ON TWO CLASSES OF WEIGHTED SOBOLEV–SLOBODETSKII SPACES IN A DIHEDRAL ANGLE

J. ROSSMANN

Department of Mathematics, University of Rostock
Universitätsplatz 1, D-O-2500 Rostock, Germany

Introduction. In this paper some problems of the theory of weighted Sobolev–Slobodetski˘ı spaces are dealt with. Weighted Sobolev spaces play an important role in the study of elliptic boundary value problems in non-smooth domains (see e.g. [2]–[4], [6]). Here two classes of weighted spaces are of special interest: the spaces $V^l_{p,\beta}(G)$ with the homogeneous norms

$$ \|u\|_{V^l_{p,\beta}(G)} = \left( \int_G \sum_{|\alpha| \leq l} r^{p(\beta-l+|\alpha|)} |D^\alpha u|^p \, dx \right)^{1/p} $$

($r = r(x)$ denotes the distance of $x$ to the set of singularities on the boundary) and the spaces $W^l_{p,\beta}(G)$ with the inhomogeneous norms

$$ \|u\|_{W^l_{p,\beta}(G)} = \left( \int_G \sum_{|\alpha| \leq l} r^{p\beta} |D^\alpha u|^p \, dx \right)^{1/p}.$$  

We restrict ourselves to the case that $G = \mathcal{D}$ is a dihedral angle. This is the model case for domains with smooth non-intersecting edges. For domains with conical points the weighted Sobolev spaces $V^l_{p,\beta}(G)$ and $W^l_{p,\beta}(G)$ were investigated e.g. in [2], [5], [8].

The present paper consists of two sections. In Section 1 the main properties of the weighted Sobolev spaces $V^l_{p,\beta}(\mathcal{D})$, $W^l_{p,\beta}(\mathcal{D})$ ($l$ an integer, $l \geq 0$, $\beta \in \mathbb{R}$, $1 < p < \infty$) and of the corresponding weighted Sobolev–Slobodetski˘ı spaces $V^s_{p,\beta}(\mathcal{D})$, $W^s_{p,\beta}(\mathcal{D})$ ($s$ real, $s \geq 0$, $\beta \in \mathbb{R}$, $1 < p < \infty$) will be investigated. It will be shown e.g. that the following imbeddings are valid:

- $V^s_{p,\beta}(\mathcal{D}) \subset V^{s'}_{p,\beta'}(\mathcal{D})$ if $s \geq s'$, $s - \beta = s' - \beta'$,
- $W^s_{p,\beta}(\mathcal{D}) \subset W^{s'}_{p,\beta'}(\mathcal{D})$ if $s \geq s'$, $s - \beta = s' - \beta'$, $\beta' > -2/p$.  

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Furthermore, we give a connection between the spaces $V^s_{p,\beta}(D)$ and $W^s_{p,\beta}(D)$. It will be proved that every function $u \in W^s_{p,\beta}(D)$ can be written as the sum of a “quasi-polynomial” and a function from $V^s_{p,\beta}(D)$ (see Theorem 5). This is a generalization of Lemma 1.3 of [6] (see also [7]) where such a representation has been shown if $s$ is an integer and $\beta + 2/p$ is not an integer. Analogous results are obtained for the space $\tilde{B}_{p,\beta}^{-s-1/p}(\Gamma^\pm)$ and $B_{p,\beta}^{-s-1/p}(\Gamma^\pm)$ of the traces of functions from $V^s_{p,\beta}(D)$ and $W^s_{p,\beta}(D)$ on the sides $\Gamma^+$ and $\Gamma^-$ of $D$, respectively.

Section 2 is concerned with two applications of the given results. In Section 2.1 we investigate conditions on $g^\pm_k \in B^{s-m^\pm_k-1/p}_{p,\beta}(\Gamma^\pm)$ under which there exists $u \in W^s_{p,\beta}(D)$ satisfying the boundary conditions

$$B^\pm_k u = g^\pm_k \quad (k = 1, \ldots, p^\pm)$$

where $\{B^+_k\}$ and $\{B^-_k\}$ are normal systems of homogeneous boundary operators of order $m^+_k$ with constant coefficients on $\Gamma^+$ and $\Gamma^-$, respectively. In the spaces $V^s_{p,\beta}$ the existence of $u$ is always ensured (see [4]). We will show that $u \in W^s_{p,\beta}(D)$ satisfying (0.3) exists if and only if the boundary data $g^\pm_k$ satisfy some compatibility conditions on the edge $M$ (see Theorem 7).

In Section 2.2 the following regularity assertion for solutions of elliptic boundary value problems will be proved: If $u \in W^{2m}_{p,\beta}(D)$ is a solution of an elliptic boundary value problem $Lu = f$ in $D$, $B^\pm_k u = g^\pm_k$ on $\Gamma^\pm$ $(k = 1, \ldots, m)$ where $f \in W^s_{p,\beta+s}(D)$, $g^\pm_k \in B^{s+2m-m^\pm_k-1/p}_{p,\beta+s}(\Gamma^\pm)$ then $u \in W^{s+2m}_{p,\beta+s}(D)$.

The corresponding result for the spaces $V^s_{p,\beta}$ has been proved in [4].

1. Weighted Sobolev–Slobodetskiï spaces in a dihedral angle

1.1. The spaces $V^s_{p,\beta}(D)$. Let $D = K \times \mathbb{R}^{n-2} = \{x = (y, z) \in \mathbb{R}^n : y \in K, z = (z_1, \ldots, z_{n-2}) \in \mathbb{R}^{n-2}\}$ be a dihedral angle in $\mathbb{R}^n$ where $K$ is a plane wedge which has the following representation in polar coordinates $r, \omega$:

$$K = \{y = (y_1, y_2) \in \mathbb{R}^2 : 0 < r < \infty, \omega \in \Omega = (-\frac{1}{2}\omega_0, \frac{1}{2}\omega_0)\}$$

$(0 < \omega_0 \leq 2\pi)$. The boundary of $D$ consists of the $(n-1)$-dimensional half-planes

$$\Gamma^\pm = \{x = (x, z) \in \mathbb{R}^n : 0 < r < \infty, \omega = \pm \frac{1}{2}\omega_0, z \in \mathbb{R}^{n-2}\}$$

and of the edge $M = \{(0,0)\} \times \mathbb{R}^{n-2}$. In the sequel we will denote the coordinates of a point $x = (x_1, \ldots, x_n)$ by $y_1, y_2, z_1, \ldots, z_{n-2}$, i.e. $y_1 = x_1, y_2 = x_2, z_j = x_{j+2}$ $(j = 1, \ldots, n-2)$.

If $s$ is a non-negative integer and $p, \beta$ are real numbers, $1 < p < \infty$, then $V^s_{p,\beta}(D)$ denotes the closure of $C^\infty_0(\overline{D} \setminus M) = \{u \in C^\infty(\overline{D}) : \text{supp } u \subset \overline{D} \setminus M, \text{ supp } u \text{ compact} \}$ with respect to the norm

$$\|u\|_{V^s_{p,\beta}(D)} = \left( \int_D \sum_{|\alpha| \leq s} r^{p(\beta-s+|\alpha|)} |D^\alpha u|^p \, dx \right)^{1/p}$$

$$(D^\alpha = D^\alpha_{x_1} \ldots D^\alpha_{x_n}, \quad D_{x_j} = (1/i)\partial_{x_j} = (1/i)\partial/\partial x_j, \quad r = |y|).$$
For non-integer positive \( s = l + \sigma \) (\( l \) an integer, \( 0 < \sigma < 1 \)) the space \( V^s_{p,\beta}(\mathcal{D}) \) is defined as the closure of \( C^\infty_0(\overline{\mathcal{D} \setminus M}) \) with respect to the norm

\[
\|u\|_{V^s_{p,\alpha}(\mathcal{D})} = \left( \int_{\mathcal{D}} \sum_{|\alpha| \leq l} |y|^{p(\beta-s+|\alpha|)} |(D^\alpha u)(x)|^p \, dx \right)^{1/p} + \int \int_{\mathcal{D} \times \mathcal{D}} \frac{|y|^\beta (D^\alpha u)(y, z) - |y'|^\beta (D^\alpha u)(y', z')|^p \, dx \, dx'}{|x - x'|^{n+p\sigma}}. \]

It can easily be shown that the norm (1.2) is equivalent to the norm

\[
\|u\| = \left( \int_{\mathcal{D}} \sum_{|\alpha| \leq l} |y|^{p(\beta-s+|\alpha|)} |(D^\alpha u)(x)|^p \, dx \right)^{1/p} + \sum_{|\alpha| = l} \int \int_{|x-x'|<|y|/2} |y|^{p\beta} |(D^\alpha u)(y, z) - (D^\alpha u)(y', z')|^p \, dx \, dx' \right)^{1/p}.
\]

**Theorem 1.** If \( s' > s \) and \( \beta' - s' = \beta - s \) then \( V^{s'}_{p,\beta}(\mathcal{D}) \) is continuously imbedded in \( V^s_{p,\beta}(\mathcal{D}) \).

**Proof.** If \( l \leq s < s' < l+1 \) (\( l \) an integer) then the imbedding immediately follows from the inequality

\[
|y|^{\beta-\beta'} = |y|^{s-s'} \leq 2^{s-s'} |x-x'|^{s-s'} \quad \text{for} \quad |x-x'| < \frac{1}{2}|y|.
\]

In the case \( l < s < s' = l+1 \) we can apply the equation

\[
(D^\alpha u)(x) - (D^\alpha u)(x') = - \int_0^1 \frac{d}{dt} (D^\alpha u)(x + t(x' - x)) \, dt
\]

\[
= (x - x') \int_0^1 (\nabla D^\alpha u)(x + t(x' - x)) \, dt
\]

to obtain

\[
\int \int_{|x-x'|<|y|/2} |y|^{p\beta} |(D^\alpha u)(x) - (D^\alpha u)(x')|^p \, dx \, dx' \right)^{1/p} \leq \int_0^1 \int \int_{|x-x'|<|y|/2} |y|^{p\beta} |x - x'|^{-n+p(l+1-s)} |(\nabla D^\alpha u)(x + t(x - x'))|^p \, dx \, dx' \, dt \]

\[
\leq \int_0^1 \int \int_{|x-x'|<|y|/2} |y|^{p\beta'} |(\nabla D^\alpha u)(x)|^p \, dx \, dx
\]

for \( |\alpha| = l \), i.e. \( \|u\|_{V^s_{p,\alpha}(\mathcal{D})} \leq C \|u\|_{V^{s'}_{p,\beta}(\mathcal{D})} \).
In the sequel let \( \zeta_\nu (\nu = \ldots, -1, 0, +1, \ldots) \) be smooth functions on \( \mathbb{R}_+ \) with support in the interval \([2^{\nu-1}, 2^{\nu+1}]\) such that

\[
\sum_{\nu=-\infty}^{\infty} \zeta_\nu (r) = 1 \quad \text{and} \quad |D_r^j \zeta_\nu (r)| < c_j 2^{-\nu j}
\]

with constants \( c_j \) independent of \( r \) and \( \nu \). If we set \( r = |y| = (y_1^2 + y_2^2)^{1/2} \) we can interpret \( \zeta_\nu \) as functions on \( \mathcal{D} \). Analogously to Lemma 1.1 in [4] the following assertion can be proved.

