}(\Omega; \mathbb{R}^N)$, u *belongs to* $W^2_{p,\mathrm{loc}}(\Omega; \mathbb{R}^N)$ *and satisfies the variational inequality* (*the obstacle problem*)

$$\text{(2.5)} \qquad u \in \mathring{W}_2^1(\Omega; \mathbb{R}^N) , \qquad F(x, u(x), \nabla u(x)) \leq 0 \quad \text{in } \Omega ,$$

$$\text{(2.6)} \qquad \langle Au, v - u\rangle \geq 0$$
$$\text{for all } v \in \mathring{W}_2^1(\Omega; \mathbb{R}^N) \text{ with } F(x, v(x), \nabla v(x)) \leq 0 \text{ a.e. in } \Omega .$$

2.9. EXAMPLES. (1) The following function F satisfies the conditions (F1)–(F11):

$$\text{(2.7)} \qquad F(x, w, p) = [C(x)p \cdot p + D(x)w \cdot w]^{(1+\alpha_2)/2} - g(x) ,$$

where the $(Nn \times Nn)$-matrix $C(x) = (c^{lm}_{ks}(x))$ and $(N \times N)$-matrix $D(x) = (d^{lm}(x))$ are real and symmetric, and the elements $c^{lm}_{ks}(x)$, $d^{lm}(x)$ ($1 \leq l, m \leq N$; $1 \leq k, s \leq n$) and the function $g(x)$ belong to the class $C^2(\overline{\Omega})$; moreover, we assume $g(x) > 0$ in $\overline{\Omega}$ and

$$\text{(2.8)} \qquad c_0|\eta|^2 \leq C(x)\eta \cdot \eta = \sum_{l,m=1}^{N} \sum_{k,s=1}^{n} c^{lm}_{ks}(x)\eta^l_k \eta^m_s \leq c_1|\eta|^2 \quad (\forall \eta \in \mathbb{R}^{Nn}) ,$$

$$\text{(2.9)} \qquad d_0|\zeta|^2 \leq D(x)\zeta \cdot \zeta = \sum_{m,l=1}^{N} d^{lm}(x)\zeta^l \zeta^m \leq d_1|\zeta|^2 \quad (\forall \zeta \in \mathbb{R}^{N}) ,$$

where $c_0, c_1, d_1 = \text{const} > 0$, $d_0 \geq 0$.

(2) The conditions (F1)–(F11) are also valid for the function

$$\text{(2.10)} \qquad F(x, w, p) = \Big[\sum_{l=1}^{N} \sum_{k=1}^{n} f^l_k(x)(p^l_k)^2\Big]^{(1+\alpha_2)/2} + \sum_{l=1}^{N} \sum_{k=1}^{n} g^l_k(x, w)|p^l_k|^{1+\gamma^l_k} - g(x) ,$$

where $f_k^l, g \in C^2(\overline{\Omega}), g_k^l \in C^2(\overline{\Omega} \times \mathbb{R}^N)$, the functions $g_k^l(x,w)$ are convex with respect to $w \in \mathbb{R}^N$, and for $x \in \overline{\Omega}$

$$g(x) > 0\,, \qquad g_k^l(x,w) \geq 0\,, \qquad g_k^l(x,0) = 0\,; \tag{2.11}$$

moreover, we assume

$$0 < f_0 \leq \min_{l,k} \min_{\overline{\Omega}} f_k^l(x) \leq \max_{l,k} \max_{\overline{\Omega}} f_k^l(x) = f_1\,, \tag{2.12}$$

$$0 \leq \min_{l,k} \gamma_k^l \leq \max_{l,k} \gamma_k^l = \gamma_0 < \alpha_2\,. \tag{2.13}$$

2.8. Remark. The assumption (2.4) is valid if the matrix $B(x)$ or the function F has a simple form. Namely, if $B(x)$ is the unit $(N \times N)$-matrix, then (2.4) holds for any convex function F with properties (F1)–(F11). (2.4) is also true for any positive-definite matrix $B(x)$ and function F such that the sets $\{(w,p) : F(x,w,p) \leq 0\}$ are balls in $\mathbb{R}^N \times \mathbb{R}^{Nn}$. In the general case, (2.4) is valid under some additional conditions. So, the function F from the first example satisfies (2.4) if

$$c_{ks}^{lm}(x) = 0 \quad \text{ if } k \neq s\,, \tag{2.14}$$

$$B(x)C_k(x) = C_k(x)B(x)\,, \qquad 1 \leq k \leq n\,. \tag{2.15}$$

3. Sketch of the proof. We keep the notation of Section 2 and assume that the conditions of Theorem 2.1 are satisfied. We first deduce some estimates for the solutions u^ε of the penalty problem. For the sake of simplicity we often omit the index ε. By C, C_1, C_2 etc. we denote different constants not depending on ε.