**Lemma 1.** The norm \( \| \cdot \|_{V_{p, \beta}^s (\mathcal{D})} \) is equivalent to the norm

\[
\| u \| = \left( \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)}^p \right)^{1/p}.
\]

Let \( B_{p, \beta}^{s-1/p} (\Gamma^\pm) \) \((s > 1/p)\) be the space of the traces of functions from \( V_{p, \beta}^s (\mathcal{D}) \) on \( \Gamma^+ \) and \( \Gamma^- \), respectively, provided with the norm

\[
\| u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)} = \sup \{ \| v \|_{V_{p, \beta}^s (\mathcal{D})} : v \in V_{p, \beta}^s (\mathcal{D}), \ v|_{\Gamma^\pm} = u \}.
\]

**Lemma 2.** The norm \( (1.5) \) is equivalent to the norm

\[
\| u \| = \left( \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)}^p \right)^{1/p}.
\]

**Proof.** Let \( v \in V_{p, \beta}^s (\mathcal{D}) \) be an extension of \( u \in B_{p, \beta}^{s-1/p} (\Gamma^\pm) \) such that

\[
\| v \|_{V_{p, \beta}^s (\mathcal{D})} \leq 2 \| u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)}.
\]

Since \( \zeta_\nu v \) is an extension of \( \zeta_\nu u \) we get

\[
\left( \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)}^p \right)^{1/p} \leq \left( \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu v \|_{V_{p, \beta}^s (\mathcal{D})}^p \right)^{1/p} \leq c \| v \|_{V_{p, \beta}^s (\mathcal{D})} \leq 2c \| u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)}.
\]

Furthermore, there exist extensions \( v_\nu \) of \( \zeta_\nu u \) such that

\[
\| v_\nu \|_{V_{p, \beta}^s (\mathcal{D})} \leq 2 \| \zeta_\nu u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)} \quad (\nu = \ldots, -1, 0, +1, \ldots). \]

Since \( w_\nu = (\zeta_{\nu-1} + \zeta_\nu + \zeta_{\nu+1}) v_\nu = \zeta_\nu u \) on \( \Gamma^\pm \) and \( w = \sum_{\nu=-\infty}^{\infty} w_\nu = u \) on \( \Gamma^\pm \) we obtain

\[
\| u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)} \leq \| u \|_{V_{p, \beta}^s (\mathcal{D})} \leq c \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu u \|_{V_{p, \beta}^s (\mathcal{D})} = c \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu \|_{\mathcal{D}} \sum_{k=-\infty}^{\nu+2} \| \zeta_{k-1} + \zeta_k + \zeta_{k+1} \|_{V_{p, \beta}^s (\mathcal{D})} \| v_k \|_{V_{p, \beta}^s (\mathcal{D})}
\]

\[
\leq c' \sum_{\nu=-\infty}^{\infty} \| v_\nu \|_{V_{p, \beta}^s (\mathcal{D})} \leq c' 2^p \sum_{\nu=-\infty}^{\infty} \| \zeta_\nu u \|_{B_{p, \beta}^{s-1/p} (\Gamma^\pm)}.
\]

\( \blacksquare \)
that
\[∥c∥∥\text{where the constants } c \text{ Applying Lemma 2 we obtain the assertion of the theorem.} \]

Using Lemmas 1 and 2 we can give the following norm equivalent to (1.5).

**Theorem 2.** Let \( p, \beta \) and \( s \) (\( 1 < p < \infty, s > 1/p \)) be real numbers and \( k \) an arbitrary integer such that \( k > s - 1/p \). Then the norm (1.5) is equivalent to the norm

\[
(1.6) \quad ∥u∥ = \left( \int_{\mathbb{R}^+ \times \mathbb{R}^{n-2}} |\xi^p(n-s+1)u(ξ)|^p \, dξ \right)^{1/p} 
+
\int_{\mathbb{R}^+ \times \mathbb{R}^{n-2}} \left( \sum_{j=0}^k (-1)^j \binom{k}{j} u\left(\frac{jξ + (k-j)ξ'}{k}\right) \right)^p \, \frac{dξ \, dξ'}{|ξ - ξ'|^{n-2+ps}}.
\]

**Proof.** Let \( u \in \dot{B}^{-1/p}_{p,\beta}(\Gamma^+) \). Define \( u_\nu(ξ) = ζ(2^\nu ξ)u(2^\nu ξ) \) \((\nu = 0, ±1, ±2, \ldots)\). Since \( \text{supp} u_\nu \subset \{ξ \in \mathbb{R}_+ \times \mathbb{R}^{n-2} : 1/2 < ξ_1 < 2 \} \) the norm of \( u \) can be estimated by the usual Besov space norm (see [11]), i.e.

\[
c_1 ∥u_\nu∥_{\dot{B}^{-1/p}_{p,\beta}(\Gamma^+)} \leq \int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} |u_\nu(ξ)|^p \, dξ 
+
\int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} \left( \sum_{j=0}^k (-1)^j \binom{k}{j} u_\nu\left(\frac{jξ + (k-j)ξ'}{k}\right) \right)^p \, \frac{dξ \, dξ'}{|ξ - ξ'|^{n-2+ps}} \leq c_2 ∥u_\nu∥_{\dot{B}^{-1/p}_{p,\beta}(\Gamma^+)}
\]

where the constants \( c_1, c_2 \) are independent of \( u \) and \( ζ \). It can be easily verified that

\[
c_1 2^{\nu(ps-p\beta-n)} ∥ζ_\nu u∥_{\dot{B}^{-1/p}_{p,\beta}(\Gamma^+)} \leq 2^{\nu(n-1)} \int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} |ζ_\nu(ξ)u(ξ)|^p \, dξ 
+
2^{\nu(ps-p\beta-n)} \int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} \left( \sum_{j=0}^k (-1)^j \binom{k}{j} \right) \times ζ_\nu\left(\frac{jξ + (k-j)ξ'}{k}\right) \, \frac{dξ \, dξ'}{|ξ - ξ'|^{n-2+ps}} \leq c_2 2^{\nu(ps-p\beta-n)} ∥ζ_\nu u∥_{\dot{B}^{-1/p}_{p,\beta}(\Gamma^+)}.
\]

Applying Lemma 2 we obtain the assertion of the theorem. ■
Remark 1. By means of other equivalent norms in the usual Besov space (see [11], Section 4.4) analogously to Theorem 2 equivalence of other weighted norms to the norm (1.5) can be proved. In particular, if $s - 1/p$ is not an integer, $s - 1/p = l + \sigma$ ($l$ an integer, $l \geq 0$, $0 < \sigma < 1$) then (1.5) is equivalent to the norm

$$\|u\|_{p, \beta} = \left( \int_{\mathbb{R}^n} |y|^{p(\beta-s)} |u(x')|^p \, dx \right)^{1/p}$$

or to the norm

$$\|u\| = \left( \int_{\mathbb{R}^n} \frac{\xi^p}{|\xi|^{n-1+\sigma}} |u(\xi)|^p \, d\xi \right)^{1/p}$$

Analogously, the norm (1.2) is equivalent to the norm

$$\|u\| = \left( \int_{\mathbb{R}^n} \frac{\xi^p}{|\xi|^{n-1+\sigma}} |D^\alpha u(\xi)|^p \, d\xi \right)^{1/p}$$

1.2. The spaces $W_{p, \beta}^s$. Let $p$, $\beta$, $s$ be real numbers, $1 < p < \infty$, $\beta > -2/p$, $s \geq 0$. If $s$ is an integer then we define the space $W_{p, \beta}^s$ as the closure of $C_0^\infty(D)$ with respect to the norm

$$(1.7) \quad \|u\|_{W_{p, \beta}^s} = \left( \sum_{|\alpha| \leq s} \int_D r^{p\beta} |D^\alpha u|^p \, dx \right)^{1/p}.$$  

If $s = l + \sigma$ ($l$ an integer, $0 < \sigma < 1$) then the space $W_{p, \beta}^s$ will be defined as the closure of $C_0^\infty(D)$ with respect to the norm

$$(1.8) \quad \|u\|_{W_{p, \beta}^s} = \left( \sum_{|\alpha| \leq l} \int_D r^{p\beta} |D^\alpha u|^p \, dx \right)^{1/p} + \left( \sum_{|\alpha| = l} \int_{|x-x'| < |y|/2} \frac{r^{p\beta} |D^\alpha u(x) - D^\alpha u(x')|^p}{|x-x'|^{n+p\sigma}} \, dx \, dx' \right)^{1/p}.$$
Furthermore, we define $W^{s}_{p,\beta}([\mathbb{R}^{n} \times \mathbb{R}^{n-2}])$ ($s \geq 0, 1 < p < \infty, \beta > -1/p$) as the closure of $C^{\infty}_{0}([\mathbb{R}^{+} \times \mathbb{R}^{n-2}])$ with respect to the norm

$$
\|u\|_{W^{s}_{p,\beta}([\mathbb{R}^{+} \times \mathbb{R}^{n-2}])} = \left( \int_{\mathbb{R}^{+} \times \mathbb{R}^{n-2}} \sum_{|\alpha| \leq s} |(D^\alpha u)(r,z)|^p \, dr \, dz \right)^{1/p}
$$

if $s$ is an integer, and

$$
\|u\|_{W^{s}_{p,\beta}([\mathbb{R}^{+} \times \mathbb{R}^{n-2}])} = \left( \int_{\mathbb{R}^{+} \times \mathbb{R}^{n-2}} \sum_{|\alpha| \leq l} |(D^\alpha u)(r,z)|^p \, dr \, dz + \int_{|\alpha|=l} \int_{(r-r', z-z') \in (r/2, z/2)} p^{\beta} |(D^\alpha u)(r,z) - (D^\alpha u)(r',z')|^p \, dr \, dz \right)^{1/p}
$$

if $s = l + \sigma$ ($l$ an integer, $l \geq 0$, $0 < \sigma < 1$).

Let $u = u(y_1, y_2, z)$ be an arbitrary function on $\mathcal{D}$. Then we denote by $\tilde{u}$ the function

$$
\tilde{u}(r, z) = \frac{1}{\omega_0} \int_{-\omega_0/2}^{\omega_0/2} u(r \cos \omega, r \sin \omega, z) \, d\omega.
$$

**Lemma 3.** If $u \in W^{s}_{p,\beta}(\mathcal{D})$ then $\tilde{u} \in W^{s}_{p,\beta+1/p}(\mathbb{R}^{+} \times \mathbb{R}^{n-2})$ and

$$
\|\tilde{u}\|_{W^{s}_{p,\beta+1/p}(\mathbb{R}^{+} \times \mathbb{R}^{n-2})} \leq c\|u\|_{W^{s}_{p,\beta}(\mathcal{D})}.
$$

If $s$ is an integer then this assertion immediately follows from the definition of the spaces $W^{s}_{p,\beta}(\mathcal{D}), W^{s}_{p,\beta}(\mathbb{R}^{+} \times \mathbb{R}^{n-2})$. For $s$ not an integer it is proved in [9].

We introduce the operator

$$(Kg)(r, z) = \chi(r) \int_{\mathbb{R}^{n-2}} g(t, r, z + \tau t) K(t, \tau) \, dt \, d\tau$$

where $\chi$ is a smooth cut-off function on $\mathbb{R}^{+}$, equal to unity in $[0, 1]$ and to zero in $(2, \infty)$ and $K(t, \tau) = \varphi(t) \psi(\tau_1) \ldots \psi(\tau_{n-2})$ is a product of smooth functions $\varphi \in C^{\infty}_{0}(\mathbb{R}^{+}), \psi \in C^{\infty}_{0}(\mathbb{R})$ satisfying

$$
\sup \varphi \subset (3/4, 5/4), \quad \int_{0}^{\infty} t^{j} \varphi(t) \, dt = \delta_{0,j},
$$

$$
\sup \psi \subset (-1/4, 1/4), \quad \int_{-\infty}^{\infty} t^{j} \psi(t) \, dt = \delta_{0,j}
$$

($j = 0, 1, \ldots, k$). Here $\delta_{0,j}$ denotes the Kronecker symbol.
Lemma 4. If \( g \in W^s_{p,\beta}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \) then \( K g \in \bigcap_{b=1}^{\infty} W^{(s)+\nu}_{p,\beta+(s)-s+\nu}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \). Furthermore,

\[
\int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^+} r^{p(\beta-s+j+|\gamma|)} |D_j^\alpha D_2^\beta (K g)(r, z)|^p \, dr \, dz \leq c \|g\|_{W^s_{p,\beta}(\mathbb{R}^+ \times \mathbb{R}^{n-2})}^p
\]

for \( j \geq 1 \) or \( |\gamma| \geq (s) + 1 \), and

\[
\int_{\mathbb{R}^{n-2}} \int_{\mathbb{R}^+} r^{p(\beta-s+(s)+1)} |D_j^\alpha D_2^\beta (K g - g)|^p \, dr \, dz \leq c \|g\|_{W^s_{p,\beta}(\mathbb{R}^+ \times \mathbb{R}^{n-2})}^p
\]

for \( |\gamma| = (s) \). Here \( (s) \) denotes the largest integer less than \( s \).

Proof. For simplicity we restrict ourselves to the case \( n = 3 \). For \( n > 3 \) the lemma can be proved analogously.

If \( j + |\gamma| = (s) \) and \( r < 1 \) then \( D_j^\alpha D_2^\beta (K g)(r, z) \) is a linear combination of terms of the form

\[
T = \int_{\mathbb{R}} \int_{\mathbb{R}_+} (D^\alpha g)(tr, z + tr) t^\mu \varphi(t) \tau^\nu \psi(\tau) \, dt \, d\tau
\]

where \( |\alpha| = j + |\gamma| \), \( \mu + \nu = j \).

Let \( j + |\gamma| \geq (s) + 1 \), \( r < 1 \). Since

\[
\int_{\mathbb{R}} \int_{\mathbb{R}_+} r^{-2} \left( \frac{t}{r} \right)^\mu \varphi \left( \frac{t}{r} \right) \left( \frac{\tau - z}{r} \right)^\nu \psi \left( \frac{\tau - z}{r} \right) \, dt \, d\tau = \int_{\mathbb{R}} \int_{\mathbb{R}_+} t^\nu \varphi(t) \tau^\nu \psi(\tau) \, dt \, d\tau
\]

is a constant, the function \( D_j^\alpha D_2^\beta K g \) can be written as a finite sum of expressions of the form

\[
T' = cr^{-2+(s)-j-|\gamma|} \int_{\mathbb{R}} \int_{\mathbb{R}_+} ((D^\alpha g)(tr, z + tr) - (D^\alpha g)(r, z))
\]

\[
\times \left( \frac{t}{r} \right)^{\mu_1} \varphi^{(\mu_2)} \left( \frac{t}{r} \right) \left( \frac{\tau - z}{r} \right)^{\nu_1} \psi^{(\nu_2)} \left( \frac{\tau - z}{r} \right) \, dt \, d\tau
\]

\[
= c r^{-s-j-|\gamma|} \int_{\mathbb{R}} \int_{\mathbb{R}_+} ((D^\alpha g)(tr, z + tr) - (D^\alpha g)(r, z))
\]

\[
\times t^{\mu_1} \varphi^{(\mu_2)}(t) \tau^{\nu_1} \psi^{(\nu_2)}(\tau) \, dt \, d\tau
\]

where \( |\alpha| = (s) \). Consequently, for \( j + |\gamma| \geq (s) + 1 \) we obtain

\[
\int_{\mathbb{R}} \int_{0}^{1} r^{p(\beta-s+j+|\gamma|)} |D_j^\alpha D_2^\beta (K g)(r, z)|^p \, dr \, dz
\]
\[ \leq c \sum_{|\alpha|=s} \int \int \int \frac{r^p}{r^p - \sigma} \int_0^1 \frac{d\tau}{d\tau} \left( \frac{\sigma r}{\sigma r + 1} \right)^{p/2} dr dz \]

\[ = c \sum_{|\alpha|=s} \int \int \int \left( \frac{r^p}{r^p - \sigma} \int_0^1 \frac{d\tau}{d\tau} \left( \frac{\sigma r}{\sigma r + 1} \right)^{p/2} dr dz \right) \]

\[ \leq c \sum_{|\alpha|=s} \int \int \int \left( \frac{r^p}{r^p - \sigma} \int_0^1 \frac{d\tau}{d\tau} \left( \frac{\sigma r}{\sigma r + 1} \right)^{p/2} dr dz \right) \]

\[ \leq c \|g\|_{W^{s,p}_{p,\beta}}(\mathbb{R}^n) \]

Analogously, the inequality (1.14) can be proved. Furthermore, it can be shown that \( r^\beta D^2 K g \in L_p(\mathbb{R}^n \times \mathbb{R}^{n-2}) \). Together with (1.13) and (1.14) this implies \( K g \in \bigcap_{\nu=1}^{\infty} W^{s+\nu}_{p,\beta+\nu-s-\nu}(\mathbb{R}^n \times \mathbb{R}^{n-2}) \).

**Remark 2.** If we interpret \( K g \) as a function on \( D \) (i.e., we define \( v(y, z) = (K g)(|y|, z) = (K g)(r, z) \)) then

\[ K g \in \bigcap_{\nu=1}^{\infty} W^{s+\nu}_{p,\beta+\nu-s-\nu-1/p}(D) \quad \text{for} \quad g \in W^{s}_{p,\beta}(\mathbb{R}^n \times \mathbb{R}^{n-2}) \]

The following lemma is a consequence of the Hardy inequality

\[ \int_0^\infty r^{\beta-p}|f(r)|^p dr \leq \left( \frac{p}{|\beta - p + 1|} \right)^p \int_0^\infty r^\beta|f(r)|^p dr, \]

which is satisfied if \( f(0) = 0, \beta < p - 1 \) or if \( f(\infty) = 0, \beta > p - 1 \).

**Lemma 5.** Let \( u \in W^{0}_{p,\beta}(D) \) (\( \beta > -2/p \)) be such that \( \nabla u \in W^{0}_{p,\beta'}(D) \) where \( \beta' > 1 - 2/p \). Then \( u \in V^{1}_{p,\beta}(D) \).

Now we can prove an imbedding analogous to Theorem 1 for the spaces \( W^{s}_{p,\beta}(D) \).

**Theorem 3.** Let \( s' \geq s, \beta' - s' = \beta - s \) and \( \beta > -2/p \). Then \( W^{s'}_{p,\beta'}(D) \) is continuously imbedded in \( W^{s}_{p,\beta}(D) \).

**Proof.** Without loss of generality we can assume that \( l \leq s < s' \leq l + 1 \) where \( l \) is a non-negative integer. We consider the following cases: (a) \( s = l, s' = l + \sigma \) \((0 < \sigma < 1)\), (b) \( s = l + \sigma, s' = l + \sigma' \) \((0 < \sigma < \sigma' < 1)\), (c) \( s = l + \sigma, s' = l + 1 \) \((0 < \sigma < 1)\).

(a) Let \( u \in W^{s'}_{p,\beta'}(D) \) and \( v_\alpha = D^\alpha u \left( |\alpha| = l \right) \). By Lemma 5 and (1.14) we have \( K_\alpha^o \in W^{1}_{p,\beta+1-\sigma}(D) = W^{1}_{p,\beta+1}(D) \subset W^{0}_{p,\beta}(D) \) and \( v_\alpha - K_\alpha^o \in W^{0}_{p,\beta}(D) \). Hence, \( v_\alpha = D^\alpha u \in W^{0}_{p,\beta}(D) \) and by Lemma 5 we obtain \( D^\alpha u \in W^{0}_{p,\beta}(D) \) for \( |\alpha| \leq l \).
(b) Analogously to (a) we obtain $K\tilde{v}_\alpha \in W^{1}_{p,\beta'+1-\sigma}(D) = W^1_{p,\beta+1-\sigma}(D)$ for $|\alpha| = l$. Since $K\tilde{v}_\alpha = 0$ for $r = |\gamma| > 2$ this implies $K\tilde{v}_\alpha \in W^1_{p,\beta+1}(D) \subset W^0_{p,\beta}(D)$ for $|\alpha| = l$. Furthermore, by (1.14), $v_\alpha - K\tilde{v}_\alpha \in W^0_{p,\beta-\sigma}(D) \cap W^0_{p,\beta}(D) \subset W^0_{p,\beta}(D)$ and analogously to (a) it follows that $D^\alpha u \in W^0_{p,\beta}(D)$ for $|\alpha| \leq l$. Using the fact that $\beta - \beta' = \sigma - \sigma'$ we get
\[
\int \int_{D \times D} \frac{|y|^{p\beta'}|D^\alpha u(x) - D^\alpha u(x')|^p}{|x-x'|^{n+p\sigma}} \frac{dx \, dx'}{|x-x'|^{n+p\sigma}} \leq c \int \int_{D \times D} \frac{|y|^{p\beta'}|D^\alpha u(x) - D^\alpha u(x')|^p}{|x-x'|^{n+p\sigma}} \frac{dx \, dx'}{|x-x'|^{n+p\sigma}}
\]
for $|\alpha| = l$. Hence, $u \in W^s_{p,\beta}(D)$.