3.1. *The estimate for* $|u^\varepsilon|_{\overline{\Omega}}$. Using the standard argument (see, e.g., [13, Ch. 8, §5]) it is easy to derive the estimate $|u^\varepsilon|_{\overline{\Omega}} \leq C$. One has to consider the function $W(x) = |u^\varepsilon(x)|^2$ at its maximum point and use conditions (A3), (G2). Therefore we further assume that the functions $\mu_i(|u^\varepsilon(x)|)$ are bounded above by constants μ_i, $i = 2, \ldots, 7$, independent of ε and $x \in \overline{\Omega}$. We also assume that $\nu_3(|u^\varepsilon(x)|) \geq \nu_3 = \text{const} > 0$ in (F5) and $R_2(|u^\varepsilon|) \leq R_2 < \infty$.

3.2. *The estimate for* $\|\nabla u^\varepsilon\|_{L_2(\Omega)}$. Using (PP1) and (L2) we can write the integral identity

$$\sum_{r=1}^{N} \int_{\Omega} \sum_{i,j=1}^{n} \left(a_{ij}(x) u^r_{x_i} v^r_{x_j} + \frac{\partial a_{ij}}{\partial x_j} u^r_{x_i} v^r - a^r(x,u,\nabla u) v^r \right) dx$$
$$= - \int_{\Omega} \beta(F(x,u,\nabla u)) \sum_{r,m=1}^{N} b^{rm}(x) u^m v^r \, dx \qquad (\forall v \in \overset{\circ}{W}{}^1_2(\Omega))\,.$$

The right-hand side is non-positive for any vector-function $v(x) = u(x)\Phi(x)$, where $\Phi(x) \geq 0$ is a scalar function. Proceeding as in [13, Ch. 8, §3] we obtain

$$\|\nabla u^\varepsilon\|_{L_2(\Omega)} = \|u\|_{\overset{\circ}{W}{}^1_2(\Omega)} \leq C\,.$$

3.3. *The estimate for* $|\nabla u^\varepsilon|_{\overline{\Omega}'}$. We denote by Ω' a subdomain of Ω such that $\overline{\Omega}' \subset \Omega$ and by ξ a cut-off function for Ω'. Define $W(x) = \xi^4 F(x, u(x), \nabla u(x))$ and evaluate $W(x_0) = \max_{x\in\overline{\Omega}} W(x)$. It is sufficient to consider the case $F(x_0, u(x_0), \nabla u(x_0)) > 0$, $|\nabla u(x_0)| > R_2$ (see (F8)). We denote by $\widetilde{C}$ constants which do not depend on ε, but may depend on $\operatorname{dist}(\overline{\Omega}', \partial\Omega)$. At x_0 we have

$$W_{x_i} = \xi^4 \frac{dF}{dx_i} + (\xi^4)_{x_i} F = 0 \quad (i = 1, \dots, n)\,, \tag{3.1}$$

$$0 \geq \sum_{i,j=1}^{n} a_{ij} W_{x_i x_j} = I_1 + I_2 + I_3 + I_4\,, \tag{3.2}$$

where

$$I_1 = \sum_{i,j=1}^{n} \xi^4 a_{ij} \Bigg[\sum_{m,l=1}^{N} \frac{\partial^2 F}{\partial w^m \partial w^l} \frac{\partial u^m}{\partial x_i} \frac{\partial u^l}{\partial x_j} + 2 \sum_{m,l=1}^{N} \sum_{k=1}^{n} \frac{\partial^2 F}{\partial w^m \partial p_k^l} \frac{\partial u^m}{\partial x_i} \frac{\partial^2 u^l}{\partial x_k \partial x_j}$$
$$+ \sum_{m,l=1}^{N} \sum_{k,s=1}^{n} \frac{\partial^2 F}{\partial p_k^l \partial p_s^m} \frac{\partial^2 u^l}{\partial x_k \partial x_i} \frac{\partial^2 u^m}{\partial x_s \partial x_j} \Bigg],$$

$$I_2 = \sum_{i,j=1}^{n} \xi^4 \Bigg\{ \sum_{l=1}^{N} \frac{\partial F}{\partial w^l} a_{ij} \frac{\partial^2 u^l}{\partial x_i \partial x_j} + \sum_{l=1}^{N} \sum_{k=1}^{n} \frac{\partial F}{\partial p_k^l} a_{ij} \frac{\partial^3 u^l}{\partial x_k \partial x_i \partial x_j} \Bigg\},$$