(c) By Lemma 5 the space $W^{l+1}_{p,\beta}(D)$ is imbedded in $W^l_{p,\beta}(D)$. Furthermore, the equation
\[
(D^\alpha u)(x) - (D^\alpha u)(x') = (x-x') \int_0^1 (\nabla D^\alpha u)(x + t(x' - x)) \, dt
\]
yields
\[
\int \int_{D \times D} \frac{|y|^{p\beta'}|D^\alpha u(x) - D^\alpha u(x')|^p}{|x-x'|^{n+p\sigma}} \frac{dx \, dx'}{|x-x'|^{n+p\sigma}} \leq c \int \int_{D} \frac{|y|^{p\beta'} \sum_{|\alpha'| = l+1} |(D^\alpha u(x))|^p}{dx}
\]
for $|\alpha| = l$. This implies $W^{l+1}_{p,\beta}(D) \subset W^s_{p,\beta}(D)$. 

**Corollary 1.** If $\beta > s - 2/p$ then
\[
W^s_{p,\beta}(D) \subset W^s_{p,\beta-s/\alpha}(D) \subset W^{s-1}_{p,\beta-s/\alpha}(D) \subset \ldots \subset W^0_{p,\beta-s}(D),
\]
i.e. $W^s_{p,\beta}(D) \subset V^s_{p,\beta}(D)$.

**Remark 3.** Analogously to Theorem 3 it can be proved that $W^s_{p,\beta'}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ is continuously imbedded in $W^s_{p,\beta}(D)$ if $s' \geq s$, $\beta' - s' = \beta - s$, $\beta > -1/p$.

**1.3. Traces of functions from $W^s_{p,\beta}(D)$ on $M$.** In [6] (Lemma 1.1) it has been proved that the trace of a function $u \in W^1_{p,\beta}(D)$ belongs to the Besov space $B^s_{p,\beta-1/2}(M)$ for $l$ an integer, $l > \beta + 2/p > 0$. Here the norm in $B^s_{p,\beta}(M)$ is defined by
\[
\|u\|_{B^s_{p,\beta}(M)} = \left( \|u\|^p_{L^p(M)} + \int_M \int_M |\Delta^k u(z)|^p \frac{dz \, d\zeta}{|\zeta|^{n-2+p\sigma}} \right)^{1/p}
\]
where $\Delta^k u(z) = \sum_{|\nu| = 0}^k (-1)^\nu \binom{k}{\nu} u(z + \nu \zeta)$.
Theorem 4. Let \( u \in W^{s}_{p,\beta}(D) \), \( s > \beta + 2/p > 0 \). Then the trace of \( u \) on \( M \) exists and belongs to the Besov space \( B^{s-\beta-2/p}_{p}(M) \).

Proof. Let \( \{u_n\} \) be a sequence of functions from \( C^\infty_0(D) \) which converges to \( u \) in \( W^{s}_{p,\beta}(D) \). Furthermore, let \( \{f_n\} \) be the sequence of the traces of \( u_n \) on \( M \), i.e. \( f_n(z) = u_n(0,0,z) = \tilde{u}_n(z) \) (the function \( \tilde{u}_n \) is defined by (1.11)). Since the functions \( \tilde{u}_n \) can be interpreted as functions from \( W^{1}_{p-1/p+1/2}(\mathbb{R}^n \times \mathbb{R}^n) \) Lemma 4 yields

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} r^{-1-p/2} |\tilde{u}_n(r,z) - (K\tilde{u}_n)(r,z)|^p \, dr \, d\zeta < \infty .
\]

Consequently, \( \tilde{u}_n(0,0,z) - (K\tilde{u}_n)(0,0,z) \) (i.e. \( f_n \)) is the trace of \( K\tilde{u}_n \) on \( M \). We denote the trace of \( K\tilde{u} \) on \( M \) by \( f \). Then by Lemma 1.1 of [6] and Lemmas 3 and 4 we get

\[
\|f - f_n\|_{B^{s-\beta-2/p}_{p}(M)} \leq c\|\tilde{K}\tilde{u} - K\tilde{u}_n\|_{W^{s+1}_{p,\beta-\alpha+1}(\mathbb{R}^n)} \leq c\|u - u_n\|_{W^{s}_{p,\beta}(\mathbb{R}^n)} .
\]

Hence, \( \{f_n\} \) converges to \( f \) in \( B^{s-\beta-2/p}_{p}(M) \).

Conversely, it can be proved that every \( f \in B^{s-\beta-2/p}_{p}(M) \) \( (s > \beta + 2/p > 0) \) can be extended to a function \( v \in W^{s}_{p,\beta}(\mathbb{R}^n) \). We define the extension operator \( \mathfrak{R} \) as follows:

\[
\mathfrak{R}(f)(r,z) = \chi(r) \int_{\mathbb{R}^n} g(z + rt)\psi(\tau_1)\ldots\psi(\tau_n-2) \, d\tau
\]

where \( \psi \in C^\infty_0(\mathbb{R}) \) satisfies (1.12). By Lemma 1.2 of [6] the operator \( \mathfrak{R} \) defines a continuous map from \( B^{s}_{p}(M) \) into \( W^{s}_{p,\alpha-2/p}(\mathbb{R}^n) \) for \( \alpha > 0 \), \( \alpha \) not an integer, \( s \) an integer, \( s + (\alpha - 2/p) > -2 \). From Theorem 3 it follows that this is also true for real \( s \), \( s + (\alpha - 2/p) > -2 \).

1.4. Connection between \( V^{s}_{p,\beta}(D) \) and \( W^{s}_{p,\beta}(D) \). In [6], [7] it has been proved that every \( u \in \mathcal{V}^{s}_{p,\beta}(D) \) is the sum of a “quasi-polynomial” and a function from \( \mathcal{V}^{s}_{p,\beta}(D) \) if \( s \) is a non-negative integer and \( \beta + 2/p \) is not an integer. We will give a similar connection between \( \mathcal{V}^{s}_{p,\beta}(D) \) and \( W^{s}_{p,\beta}(D) \) without any restrictions on \( s \) and \( \beta \).

Let \( u \in W^{s}_{p,\beta}(D) \) \( (\beta > -2/p) \). We denote the derivatives \( \partial_{y_1}^i \partial_{y_2}^j u \) \( (i + j \leq \langle s \rangle) \) by \( u_{ij} \). By the properties of the operator \( K \) the following lemma can be easily proved (see [9]).

Lemma 6. If \( u \in W^{s}_{p,\beta}(D) \) then the following inequality holds for \( i + j + \langle \alpha \rangle \leq \langle s \rangle \):

\[
\int_{\mathcal{D}} r^{p(\beta + i + j + \langle \alpha \rangle)} \left| \partial_{y_1}^i \partial_{y_2}^j \partial_{z}^\alpha u - \sum_{\mu + \nu = 0}^{(s) - i - j - \langle \alpha \rangle} (\partial_{z}^\mu K\tilde{u}_{i + \mu, j + \nu} \tilde{u}_{s - i - j - \langle \alpha \rangle} \frac{y_1^\mu y_2^\nu}{\mu! \nu!} \right|^p \, dx \leq c\|u\|^{p}_{W^{s}_{p,\beta}(\mathbb{R}^n)} .
\]
Lemma 6 implies the following corollary.

**Corollary 2.** If \( u \in W^s_{p,\beta}(\mathcal{D}) \) then
\[
u - \sum_{i+j=0}^{(s)} (Ku_{ij})(r, z) \frac{y_i^j y_2^j}{ij!} \in V^s_{p,\beta}(\mathcal{D}).
\]

**Proof.** By Lemma 4 we have \( \partial_y^\mu \partial_z^\nu \partial^\alpha (Ku_{ij}) \in V^0_{p,\beta-s+i+j+\mu+\nu+|\alpha|}(\mathcal{D}) \) if \( \mu + \nu \geq 1 \) or \( |\alpha| \geq s-i-j \). Therefore, from Lemma 6 it follows that
\[
\int \partial_y^\mu \partial_z^\nu \partial^\alpha \left( u - \sum_{i+j=0}^{(s)} (Ku_{ij}) \frac{y_i^j y_2^j}{ij!} \right)^p dx < \infty
\]
for \( \mu + \nu + |\alpha| \leq \langle s \rangle \), which yields the desired conclusion. 

By Corollary 2 for every \( u \in W^s_{p,\beta}(\mathcal{D}) \) there exist \( v_{ij} \in W^s_{p,\beta+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \) such that
\[
u - \sum_{i+j=0}^{(s)} (Kv_{ij})(r, z) \frac{y_i^j y_2^j}{ij!} \in V^s_{p,\beta}(\mathcal{D}).
\]

Now we investigate the question whether the \( v_{ij} \) are (in some sense) uniquely determined by \( u \). Similarly to [8] we introduce the following equivalence relation in \( W^s_{p,\beta+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \) \( (s > 0, \beta > -2/p) \):
\[
(1.16) \quad f \overset{s, p}{\sim} g \Leftrightarrow \int_{\mathbb{R}^+ \times \mathbb{R}^{n-2}} r^{p(\beta-s)+1} |(Kf)(r, z) - (Kg)(r, z)|^p dr dz < \infty.
\]

Another characterization is the following:

\[
(1.17) \quad f \overset{s, p}{\sim} g \Leftrightarrow K(f - g) \in \bigcap_{\nu=0}^{\infty} V^\nu_{p,\beta-s+\nu+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}).
\]

**Proof.** If \( f, g \in W^s_{p,\beta+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \) and \( f \overset{s, p}{\sim} g \) then by Lemma 4
\[
K(f - g) \in \bigcap_{\nu=0}^{\infty} W^\nu_{p,\beta-s+\nu+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \cap V^0_{p,\beta-s+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}),
\]
and (1.17) follows from \( W^s_{p,\beta}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \cap V^0_{p,\beta-s}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \subset V^s_{p,\beta}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \) (cf. Remark 1).

**Remark 4.** Let \( f, g \in W^s_{p,\beta+1/p}(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \).

(a) If \( \beta - s < -2/p \) then \( f \overset{s, p}{\sim} g \Leftrightarrow f|_M = g|_M \).

(b) If \( \beta - s = -2/p \) then
\[
f \overset{s, p}{\sim} g \Leftrightarrow \int_{\mathbb{R}^+ \times \mathbb{R}^{n-2}} r^{p(\beta-s)+1} |f(r, z) - g(r, z)|^p dr dz < \infty.
\]
(c) If $\beta - s > -2/p$ then $f^{\beta - s,p} \sim g^{\beta - s,p}$. 

Proof. (a) If $\beta - s < -2/p$ then the trace of $f-g$ on $M$ exists and coincides with that of $K(f-g)$. The Hardy inequality and Lemma 4 imply

$$\int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} r^p(\beta - s + 1)|K(f-g) - (f-g)|_M^p \, dr \, dz \\
\leq c \int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} r^p(\beta - s + 1) \left| \frac{\partial}{\partial r} K(f-g) \right|^p \, dr \, dz \leq c \|f-g\|_{W^{s,p}_{p,\beta+1/p}}^p (\mathbb{R}_+ \times \mathbb{R}^{n-2}).$$

Hence, from (1.16) it follows that

$$\int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} r^p(\beta - s + 1)(f-g)|_M^p \, dr \, dz < \infty,$$

i.e. $f|_M = g|_M$. Conversely, if $f|_M = g|_M$ then (1.16) follows from (1.18).

(b) If $\beta - s = -2/p$ then $W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \subset W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ (see Remark 3) and Lemma 4 yields

$$\int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} r^p(\beta - s + 1)|Kf - f|^p \, dr \, dz \\
\leq c \|f\|^p_{W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})} \leq c \|f\|^p_{W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})}.$$

This proves the assertion.

(c) If $\beta - s > -2/p$ then $W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ is imbedded in $V^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ (cf. Corollary 1, Remark 3). Consequently, (1.16) is satisfied for all $f, g \in W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$.

Lemma 7. Let $f \in W^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$, $\beta - s \leq -1 - 2/p$. Then $f^{\beta - s,p} \sim f$ iff $\partial \beta/\partial z_i f^{\beta - s+1,p} \sim 0$ for some $i \in \{1, \ldots, n-2\}$.

Proof. 1) If $f^{\beta - s,p} \sim 0$ then $Kf \in V^s_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ and

$$Kf = z_i^{\beta - s+1,p} \sim 0,$$

i.e. $\partial f/\partial z_i f^{\beta - s+1,p} \sim 0$.

2) Let $\partial f/\partial z_1 f^{\beta - s+1,p} \sim 0$. If $\beta - s < -1 - 2/p$ this means $\partial f/\partial z_1 f|_M = \partial f/\partial z_1 |_M = 0$ where $f|_M = B^p_{\beta - s+2}(M)$. Hence, $f|_M = 0$, i.e. $f^{\beta - s,p} \sim 0$. If $\beta - s = -1 - 2/p$ we first assume that $f(r,z) = 0$ for $r > 1$ and $|z| > 1$. Using the Hardy inequality we get

$$\int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} r^{-1}|f(0,z)|^p \, dr \, dz \leq c \int_{\mathbb{R}_+ \times \mathbb{R}^{n-2}} (r^{-1}|f(r,z)|^p + r^{-2}|f(r,z) - f(0,z)|^p) \, dr \, dz$$
Hence, \( f|_M = 0 \), i.e. \( f \stackrel{\beta-s,p}{\sim} 0 \). If \( f \) has an arbitrary support then we show that \( (\varphi f)|_M = 0 \) for every cut-off function \( \varphi \) with compact support. 

**Theorem 5.** Let \( u, v_{ij} \) (\( i + j \leq \langle s \rangle \)) be arbitrary functions from \( W^{s}_{p,\beta}(\mathcal{D}) \) and \( W^{s+\mu+p}_{p,\beta+1,p}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \), respectively. Then

\[
(1.19) \quad u - \sum_{i+j=0}^{\langle s \rangle} \frac{1}{i!j!} (K v_{ij})(r,z) y_i^j y_2^j \in V^{s}_{p,\beta}(\mathcal{D})
\]

if and only if \( v_{ii} \stackrel{\beta-\nu+i+j,p}{\sim} \hat{u}_{ii} \) for \( i + j \leq \langle s \rangle \).

**Proof.** 1) Let \( v_{ij} \stackrel{\beta-s+i+j,p}{\sim} \hat{u}_{ij} \). Then \( \mathcal{K}(v_{ij} - \hat{u}_{ij}) \in V^{(s)+1}_{p,\beta-s+(i+j)+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \) for \( i + j \leq \langle s \rangle \). Interpreting \( \mathcal{K}(v_{ij} - \hat{u}_{ij}) \) as functions on \( \mathcal{D} \) we get \( \mathcal{K}(v_{ij} - \hat{u}_{ij}) \in V^{(s)+1}_{p,\beta-s+(i+j)+1}(\mathcal{D}) \) and \( \mathcal{K}(v_{ij} - \hat{u}_{ij}) y_i^j y_2^j \in V^{(s)+1}_{p,\beta-s+(i+j)+1}(\mathcal{D}) \subset V^{s}_{p,\beta}(\mathcal{D}) \). Using Corollary 2 we obtain (1.19).

2) Assume that (1.19) is satisfied. Then by Corollary 2

\[
(1.20) \quad \int_{\mathcal{D}} r^{p(\beta-s+\mu+p)} D^\nu y_1 D^\nu y_2 \sum_{i+j=0}^{\langle s \rangle} \mathcal{K}(\hat{u}_{ij} - v_{ij}) \frac{y_i^j y_2^j}{i!j!} \, dx < \infty .
\]

Since \( D^\nu y_1 \mathcal{K}(\hat{u}_{ij} - v_{ij}) \in V^0_{p,\beta-s+i+j+|\nu|}(\mathcal{D}) \) for \( |\nu| \geq 1 \) (see Lemma 4), (1.20) yields

\[
(1.21) \quad \int_{\mathcal{D}} r^{p(\beta-s+\mu+p)} \left( \sum_{i+j=0}^{\langle s \rangle} \mathcal{K}(\hat{u}_{i+\mu,j+\nu} - v_{i+\mu,j+\nu}) \frac{y_i^j y_2^j}{i!j!} \right) \, dx < \infty .
\]

For \( \mu + \nu = \langle s \rangle \), (1.21) implies \( \mathcal{K}(\hat{u}_{\mu\nu} - v_{\mu\nu}) \in V^0_{p,\beta-s+\mu+\nu}(\mathcal{D}) \), i.e. \( \hat{u}_{\mu\nu} \stackrel{\beta-s+i+j,p}{\sim} v_{\mu\nu} \). Then from (1.21) it follows that

\[
\int_{\mathcal{D}} r^{p(\beta-s+\mu+p)} \left( \sum_{i+j=0}^{\langle s \rangle-1} \mathcal{K}(\hat{u}_{i+\mu,j+\nu} - v_{i+\mu,j+\nu}) \frac{y_i^j y_2^j}{i!j!} \right) \, dx < \infty .
\]

For \( \mu + \nu = \langle s \rangle - 1 \) this yields \( \hat{u}_{\mu\nu} \stackrel{\beta-s+i+j,p}{\sim} v_{\mu\nu} \). Analogously, by induction on \( \mu + \nu \) we show that \( \hat{u}_{\mu\nu} \stackrel{\beta-s+i+j,p}{\sim} v_{\mu\nu} \) for \( \mu + \nu \leq \langle s \rangle - 2 \).
1.5. **Connection between the spaces of traces.** We denote by $B_{p,\beta}^{s-1/p}(\Gamma^\pm)$ ($s > 1/p$, $\beta > -2/p$) the space of the traces of functions from $W_{p,\beta}^s(D)$ on the sides $\Gamma^+$ and $\Gamma^-$, respectively, provided with the norm
\[
\|u\|_{B_{p,\beta}^{s-1/p}(\Gamma^\pm)} = \inf\{\|v\|_{W_{p,\beta}^s(D)} : v \in W_{p,\beta}^s(D), \ v|_{\Gamma^\pm} = u\}.
\]
By Corollary 1, $B_{p,\beta}^{s-1/p}(\Gamma^\pm) \subset \tilde{B}_{p,\beta}^{s-1/p}(\Gamma^\pm)$ for $\beta > s - 2/p$. Let $u \in B_{p,\beta}^{s-1/p}(\Gamma^+)$ and let $v \in W_{p,\beta}(D)$ be an extension of $u$. Then by Theorem 5, $u$ has the representation
\[
u = v|_{\Gamma^+} = \left( \sum_{i+j \leq \langle s \rangle} (\mathcal{K}v)_{ij} \frac{y_i y_j}{i! j!} + v' \right)_{|\Gamma^+} = \sum_{k=0}^{\langle s \rangle} (\mathcal{K}v_k)r_k^k + u'
\]
where
\[
v_k = \sum_{i=0}^{k} \binom{k}{i} \left( \cos \frac{\omega_0}{2} \right)^i \left( \sin \frac{\omega_0}{2} \right)^{k-i} v_{k-i} \in W_{p,\beta+1/p}^{s-k}(\mathbb{R}^+ \times \mathbb{R}^{n-2})
\]
and $u' \in \tilde{B}_{p,\beta}^{s-1/p}(\Gamma^+)$.  

**Lemma 8.** Let $v_k \in W_{p,\beta+1/p}^{s-k}(\mathbb{R}^+ \times \mathbb{R}^{n-2})$ ($k = 0, 1, \ldots, \langle s \rangle$). Then
\[
\sum_{k=0}^{\langle s \rangle} (\mathcal{K}v_k)r_k^k \in V_{p,\beta-s+1/p}^0(\mathbb{R}^+ \times \mathbb{R}^{n-2}) \iff v_k \overset{\beta-s+k,p}{\sim} 0.
\]
**Proof.** 1) If $v_k \overset{\beta-s+k,p}{\sim} 0$ then $\mathcal{K}v_k \in V_{p,\beta-s+k+1/p}^0(\mathbb{R}^+ \times \mathbb{R}^{n-2})$ and $(\mathcal{K}v_k)r_k^k \in V_{p,\beta-s+1/p}^0(\mathbb{R}^+ \times \mathbb{R}^{n-2})$.

2) If $\sum_{k=0}^{\langle s \rangle} (\mathcal{K}v_k)r_k^k/k! \in V_{p,\beta-s+1/p}^0(\mathbb{R}^+ \times \mathbb{R}^{n-2})$ then from the properties of $\mathcal{K}$ (see Lemma 4) it follows that $\sum_{k=0}^{\langle s \rangle} (\mathcal{K}v_k)r_k^k/k! \in V_{p,\beta-s+1/p}^0(\mathbb{R}^+ \times \mathbb{R}^{n-2})$, i.e.
\[
\int_{\mathbb{R}^+ \times \mathbb{R}^{n-2}} r^{p(\beta-s+\mu)+1} \left| D^p \sum_{k=0}^{\langle s \rangle} (\mathcal{K}v_k)r_k^k/k! \right|^p dz < \infty
\]
for $\mu = 0, 1, \ldots, \langle s \rangle$. Analogously to the proof of Theorem 5, this implies that $v_k \overset{\beta-s+k,p}{\sim} 0$. \qed

As a consequence of Lemma 8 we obtain the following theorem.