$$I_3 = \sum_{i,j=1}^{n} \xi^4 \Bigg\{ a_{ij} \frac{\partial^2 F}{\partial x_i \partial x_j} + 2 \sum_{l=1}^{N} a_{ij} \frac{\partial^2 F}{\partial x_i \partial w^l} \frac{\partial u^l}{\partial x_j} + 2 \sum_{l=1}^{N} \sum_{k=1}^{n} a_{ij} \frac{\partial^2 F}{\partial x_i \partial p_k^l} \frac{\partial^2 u^l}{\partial x_k \partial x_j} \Bigg\},$$

$$I_4 = \sum_{i,j=1}^{n} \Bigg[a_{ij} (\xi^4)_{x_i x_j} F + a_{ij} (\xi^4)_{x_i} \Bigg[\frac{\partial F}{\partial x_j} + \frac{\partial F}{\partial w^l} \frac{\partial u^l}{\partial x_j} + \frac{\partial F}{\partial p_k^l} \frac{\partial^2 u^l}{\partial x_k \partial x_j} \Bigg] \Bigg].$$

Represent I_1 in the form

$$\sum_{r,t=1}^{N(n+1)} \sum_{i,j=1}^{n} a_{ij}(x_0) d^{rt}(x_0) q_i^r q_j^t\,,$$

where $d^{rt}(x_0)$ is the $N(n+1) \times N(n+1)$-matrix of the second derivatives of $F(x, w, p)$ with respect to $(w, p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ at $(x_0, u(x_0), \nabla u(x_0))$. By $q = (q_i^r)$ we denote the $Nn(n+1)$-dimensional vector with components $q = (q_1, \dots, q_n)$, $q_i = (\zeta_i, \eta_i)$, where

$$\zeta_i = (u^1_{x_i}, \dots, u^N_{x_i}) \in \mathbb{R}^N\,, \qquad \eta_i = (u^1_{x_i x_1}, \dots, u^1_{x_i x_n}; \dots; u^N_{x_i x_1}, \dots, u^N_{x_i x_n}) \in \mathbb{R}^{Nn}\,.$$

Since the matrices $a_{ij}(x_0)$ and $d^{rt}(x_0)$ are positive-definite (see (A2) and (F5)), we obtain

$$\sum_{r,t=1}^{N(n+1)} \sum_{i,j=1}^{n} a_{ij} d^{rt} q_i^r q_j^t \geq \nu_1 \nu_3 [1 + |\nabla u|^2]^{(\alpha_2 - 1)/2} \sum_{i=1}^{n} (|\eta_i|^2 + |\zeta_i|^2)$$
$$\geq \nu_0 (1 + |\nabla u|^2)^{(\alpha_2 - 1)/2} |u_{xx}|^2\,,$$

where $\nu_0 = \nu_1\nu_3 > 0$. Thus,

$$I_1 \geq \nu_0(1+|\nabla u|^2)^{(\alpha_2-1)/2}|u_{xx}|^2\xi^4 \,. \tag{3.3}$$

In I_2, there are terms with third order derivatives of u. Therefore we differentiate the lth equation of the system (PP1) with respect to x_k, multiply the result by $\partial F/\partial p_k^l$ and sum over k, l to obtain

$$\sum_{i,j=1}^{n}\sum_{l=1}^{N}\sum_{k=1}^{n}\Big[a_{ij}u^l_{x_ix_jx_k} + (a_{ij})_{x_k}u^l_{x_ix_j} + \frac{da^l(x,u,\nabla u)}{dx_k}\Big]\frac{\partial F}{\partial p_k^l}$$
$$= \sum_{i,j=1}^{n}\sum_{l=1}^{N}\sum_{k=1}^{n}\Big[\beta'(F)\frac{dF}{dx_k}\sum_{r=1}^{N}b^{lr}u^r + \beta(F)\sum_{r=1}^{N}(b^{lr}_{x_k}u^r + b^{lr}u^r_{x_k})\Big]\frac{\partial F}{\partial p_k^l}\,.$$

Using the lth equation of (PP1) to express $a_{ij}u^l_{x_ix_j}$ we can write

$$I_2 - J$$
$$= \xi^4\Big[\sum_{i,j=1}^{n}\sum_{l=1}^{N}\sum_{k=1}^{n}\Big(\frac{\partial F}{\partial w^l}a^l(x,u,\nabla u) - \frac{\partial F}{\partial p_k^l}(a_{ij})_{x_k}u^l_{x_ix_j} - \frac{\partial F}{\partial p_k^l}\frac{da^l(x,u,\nabla u)}{dx_k}\Big)\Big],$$

where

$$J = \xi^4\sum_{l,r=1}^{N}\Big[\frac{\partial F}{\partial w^l}b^{lr}u^r + \sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}b^{lr}u^r_{x_k}\Big]\beta(F)$$
$$+ \xi^4\beta'(F)\sum_{l,r=1}^{N}\sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}\frac{dF}{dx_k}b^{lr}u^r + \xi^4\beta(F)\sum_{l,r=1}^{N}\sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}(b^{lr})_{x_k}u^r$$
$$= J_1 + J_2 + J_3\,.$$

Without loss of generality one can assume that $\varepsilon < 1$, $F(x_0, u, \nabla u) > 1$ and $|\nabla u(x_0)|$ is sufficiently large. Due to the choice of the penalty function β we have

$$\beta(F) = \frac{1}{\varepsilon}\beta_0(F) \geq F^{1+\alpha_0} \geq \widetilde{\nu}_4|\nabla u|^{(1+\alpha_2)(1+\alpha_0)}\,, \tag{3.4}$$

$$F\beta'(F) = \frac{1}{\varepsilon}F\beta_0(F) = (1+\alpha_0)\frac{1}{\varepsilon}F^{1+\alpha_0} = (1+\alpha_0)\beta(F)\,. \tag{3.5}$$

To estimate J_1 we note that

$$\nabla_w G\cdot\nabla_w F = \sum_{l,r=1}^{N}\frac{\partial F}{\partial w^l}b^{lr}w^r$$

and apply the assumption (2.4) of Theorem 2.1 with $w = u(x_0)$ and $p = \nabla u(x_0)$ to obtain

$$J_1 \geq \xi^4\beta(F)[\nu_5|\nabla u|^{1+\alpha_2} - \mu_7|\nabla u|^{1+\alpha_2-\gamma_2}]\,, \qquad \gamma_2 > 0\,.$$

Hence for sufficiently large $|\nabla u(x_0)|$ we have

$$J_1 \geq \frac{\nu_5}{2}\xi^4\beta(F)|\nabla u|^{1+\alpha_2}\,.$$

Further, by (3.1) we have $\xi^4 dF/dx_k = -(\xi^4)_{x_k} F$ $(k = 1, \dots, n)$. Hence by (3.5)

$$\begin{aligned} J_2 &= -F\beta'(F) \sum_{l,r=1}^{N} \sum_{k=1}^{n} \frac{\partial F}{\partial p_k^l} b^{lr} u^r (\xi^4)_{x_k} \\ &= -(1+\alpha_0)\beta(F) \sum_{l,r=1}^{N} \sum_{k=1}^{n} b^{lr} u^r \frac{\partial F}{\partial p_k^l} (\xi^4)_{x_k} \\ &\geq -C\beta(F)[|\nabla u|^{\alpha_2} + 1] \sum_{k=1}^{n} |(\xi^4)_{x_k}| \,. \end{aligned}$$

Here we have used the inequality (F10). By analogy we deduce the bound

$$J_3 \geq -\xi^4 \beta(F) C[|\nabla u|^{\alpha_2} + 1] \,.$$

It is obvious that $J_1 + J_3 \geq (\nu_5/4)\xi^4\beta(F)|\nabla u|^{1+\alpha_2}$ for large $|\nabla u|$. Note that if

$$\xi |\nabla u(x_0)| \leq C \sum_{k=1}^{n} |\xi_{x_k}| = \widetilde{C}_1$$

then $\xi^4 |\nabla u| \leq \widetilde{C}$. If

$$\xi |\nabla u(x_0)| > C \sum_{k=1}^{n} |\xi_{x_k}| = \widetilde{C}_1 \,,$$

we obtain by (3.4)

$$J = J_1 + J_2 + J_3 \geq \frac{\nu_5}{8} \xi^4 \beta(F) |\nabla u|^{1+\alpha_2} \geq \nu_6 \xi^4 |\nabla u|^{(1+\alpha_2)(2+\alpha_0)} \,. \tag{3.6}$$

Now, we write out the derivative da^l/dx_k and use the conditions (A5) and (A6) to get

$$\begin{aligned} \left| \frac{da^l}{dx_k} \right| &= \left| \frac{\partial a^l}{\partial x_k} + \frac{\partial a^l}{\partial w^l} \frac{\partial u^m}{\partial x_k} + \sum_{l=1}^{N} \sum_{s=1}^{n} \frac{\partial a^l}{\partial p_k^l} \frac{\partial^2 u^l}{\partial x_k \partial x_s} \right| \\ &\leq C[|\nabla u|^{1+\alpha_1} + |\nabla u|^{2+\alpha_1} + |\nabla u|^{\alpha_1} |u_{xx}|] \,. \end{aligned} \tag{3.7}$$