**Theorem 6.** Let $u \in B_{p,\beta}^{s-1/p}(\Gamma^+)$. Then there exist $v_k \in W_{p,\beta+1/p}^{s-k}(\mathbb{R}^+ \times \mathbb{R}^{n-2})$ ($k = 0, 1, \ldots, \langle s \rangle$) such that
\[
u - \sum_{k=0}^{\langle s \rangle} (\mathcal{K}v_k)(r,z)r_k^k \in \tilde{B}_{p,\beta}^{s-1/p}(\Gamma^+).
\]
The functions $v_k$ in (1.22) are uniquely determined in the following sense: if

$$w_k \in W^{s-k}_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \ (k = 0, 1, \ldots, \langle s \rangle)$$

then

$$u - \sum_{k=0}^{\langle s \rangle} (Kw_k)(r, z) \frac{r^k}{k!} \in \tilde{B}^{s-1/p}_{p,\beta}(I^+) \iff w_k \sim v_k.$$

In particular, $v_k \sim \partial^k u$ for $k \leq \langle s-1/p \rangle$ and $v_k \sim 0$ for $k > \lfloor s-\beta-2/p \rfloor$ (here $[x]$ denotes the largest integer less than or equal to $x$, i.e. $[x] = -(-x) - 1$).

**Proof.** It remains to show that $v_k \sim \partial^k u$ for $k \leq \langle s-1/p \rangle$. Differentiating (1.22) we get

$$\partial^k u \left( u - \sum_{k=0}^{\langle s \rangle} (Kw_k) \frac{r^k}{k!} \right) \in \tilde{B}^{s-\nu-1/p}_{p,\beta}(I^+).$$

Since $\partial^j KW_k \in V^{s-k-j}_{p,\beta+1/p}(I^+)$ for $j \geq 1$ (see Lemma 4) this implies

$$(1.23) \quad \partial^k u \left( u - \sum_{k=0}^{\langle s \rangle} (Kw_k) \frac{r^k}{k!} \right) \in \tilde{B}^{s-\nu-1/p}_{p,\beta}(I^+).$$

It can be easily verified that $K(r^k v) \in V^{s-k-j}_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ for $v \in W^{s}_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ and positive integers $k$. Then (1.23) yields that $K(\partial^k u) = K Kw_k \in V^{s-\nu}_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$, i.e. $\partial^k u \sim v_k \sim v_k$. $\blacksquare$

The following lemma gives a connection between the spaces $B^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ and $W^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$.

**Lemma 9.** Let $s-1/p$ be non-integer, $s > 1/p$ and $\beta > -1/p$. Then

$$B^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) = W^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2}).$$

**Proof.** 1) If $u \in B^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ then there exist $v_k \in W^{s-k}_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ such that

$$u = \sum_{k=0}^{\langle s \rangle} (Kw_k) \frac{r^k}{k!} \in B^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) = W^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2}).$$

Since $(Kw_k)r^k \in W^{s-k-j}_{p,\beta+1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$ we get $u \in W^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$.

2) Let $u \in W^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2})$. Then analogously to Theorem 5 it can be shown that

$$v = u - \sum_{k=0}^{\langle s-1/p \rangle} (Kw_k) \frac{r^k}{k!} \in V^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \sim B^{s-1/p}_{p,\beta}(\mathbb{R}_+ \times \mathbb{R}^{n-2}).$$
where \( u_k = \partial^k u \in W^{s-k-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \). Without loss of generality we can assume that \( \Gamma^+ \) is the half-plane \( \Gamma^+ = \{ x = (y, z) : y_1 > 0, \, y_2 = 0, \, z \in \mathbb{R}^{n-2} \} \) which can be identified with \( \mathbb{R}_+ \times \mathbb{R}^{n-2} \). If \( \nu' \in V_{p,\beta}(D) \) is an extension of \( u \) then

\[
u' = \nu' + \sum_{k=0}^{(s-1)/p} (Ku_k) \frac{y_k}{k!}\]

is an extension of \( u \) which lies in \( W^{s,p}(D) \). Hence, \( u \in B^{s-1/p}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \).

2. Applications to boundary value problems

2.1. Compatibility conditions for boundary data on \( \Gamma^\pm \). For the spaces \( V^{s}_{p,\beta}(D) \) the following lemma has been proved (see [4], [9]).

**Lemma 10.** Let \( B^\pm_k (k = 1, \ldots, p^\pm) \) be homogeneous differential operators with constant coefficients of order \( m^\pm \). Assume that \( \{B^\pm_k\} \) and \( \{B^\pm_k\} \) are normal on \( \Gamma^+ \) and \( \Gamma^- \), respectively. Then for all \( g^\pm_k \in B^\pm-m^\pm_{k-1/p}(\Gamma^\pm) \) there exists \( u \in V^{s}_{p,\beta}(D) \) such that \( B^\pm_k u = g^\pm_k \) on \( \Gamma^\pm \) for \( k = 1, \ldots, p^\pm \).

We will investigate conditions on \( g^\pm_k (g^\pm_k \in B^\pm-m^\pm_{k-1/p}(\Gamma^\pm)) \) under which there exists \( u \in W^{s}_p(D) \) satisfying the boundary conditions

\[
B^\pm_k u = \sum_{\mu^s + \nu = m^\pm_k} b^\pm_{k,\mu,\nu} D^\mu u D^\nu u = g^\pm_k \quad \text{on} \quad \Gamma^\pm
\]

\((k = 1, \ldots, p^\pm)\). Here we restrict ourselves to the case \( n = 3 \).

By Theorem 6, \( B^\pm_k u |_{\Gamma^\pm} = g^\pm_k \) (\( u \in W^{s}_p(D) \), \( g^\pm_k \in B^\pm-m^\pm_{k-1/p}(\Gamma^\pm) \)) belongs to \( B^\pm-m^\pm_{k-1/p}(\Gamma^\pm) \) iff

\[
\partial^j(B^\pm_k u |_{\Gamma^\pm} = g^\pm_k) \simeq_{j,p} 0
\]

for \( j = 0, 1, \ldots, [s - \beta - 2/p] - m^\pm \). Using the representation

\[
u = \sum_{i+j=0} (Ku) \frac{y^i y^j}{i! j!} + u'
\]

where \( u \in W^{s-i-j}_p(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \), \( u' \in V^{s}_{p,\beta}(D) \), \( D^\mu D^\nu K u \in V^{s-i-j-\mu-\nu}_p(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \) if \( \mu \geq 1 \) or \( \nu \geq s - \beta - 2/p - i - j \) (see Theorem 5) we get the following equivalences:

\[
\sum_{\mu^s + \nu = m^\pm_k} (-i)^{\mu^s + \nu} b^\pm_{k,\mu,\nu} \sum_{\sigma=0}^{j} \binom{j}{\sigma} \left( \cos \frac{\omega_0}{2} \right)^\sigma \left( \pm \sin \frac{\omega_0}{2} \right)^{j-\sigma} \\
\times D^\nu u_{\mu+\nu+j-\sigma} \simeq_{\beta-s+m^\pm_{k-1/p}+j,p} \partial^j g^\pm_k
\]
(j = 0, 1, ... , [s − β − 2/p] − m_k^±). By Lemma 7 the system (2.3) is equivalent to

\[\sum_{\mu + \nu + \gamma = m_k^\pm} (-i)^{\mu + \nu} b_{k, \mu \nu}^\pm \sum_{\sigma = 0}^j \left( \begin{array}{c} j \\ \sigma \end{array} \right) \left( \cos \frac{\omega_0}{2} \right)^\sigma \left( \pm \sin \frac{\omega_0}{2} \right)^{j-\sigma} \times D_z^{[s-\beta-2/p]-m_k^\pm-j} g_{\mu+\nu+\gamma+j-\sigma} \sim (2)

\]

\[D_z^{[s-\beta-2/p]-m_k^\pm-j} g_k^\pm \quad (j = 0, 1, ... , [s − β − 2/p] − m_k^\pm). \] (2.4) can be written as a linear system of equations of the form

\[\sum_{\mu + \nu + \gamma = j} c_{\mu, \nu, \gamma, k}^\pm v_{\mu \nu}^{\gamma} \sim D_z^{[s-\beta-2/p]-m_k^\pm-j} \partial_i g_k^\pm \quad (j = 0, 1, ... , [s − β − 2/p] − m_k^\pm; k = 1, ... , p^\pm) \] for the functions v_{\mu \nu} = D_z^{s-\beta-2/p}-\mu-\nu a_{\mu \nu}. In general, the equations of (2.5) are not linearly independent. Therefore, (2.5) is solvable iff the right-hand sides D_z^{[s-\beta-2/p]-m_k^\pm-j} \partial_i g_k^\pm of (2.5) satisfy some linear compatibility conditions

\[\sum_{k=1}^{p^\pm} \sum_{j=0}^{[s-\beta-2/p]-m_k^\pm} d_{k, j, \tau}^\pm D_z^{[s-\beta-2/p]-m_k^\pm-j} \partial_i g_k^\pm = 0 \quad (\tau = 1, ... , t). \]

The number t of the compatibility conditions depends on the rank of the coefficient matrix of (2.5). If s − β − 2/p is not an integer then (2.6) is equivalent to the equations

\[\sum_{k=1}^{p^\pm} \sum_{j=0}^{[s-\beta-2/p]-m_k^\pm} d_{k, j, \tau}^\pm D_z^{[s-\beta-2/p]-m_k^\pm-j} \partial_i g_k^\pm |_{M} = 0 \quad (\tau = 1, ... , t). \]

Theorem 7. Suppose that g_k^\pm ∈ B_{p, \beta}^{s-m_k^\pm-1/p}(\Gamma^\pm) satisfy the compatibility conditions (2.6). Then there exists u ∈ W_{p, \beta}(\D) such that B_{k, \beta}^{s} u = g_k^\pm on \Gamma^\pm for k = 1, ... , p^\pm. If s − β − 2/p is not an integer then

\[\|u\|_{W_{p, \beta}(\D)} \leq c \sum_{k=1}^{p^\pm} \|g_k^\pm\|_{B_{p, \beta}^{s-m_k^\pm-1/p}(\Gamma^\pm)}. \]