In view of (A4), (F9)–(F11), (3.7) and the assumption $|\nabla u(x_0)| > 1$ we obtain

$$I_2 - J \geq -\xi^4 C[|\nabla u|^{2+\alpha_1+\alpha_2} + |\nabla u|^{\alpha_1+\alpha_2} |u_{xx}|] \,. \tag{3.8}$$

For I_3 we use (F9)–(F11) to obtain

$$I_3 \geq \xi^4 C[|\nabla u|^{2+\alpha_2-\delta} + |\nabla u|^{\alpha_2} |u_{xx}|] \,. \tag{3.9}$$

Thus, (3.3), (3.6), (3.8) and (3.9) imply

$$\begin{aligned} I_1 + I_2 + I_3 \geq \xi^4 &\left[\nu_0 (1 + |\nabla u|^2)^{(\alpha_2-1)/2} |u_{xx}|^2 + \frac{\nu_6}{2} |\nabla u|^{(1+\alpha_2)(2+\alpha_0)} \right] \\ &- C\xi^4 [|\nabla u|^{2+\alpha_1+\alpha_2} + |\nabla u|^{\alpha_1+\alpha_2} |u_{xx}|] \,. \end{aligned} \tag{3.10}$$

Apply the Cauchy inequality to obtain

$$|\nabla u|^{\alpha_1+\alpha_2}|u_{xx}| \le \frac{\nu_0}{4}(1+|\nabla u|^2)^{(\alpha_2-1)/2}|u_{xx}| + C|\nabla u|^{1+2\alpha_1+\alpha_2}\,.$$

By (PP3)

$$\gamma = (1+\alpha_2)(2+\alpha_0) > \max\{1+2\alpha_1+\alpha_2, 2+\alpha_1+\alpha_2\}\,.$$

Therefore for sufficiently large $|\nabla u(x_0)|$ we deduce from (3.10)

$$I_1+I_2+I_3 \ge \xi^4\left[\frac{\nu_0}{2}(1+|\nabla u|^2)^{(\alpha_2-1)/2}|u_{xx}|^2 + \frac{\nu_6}{2}|\nabla u|^{(1+\alpha_2)(2+\alpha_0)}\right]. \tag{3.11}$$

To estimate I_4 we note that

$$|a_{ij}(\xi^4)_{x_ix_j}F| \le \xi^2 F|12a_{ij}\xi_{x_i}\xi_{x_j} + 4\xi a_{ij}\xi_{x_ix_j}| \le \xi^2\widetilde{C}|\nabla u|\,.$$

Taking into account the last inequality and (F9)–(F11) we obtain for sufficiently large $|\nabla u(x_0)|$

$$I_4 \ge -\widetilde{C}\xi^2|\nabla u| - \widetilde{C}\xi^3(|\nabla u|^{2+\alpha_2-\delta} + |\nabla u|^{\alpha_2}|u_{xx}|)\,.$$

By the Cauchy inequality

$$\xi^3|\nabla u|^{\alpha_2}|u_{xx}| \le \frac{\nu_0}{4}\xi^4(1+|\nabla u|^2)^{(\alpha_2-1)/2}|u_{xx}|^2 + C\xi^2|\nabla u|^{1+\alpha_2}\,;$$

therefore,

$$I_4 \ge -\frac{\nu_0}{4}\xi^4(1+|\nabla u|^2)^{(\alpha_2-1)/2}|u_{xx}|^2 - \widetilde{C}_0\xi^2|\nabla u|^{2+\alpha_2-\delta}\,,$$

and consequently,

$$\sum_{i=1}^4 I_i \ge \xi^2|\nabla u|^{2+\alpha_2-\delta}\left[\xi^2\frac{\nu_6}{4}|\nabla u|^\gamma - \widetilde{C}_0\right].$$

Now we have two possibilities: 1) $\xi^2|\nabla u(x_0)|^\gamma \le 8\widetilde{C}_0/\nu_6$ and by (F9) we have $W(x_0) \le \widetilde{C}$; 2) $\sum_{i=1}^4 I_i \ge \xi(\nu_6/8)|\nabla u|^{(2+\alpha_0)(1+\alpha_2)} > 0$; in this case by (3.2) the last inequality is not true.

Thus, for any subdomain Ω' with $\overline{\Omega}' \subset \Omega$, the estimate $\max_{x\in\overline{\Omega}'}|\nabla u^\varepsilon(x)| \le \widetilde{C}$ holds.