If s − β − 2/p is an integer then

\[\|u\|_{W_{p, \beta}(\D)} \leq c \left( \sum_{k=1}^{p^\pm} \sum_{k=1}^{p^\pm} \|g_k^\pm\|_{B_{p, \beta}^{s-m_k^\pm-1/p}(\Gamma^\pm)} + \sum_{\tau=1}^{t} \left( \int_0^1 \int_{\R} r^{-1} \right) \sum_{k=1}^{p^\pm} \sum_{j=0}^{[s-\beta-2/p]-m_k^\pm} d_{k, j, \tau}^\pm D_z^{[s-\beta-2/p]-m_k^\pm-j} \partial_i g_k^\pm dz \, dr \right)^{1/p}. \]
Proof. Under the conditions (2.6) the system (2.5) has a solution $V = (v_{\mu \nu})_{\mu + \nu \leq s - \beta + 2/p}$ where the functions $v_{\mu \nu} = D_z^{s - \beta - 2/p - \mu - \nu} a_{\mu \nu}$ are linear combinations of the right-hand sides $D_z^{s - \beta - 2/p - m_k^+} \partial_\tau g_k^\pm$. Furthermore, the $v_{\mu \nu}$ satisfy

$$\sum_{\mu + \nu = j} v_{\mu, \nu, j, k}^\pm v_{\mu \nu} = D_z^{s - \beta - 2/p - m_k^+ - j} \partial_\tau g_k^\pm$$

$$+ \sum_{t=1}^t \sum_{\pm} \sum_{j', k'} d_{k', j'}^\pm D_z^{s - \beta - 2/p - m_k^+ - j'} \partial_\tau g_{k'}^\pm$$

with some complex numbers $\alpha_t$. If we set

$$w = \sum_{i+j=0} (\mathcal{K} \hat{u}_{ij}) \frac{y_i y_j}{i! j!}$$

then $B_k^\pm w - g_k^\pm \in B_z^{s - m_k^+ - 1/p}(\Gamma^\pm)$ and

$$\|B_k^\pm w - g_k^\pm\|_{B_z^{s - m_k^+ - 1/p}(\Gamma^\pm)} \leq c \left( \|g_k^\pm\|_{B_z^{s - m_k^+ - 1/p}(\Gamma^\pm)} \right)$$

$$+ \sum_{t=1}^t \int_0^1 \int_0^R r^{p(\beta - s + |s - \beta - 2/p|) + 1} \left| \sum_{j, k'} d_{j, k'}^\pm D_z^{s - \beta - 2/p - m_k^+ - j} \partial_\tau g_{k'}^\pm \right| dz \, dr.$$

If $s - \beta - 2/p$ is not an integer then there is a constant $c$ such that the integral in (2.8) is at most $c \sum_k \|g_k^\pm\|_{B_z^{s - m_k^+ - 1/p}(\Gamma^\pm)}$. By Lemma 9 there exists $w' \in V_{s, \beta}(\mathcal{D})$ satisfying the equations $B_k^\pm w' = g_k^\pm - B_k^\pm w$ on $\Gamma^\pm$ ($k = 1, \ldots, p^\pm$) and the estimate

$$\|w'\|_{V_{s, \beta}(\mathcal{D})} \leq c \sum_k \sum_{\pm} \|B_k^\pm w - g_k^\pm\|_{B_z^{s - m_k^+ - 1/p}(\Gamma^\pm)}.$$

Then $u = w + w'$ satisfies the conditions of the theorem. ■

Example. Let $B^+ u = u$, $B^- u = u$ and let $g^+ \in W_p^{1-1/p}(\Gamma^+)$, $g^- \in W_p^{1-1/p}(\Gamma^-)$ be given functions on $\Gamma^+$ and $\Gamma^-$, respectively.

(a) In the case $p < 2$ for arbitrary $g^+$, $g^-$ there exists $u \in W_p^1(\mathcal{D})$ such that

$$\|u\|_{W_p^1(\mathcal{D})} \leq c(\|g^+\|_{W_p^{1-1/p}(\Gamma^+)} + \|g^-\|_{W_p^{1-1/p}(\Gamma^-)}).$$

In this case the space $W_p^1(\mathcal{D}) = W_{p, 0}(\mathcal{D})$ is imbedded in $V_{p, 0}(\mathcal{D})$.

(b) In the case $p = 2$ the compatibility condition

$$\int_0^1 \int_0^R r^{-1} |g^+(r, z) - g^-(r, z)|^2 \, dz \, dr < \infty$$

The Sobolev-Slobodeckiǐ spaces are...
is necessary and sufficient for the existence of \( u \in W^1_2(D) \) satisfying \( u|_{\Gamma^\pm} = g^\pm \).

This \( u \) can be chosen such that

\[
\|u\|_{W^1_2(D)}^2 \leq c \left( \|g^+\|_{W^{1/2}_1(\Gamma^+)}^2 + \|g^-\|_{W^{1/2}_1(\Gamma^-)}^2 \right) + \int_0^1 \int |g^+(r, z) - g^-(r, z)|^2 r^{-1} dz \, dr 
\]

(c) If \( p > 2 \) then the condition \( g^+|_M = g^-|_M \) is necessary and sufficient for the existence of \( u \) satisfying \( u|_{\Gamma^\pm} = g^\pm \) and the inequality (2.9).

For inhomogeneous operators \( B_k^\pm \) with variable coefficients it is more difficult to give a general description of the compatibility conditions on the edge \( M \). Therefore, we will restrict ourselves to the case that the coefficients of \( B_k^\pm \) do not depend on the variable \( z \), i.e.

\[
B_k^\pm = \sum_{\mu + \nu + \gamma \leq m_k^\pm} b^\pm_{k,\mu\nu\gamma}(y_1, y_2) D^{\mu}_{y_1} D^{\nu}_{y_2} D^{\gamma}_z.
\]

Let \( u \) and \( g_k^\pm \) be arbitrary functions from \( W^{s_1}_{p,\beta}(D) \) and \( B_{p,\beta}^{s-m_k^\pm-1/p}(\Gamma^\pm) \), respectively. Then analogously to (2.3) it can be verified that \( B_k^\pm u|_{\Gamma^\pm} = g_k^\pm \in B_{p,\beta}^{s-m_k^\pm-1/p}(\Gamma^\pm) \) iff

\[
(2.10) \quad \sum_{\mu + \nu + \gamma \leq m_k^\pm} (-i)^{\mu+\nu} \sum_{\tau=0}^{j} \left( \left. \partial_r^{j-\tau} b^\pm_{k,\mu\nu\gamma}(r \cos \frac{\omega_0}{2}, \pm r \sin \frac{\omega_0}{2}) \right|_{r=0} \right) \times \sum_{\sigma=0}^{\tau} \left( \left. \partial_r^{\sigma} \right|_{r=0} \right)_2 \left( \frac{\omega_0}{2} \right) \left( \pm \frac{\sin \omega_0}{\gamma} \right)^{\tau-\sigma} D^\sigma_{\mu+\sigma,\nu+\tau-\sigma} u^{(\mu)}_{\mu+\sigma,\nu+\tau-\sigma} \left|_M \right.
\]

\[
(2,10) \quad (j = 0, 1, \ldots, [s - \beta - 2/p] - m_k^\pm). \text{ Here } u^{(\mu)}_{\mu} \text{ denotes the trace of } \partial_\mu u \text{ on } M. \text{ If } s - \beta - 2/p \text{ is an integer and } j = [s - \beta - 2/p] - m_k^\pm \text{ then (2.10) has to be interpreted in the generalized sense, i.e.}
\]

\[
(2.11) \quad \sum_{\mu + \nu + \gamma \leq m_k^\pm} (-i)^{\mu+\nu} \sum_{\tau=0}^{j} \left( \left. \partial_r^{j-\tau} b^\pm_{k,\mu\nu\gamma}(r \cos \frac{\omega_0}{2}, \pm r \sin \frac{\omega_0}{2}) \right|_{r=0} \right) \times \sum_{\sigma=0}^{\tau} \left( \left. \partial_r^{\sigma} \right|_{r=0} \right)_2 \left( \frac{\omega_0}{2} \right) \left( \pm \frac{\sin \omega_0}{\gamma} \right)^{\tau-\sigma} D^\sigma_{\mu+\sigma,\nu+\tau-\sigma} u^{(\mu)}_{\mu+\sigma,\nu+\tau-\sigma} \left|_M \right.
\]

By means of the equations (2.10) one can prove the following lemma which was used in [10].

**Lemma 11.** Let \( \chi \) be a smooth function on \( \partial D \) with compact support such that \( D^\alpha_{\gamma} \chi = 0 \) on \( M \) for \( |\alpha| \geq 1 \) (the last condition is satisfied, e.g., if \( \chi(y, z) = \chi_1(y)\chi_2(z) \) and \( \chi_1 = c \) in a neighbourhood of \( y = 0 \)). If \( u \in W^{s}_{p,\beta}(D) \) and \( B_k^\pm u = 0 \) on \( \Gamma^\pm \cap \text{supp } \chi \) then there exists \( v \in W^{s}_{p,\beta}(D) \) such that \( \text{supp } v \subset \text{supp } \chi \),
\[ \partial v / \partial z \in W^s_{p, \beta}(D), \quad B_k^\pm v = B_k^\pm (\chi u) \text{ on } \Gamma^\pm \text{ and} \]

\[ \| v \|_{W^s_{p, \beta}(D)} + \left\| \frac{\partial v}{\partial z} \right\|_{W^s_{p, \beta}(D)} \leq c \| u \|_{W^s_{p, \beta}(D)}. \]

In order to prove this we need the following two lemmas.

**Lemma 12.** Consider the differential equation

\[ (2.12) \quad p_1(D) u_1 + p_2(D) u_2 + \ldots + p_n(D) u_n = 0 \]

for the functions \( u_j \in W^{s+t_j}(\mathbb{R}) \) (\( t_j \) an integer, \( t_j \geq 0 \)) where \( p_j(D) \) are differential operators with ord \( p_j \leq \mu \) and constant coefficients. Then there exist non-negative integers \( \tau_1, \ldots, \tau_{n-1} \) and differential operators \( q_{ij}(D) \) (\( i = 1, \ldots, n; j = 1, \ldots, n-1 \)) with ord \( q_{ij} \leq \tau_j - t_i \) \( q_{ij} = 0 \) if \( \tau_j < t_i \) for the general solution \( U = (u_1, \ldots, u_n)^T \in \prod W^{s+t_j}(\mathbb{R}) \) of (2.12) can be written in the form

\[ (2.13) \quad \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} q_{1,1} & \ldots & q_{1,n-1} \\ \vdots & \ddots & \vdots \\ q_{n,1} & \ldots & q_{n,n-1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \end{pmatrix} \]

where \( \varphi_j \) are arbitrary functions from \( W^{s+t_j}(\mathbb{R}) \) (\( j = 1, \ldots, n-1 \)).

**Proof.** We use induction on min ord \( p_j = \mu \).

1) Let first \( \mu = 0 \). Without loss of generality, we can assume that \( t_1 - \text{ord} p_n = \ldots = t_1 - \text{ord} p_1 = t_1 - \text{ord} p_1 < \ldots < t_1 - \text{ord} p_1 \) and ord \( p_n \leq \text{ord} p_1 \leq \ldots \leq \text{ord} p_1 \).