3.4. *The existence of the limit function u.* Because of the estimates deduced above one can extract a subsequence from $\{u^\varepsilon\}$ (which we denote again by u^ε) such that $u^{\varepsilon l}$ converges weakly in $\overset{\circ}{W}{}^1_2(\Omega)$ to u^l ($l = 1,\ldots,N$). Moreover, $u^l \in W^1_{\infty,\mathrm{loc}}(\Omega)$ ($l = 1,\ldots,N$) and, because of the imbedding theorems, $u^{\varepsilon l} \to u^l$ uniformly on compact sets $\overline{\Omega}'$ with $\overline{\Omega}' \subset \Omega$.

3.5. *The estimate for $\xi^2\beta_\varepsilon(F(x,u^\varepsilon,\nabla u^\varepsilon))|_{\overline{\Omega}'_0}$.* Let $\xi \in C_0^\infty(\Omega)$, $\operatorname{supp}\xi \subset \Omega_u = \{x\in\Omega : u(x) \ne 0\}$. Since u is continuous in Ω, there exists a constant $\delta > 0$ such that $(\forall x \in \operatorname{supp}\xi)$ $|u(x)| > \delta$. Since $u^\varepsilon \to u$ uniformly on $\overline{\Omega}'$, we have $|u^\varepsilon(x)| > \delta/2$ for sufficiently small ε. Suppose that $W(x) = \xi^2\beta(F)$ has its maximum at $x_0 \in \operatorname{supp}\xi$. Now we show that $W(x_0)$ is bounded above by a constant which depends on δ and $\operatorname{dist}(\operatorname{supp}\xi, \partial\Omega)$, but does not depend on ε. Assume that $W(x_0) \neq 0$. At x_0 we have

$$W_{x_i} = (\xi^2)_{x_i}\beta(F) + \xi^2 \frac{d\beta}{dx_i} = 0 \qquad (i = 1, \dots, n)\,, \tag{3.12}$$

$$0 \geq \sum_{i,j=1}^{n} a_{ij} W_{x_i x_j} = \sum_{i,j=1}^{n} \left[a_{ij}(\xi^2)_{x_i x_j}\beta + 2a_{ij}(\xi^2)_{x_i}\frac{d\beta}{dx_j} + a_{ij}\xi^2 \frac{d^2\beta}{dx_i dx_j} \right]. \tag{3.13}$$

Since $\beta(0) = 0$ and $\beta(\tau)$ is a convex function, we have $\beta(\tau) \leq \tau\beta'(\tau)$. Substituting, in (3.13), $F\beta'(F)$ for $\beta(F)$ we obtain the lower bound $-\widetilde{C}\beta'$ for the first summand of (3.13). Writing out the expression for the derivative $d\beta/dx_i$ and taking into account the estimates for F and its partial derivatives, we obtain

$$\begin{aligned} \sum_{i,j=1}^{n} 2a_{ij}(\xi^2)_{x_i}\frac{d\beta}{dx_j} &= \sum_{i,j=1}^{n} 2a_{ij}(\xi^2)_{x_i}\beta' \left[\frac{\partial F}{\partial x_j} + \sum_{m=1}^{N} \frac{\partial F}{\partial w^m} u^m_{x_j} + \sum_{l=1}^{N}\sum_{k=1}^{n} \frac{\partial F}{\partial p^l_k} u^l_{x_k x_j} \right] \\ &\geq \beta'[-\widetilde{C} - \widetilde{C}\xi|u_{xx}|]\,. \end{aligned}$$

Since $\beta'' \geq 0$ and the matrix a_{ij} is positive-definite, the third summand in (3.13) is bounded below by $\beta'\xi^2 a_{ij} d^2F/dx_i dx_j$. Thus, (3.13) implies

$$0 \geq \sum_{i,j=1}^{n} a_{ij} W_{x_i x_j} \geq \beta'[-\widetilde{C} - \widetilde{C}\xi|u_{xx}|] + \beta'\xi^2 \sum_{i,j=1}^{n} a_{ij}\frac{d^2F}{dx_i dx_j}\,. \tag{3.14}$$

We estimate the last term in (3.14) as follows:

$$\begin{aligned} \beta'\xi^2 \sum_{i,j=1}^{n} a_{ij}\frac{d^2F}{dx_i dx_j} &= \beta'\xi^2 \sum_{i,j=1}^{n} a_{ij} \left[\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{m,l=1}^{N} \frac{\partial^2 F}{\partial w^m \partial w^l} u^m_{x_i} u^l_{x_j} \right. \\ &\quad + \sum_{l,m=1}^{N}\sum_{k,s=1}^{n} \frac{\partial^2 F}{\partial p^l_k \partial p^m_s} u^l_{x_k x_i} u^m_{x_s x_j} + \sum_{m=1}^{N} \frac{\partial^2 F}{\partial x_i \partial w^m} u^m_{x_j} \\ &\quad \left. + \sum_{l,m=1}^{N}\sum_{k=1}^{n} \frac{\partial^2 F}{\partial w^m \partial p^l_k} u^m_{x_i} u^l_{x_k x_j} + \sum_{m=1}^{N} \frac{\partial F}{\partial w^m} u^m_{x_i x_j} + \sum_{l=1}^{N}\sum_{k=1}^{n} \frac{\partial F}{\partial p^l_k} u^l_{x_k x_i x_j} \right] \\ &\geq \beta'\xi^2 \left[\nu_0|u_{xx}|^2 - \widetilde{C}|u_{xx}| - \widetilde{C} + \sum_{i,j=1}^{n}\sum_{m=1}^{N} \frac{\partial F}{\partial w^m} a_{ij} u^m_{x_i x_j} \right. \\ &\quad \left. + \sum_{i,j=1}^{n}\sum_{l=1}^{N}\sum_{k=1}^{n} \frac{\partial F}{\partial p^l_k} a_{ij} u^l_{x_k x_i x_j} \right]. \end{aligned} \tag{3.15}$$

Here $\nu_0 \leq \nu_1\nu_3(1 + |\nabla u^\varepsilon|^2)^{(\alpha_2 - 1)/2}$. Because of the estimate from 3.3 we can assume that $\nu_0 > 0$ does not depend on ε. Now, we differentiate the system

(PP1) with respect to x_k and sum over k. By (A5), (A6) and (2.4) we obtain

$$0 \ge \beta'(F)\xi^2[\nu_0|u_{xx}|^2 - \widetilde{C}|u_{xx}| - \widetilde{C}] + \beta'(F)\sum_{l=1}^{N}\sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}\left(\xi^2\frac{d\beta}{dx_k}\right)\sum_{s=1}^{N} b^{ls}u^s\,. \tag{3.16}$$

By (3.12) the last summand in (3.16) can be rewritten as follows:

$$\beta'(F)\sum_{l=1}^{N}\sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}\left(\xi^2\frac{d\beta}{dx_k}\right)\sum_{s=1}^{N} b^{ls}u^s = -\beta'(F)\sum_{l=1}^{N}\sum_{k=1}^{n}\frac{\partial F}{\partial p_k^l}(a_{ij}u^l_{x_ix_j} + a^l)(2\xi\xi_{x_k}) \ge -\beta'(F)[\widetilde{C}\xi|u_{xx}| + \widetilde{C}]\,.$$

Thus, it follows from (3.14) and (3.16) that

$$0 \ge \sum_{i,j=1}^{n} a_{ij}W_{x_ix_j} \ge \beta'(F)[\nu_0\xi^2|u_{xx}|^2 - \widetilde{C}\xi|u_{xx}| - \widetilde{C}]\,.$$

We apply the Cauchy inequality to obtain

$$0 \ge \sum_{i,j=1}^{n} a_{ij}W_{x_ix_j} \ge \beta'(F)[\nu_0\xi^2|u_{xx}|^2 - \widetilde{C}]\,.$$

Recalling our assumption $W(x_0) \ne 0$ (see 3.5) and, consequently, $\beta'(F) > 0$, we conclude that $\xi^2|u_{xx}| \le \widetilde{C}$ at x_0. Hence

$$\xi^4\beta^2|u^\varepsilon|^2 = \xi^4|Lu^\varepsilon|^2 = \xi^4\sum_{i,j=1}^{n}\sum_{l=1}^{N}(a_{ij}u^{\varepsilon l}_{x_ix_j} + a^l)^2 \le \widetilde{C}\,.$$

Since $|u^\varepsilon(x_0)| > \delta/2$, the desired estimate is obtained.

3.6. Proof of Theorem 2.1. Because of the estimates obtained and the imbedding theorems there exists a subsequence $\{u^\varepsilon\}$ such that

$$u^{\varepsilon l} \to u^l \quad \text{uniformly in } \overline{\Omega}' \subset \Omega\,, \tag{3.17}$$

$$\nabla u^{\varepsilon l} \to \nabla u^l \quad \text{uniformly in } \overline{\Omega}'_u \subset \Omega_u\,, \tag{3.18}$$

$$u^{\varepsilon l}_{xx} \to u^l_{xx} \quad \text{weakly in } L_q(\Omega'_u)\ (1 < q < \infty)\,, \tag{3.19}$$

where $u = (u^1, \ldots, u^N)$ is the limit function from 3.4. Moreover,

$$u \in C^{0,1}(\Omega;\mathbb{R}^N) \cap C^1(\Omega_u;\mathbb{R}^N) \cap W^2_{2,\mathrm{loc}}(\Omega_u;\mathbb{R}^N)\,,$$

i.e., u satisfies (QV1). By (3.17) and (3.18) we have $\beta_0(F(x,u^\varepsilon,\nabla u^\varepsilon)) \to \beta_0(F(x,u,\nabla u))$ for $x \in \Omega_u$. Since $\beta_\varepsilon(\tau) = (1/\varepsilon)\beta_0(\tau)$, the estimate from 3.5 provides $\beta_0(F(x,u,\nabla u)) = 0$. Hence $F(x,u,\nabla u) \le 0$ for $x \in \Omega_u$, i.e., the inequality (QV3) is valid. To verify the quasivariational inequality (QV2) let us note that from (3.17), (3.18) we have

$$\langle Au^\varepsilon, \eta(v - u^\varepsilon)\rangle \to \langle Au, \eta(v - u)\rangle$$

for η from (QV2) and $v \in \overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^N)$. It follows from (PP1) that

$$\langle Au^\varepsilon, \eta(v-u^\varepsilon)\rangle \le 0 \tag{3.20}$$

for any $v \in \overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^N)$ such that $v(x) \in M(x,u^\varepsilon(x))$ a.e. in Ω. Now we set $\delta = \min_{x\in\operatorname{supp}\eta}|u(x)|$. We have $\delta > 0$ and $G(x,u(x)) \ge \nu_4\delta^2$ for $x \in \operatorname{supp}\eta$. Let v be an arbitrary function in $\overset{\circ}{W}{}_2^1(\Omega;\mathbb{R}^N)$ such that $v(x) \in M(x,u(x))$ a.e. in Ω. Define $v_n(x) = (1-1/n)v(x)$. Note that $v_n(x) \in M(x,u(x))$ and

$$G(x,v_n) = B(x)v_n\cdot v_n = \left(1-\frac{1}{n}\right)^2 G(x,v) \le \left(1-\frac{1}{n}\right)^2 G(x,u)\,.$$

By (3.17) we also have $G(x,v_n) \le G(x,u^\varepsilon)$ for $x \in \operatorname{supp}\eta$ if ε is sufficiently small. Hence we obtain (QV2) from (3.20) as $\varepsilon \to 0$ and $n \to \infty$. To check (QV4) one has to use (3.17) and (3.18).

3.7. Proof of Theorem 2.2. Proceeding as in [13, Ch. 8] with the help of the barrier technique one can obtain the conclusion of the theorem.

3.8. Proof of Theorem 2.3. The proof of the estimates for $|u^\varepsilon|_{\overline{\Omega}}$, $\|u^\varepsilon\|_{\overset{\circ}{W}{}_2^1(\Omega)}$ and $|\nabla u|_{\partial\Omega}$ is quite similar to that in Theorems 2.1 and 2.2. To deduce the estimate for $|\nabla u|_{\overline{\Omega}}$ consider the function $W(x) = |\nabla u|^2$ at its maximum point $x_0 \in \Omega$. The inequality

$$\begin{aligned} 0 \ge \Delta W &= 2u^l_{x_kx_j}u^l_{x_kx_j} + 2u^l_{x_k}u^l_{x_kx_jx_j} \\ &\ge |u_{xx}|^2 - 2\beta' f_{x_k}u^l_{x_k}u^l + 2\beta'(u^l u^l_{x_k})^2 + 2\beta|\nabla u|^2 \end{aligned}$$

provides the desired estimate.

Note that $\beta_\varepsilon(F(x,u^\varepsilon)) = 0$ if $|u^\varepsilon(x)| \le f_0$ (see (O3)). Taking this into account and proceeding as in 3.5 we can establish the estimate $\beta_\varepsilon(F(x,u^\varepsilon)) \le C$ for all $x \in \overline{\Omega}$. Arguing as in the proof of Theorem 2.1 we deduce that $F(x,u(x)) \le 0$ in $\overline{\Omega}$ and $Lu(x) = \lambda(x)u(x)$ where $\lambda(x) \ge 0$. Moreover, $\lambda(x) = 0$ if $F(x,u(x)) < 0$. Thus, we conclude that the solution u of (QV1)–(QV4) (or (U1)–(U5)) satisfies the obstacle problem of type (1.19).

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