1(a) If ord \( p_n = 0 \), \( p_n \neq 0 \), then the general solution \( U = (u_1, \ldots, u_n)^T \in \prod W^{s+t_j}(\mathbb{R}) \) can be written in the form

\[ u_1 = q_{1,1} \varphi_1 = \varphi_1, \ldots, u_{n-1} = q_{n-1,n-1} \varphi_{n-1} = \varphi_{n-1}, \quad u_n = -\frac{1}{p_n} (p_1 \varphi_1 + \ldots + p_{n-1} \varphi_{n-1}) \quad (\varphi_j \in W^{s+t_j}(\mathbb{R})). \]

1(b) If ord \( p_n > 0 \) then there exists \( k \in \{1, \ldots, l-1\} \) such that ord \( p_k = 0 \). In this case we use induction on \( d = t_k - \text{ord} p_k - (t_n - \text{ord} p_n) \).

Assume that the assertion is true if \( t_k - \text{ord} p_k - (t_n - \text{ord} p_n) < d \). Using the representation \( p_j = p_j p_j' + r_j \) \( (j = l, \ldots, n-1) \) where ord \( p_j' = \text{ord} p_j - \text{ord} p_n \) and ord \( r_j < \text{ord} p_n \) we obtain

\[ (2.14) \quad p_1 u_1 + \ldots + p_{l-1} u_{l-1} + r_l u_l + \ldots + r_{n-1} u_{n-1} + p_n (u_n + p_j u_j + \ldots + p_{n-1} u_{n-1}) = 0. \]

Since \( t_j - \text{ord} p_j \geq t_n - \text{ord} p_n + 1 \) for \( j = 1, \ldots, l-1 \) and \( t_j - \text{ord} r_j \geq t_n - \text{ord} p_n + 1 \) for \( j = l, \ldots, n-1 \) we get

\[ p_1 u_1 + \ldots + p_{l-1} u_{l-1} + r_l u_l + \ldots + r_{n-1} u_{n-1} \in W^{s+t_n+1-\text{ord} p_n}(\mathbb{R}). \]
Hence, \( v_n = u_n + p'_1u_1 + \ldots + p'_{n-1}u_{n-1} \in W_p^{s+t_n'}(\mathbb{R}) \) where \( t_n' = t_n + 1 \), i.e. \( t_k - \text{ord} p_k - (t_n' - \text{ord} p_n) = d - 1 \). Then by the inductive assumption there exist non-negative integers \( \tau_1, \ldots, \tau_{n-1} \) and differential operators \( q_{ij} \) (\( i = 1, \ldots, n-1 \)), \( q_{nj}' \) with \( \text{ord} q_{ij} \leq \tau_j - t_i \), \( \text{ord} q_{nj}' \leq \tau_j - t_n' \) such that the general solution \((u_1, \ldots, u_{n-1}, v_n) \in \prod_{i=1}^{n-1} W_p^{s+t_j}(\mathbb{R}) \times W_p^{s+t_n'}(\mathbb{R}) \) of (2.14) is

\[
u_i = \sum_{j=1}^{n-1} q_{ij} \psi_j, \quad v_n = \sum_{j=1}^{n-1} q_{nj}' \varphi_j \quad (\varphi_j \in W_p^{s+\tau_j}(\mathbb{R})).
\]

Hence,

\[
u_i = \sum_{j=1}^{n-1} q_{ij} \psi_j,
\]

\[
u_n = v_n - p'_1u_1 - \ldots - p'_{n-1}u_{n-1} = \sum_{j=1}^{n-1} \left( q_{nj}' - \sum_{i=1}^{n-1} p'_i q_{ij} \right) \varphi_j.
\]

2) Let \( \mu = \min_j \text{ord} p_j \geq 1 \). Without loss of generality we assume that \( \text{ord} p_n \leq \ldots \leq \text{ord} p_1 \). Then each \( p_j \) has a representation \( p_j = p_n' + r_j \) where \( \text{ord} p_n' = \text{ord} p_j - \text{ord} p_n \), \( \text{ord} r_j < \text{ord} p_j \). If \( r_j = 0 \) for \( j = 2, \ldots, n \) then (2.12) is equivalent to \( u_1 + p'_2u_2 + \ldots + p'_n u_n = 0 \) and the assertion of the lemma follows from the first part of the proof. If e.g. \( r_n \neq 0 \) then we write (2.12) in the form

\[
\begin{align*}
p_1 u_1 + \ldots + p_{n-2} u_{n-2} + r_{n-1} u_{n-1} + p_n (u_n + p_{n-1} u_{n-1}) &= 0.
\end{align*}
\]

Since \( r_{n-1} < \mu \) we can suppose that there exist integers \( \tau_1', \ldots, \tau_{n-1}' \) and operators \( q_{ij}' \), \( q_{nj}'' \) such that the general solution of (2.15) is

\[
u_i = \sum_{j=1}^{n-1} q_{ij}' \psi_j \quad (i = 1, \ldots, n-1), \quad u_n + p'_{n-1} u_{n-1} = \sum_{j=1}^{n-1} q_{nj}'' \psi_j
\]

(\( \psi_j \in W_p^{s+t_j'}(\mathbb{R}) \)). From the last equation it follows that

\[
u_n + \sum_{j=1}^{n-1} \left( p'_n q_{n-1,j} - q_{nj}'' \right) \psi_j = 0.
\]

Hence, by the first part of the proof there exist integers \( \tau_1, \ldots, \tau_{n-1} \) and operators \( q_{ij}'' \) (\( \text{ord} q_{ij}'' \leq \tau_j - \tau_i' \) for \( i = 1, \ldots, n-1 \); \( \text{ord} q_{nj}'' \leq \tau_j - t_n \)) such that the general solution of (2.16) can be written in the form

\[
\psi_i = \sum_{j=1}^{n-1} q_{ij}'' \varphi_j \quad (i = 1, \ldots, n-1), \quad u_n = \sum_{j=1}^{n-1} q_{nj}'' \varphi_j
\]

(\( \varphi_j \in W_p^{s+\tau_j}(\mathbb{R}) \)). This implies the assertion of the lemma. ■

By induction on the number of equations Lemma 12 can be easily generalized to systems of differential equations with constant coefficients.
Lemma 13. Consider a system of ordinary differential equations
\begin{equation}
    p_{11}(D_z)u_1 + \ldots + p_{1n}(D_z)u_n = 0, \\
    \vdots \\
    p_{m1}(D_z)u_1 + \ldots + p_{mn}(D_z)u_n = 0,
\end{equation}
where \( p_{ij} \) are differential operators with constant coefficients and \( \text{ord} p_{ij} \leq s_i + t_j \). Then there exist an integer \( l, 0 \leq l \leq n - 1 \), integers \( \tau_1, \ldots, \tau_l \) and differential operators \( q_{ij} \ (i = 1, \ldots, n; j = 1, \ldots, l) \) with \( \text{ord} q_{ij} \leq \tau_j - t_i \) and constant coefficients such that the general solution \( (u_1, \ldots, u_n) \in \prod W_{s+i}^p(\mathbb{R}) \) can be written in the form
\begin{equation}
    u_i = \sum_{j=1}^{l} q_{ij} \varphi_j,
\end{equation}
where \( \varphi_j \) are arbitrary functions from \( W_{s+i}^p(\mathbb{R}) \).

Proof. Suppose that the assertion is true for every system of \( m - 1 \) equations. By Lemma 12 the general solution of the last equation of (2.17) has a representation
\begin{equation}
    u_i = \sum_{j=1}^{n-1} q'_{ij} \psi_j \ (i = 1, \ldots, n-1) \quad \text{where} \quad \psi_j \in W_{s+i}^{p+\tau_j}(\mathbb{R}), \quad \text{ord} \ q'_{ij} \leq \tau_j - t_i.
\end{equation}
Replacing \( u_j \) by \( \sum q'_{ij} \psi_j \) in the first \( m - 1 \) equations of (2.17) we get the system of \( m - 1 \) equations
\begin{equation}
    \sum_{j=1}^{n-1} \left( \sum_{k=1}^{n} p_{ik} q'_{kj} \right) \psi_j = 0 \quad (i = 1, \ldots, m - 1).
\end{equation}
Applying the inductive assumption to this system we get the assertion of the lemma.

Remark 5. For given \( u_j \in W_{s+i}^{p+\tau_j}(\mathbb{R}) \) the functions \( \varphi_j \) in Lemma 13 can be chosen such that
\begin{equation}
    \sum_{j=1}^{l} \| \varphi_j \|_{W_{s+i}^{p+\tau_j}(\mathbb{R})} \leq c \sum_{j=1}^{n} \| u_j \|_{W_{s+i}^{p+\tau_j}(\mathbb{R})}
\end{equation}
where the constant \( c \) is independent of \( u_1, \ldots, u_n \).

Proof of Lemma 11. First, suppose \( s - \beta - 2/p \) is not an integer. Then the traces \( u_{ij}^{(M)} \) of \( \partial_y \partial_{\gamma_j} u \) on \( M \) satisfy the differential equations (2.10) where \( g_{ij}^k = 0 \). We write (2.10) in the form
\begin{equation}
    \mathcal{P} U = 0
\end{equation}
where \( \mathcal{P} \) is a matrix of ordinary differential operators with constant coefficients and \( U \) is the vector composed of the functions \( u_{ij}^{(M)} \in W_{s+i-j-\beta-2/p}(M) \) \((i + j \leq (s - \beta - 2/p))\). By Lemma 13 there exist a vector \( \Phi \) of functions \( \varphi_1, \ldots, \varphi_l \) and a matrix \( Q \) of differential operators such that \( U = Q \Phi \). Another solution of (2.20)
We define the functions $B_{\pm}$.

Furthermore, from the representation (2.21) and Remark 5 it follows that

$$\mathcal{P}(V - \chi|MU) = \mathcal{P}(\chi|M\Phi) - \mathcal{P}Q(\chi|MU) - \mathcal{P}(\chi|MU) = 0.$$  

Furthermore, from the representation (2.21) and Remark 5 it follows that $v_{ij} \in W^{s-i+j+1-\beta-2/p}_p(M)$ and

$$\|v_{ij}\|_{W^{s-i+j+1-\beta-2/p}_p(M)} \leq c \sum_{\mu,\nu=0}^{(s-\beta-2/p)} \|u_{\mu\nu}\|_{W^{s-\mu-\nu-\beta-2/p}_p(M)}.$$

We define the functions $\tilde{v}$, $g^\pm_k$ as follows:

$$\tilde{v} = \sum_{i+j=0}^{(s-\beta-2/p)} (\partial v_{ij}) \frac{\partial^j\tilde{g}^1}{\partial y^j}, \quad g^\pm_k = B^\pm_k(\tilde{v} - \chi u).$$

Since $v_{ij} \in W^{s-i+j+1-\beta-2/p}_p(M)$ and $B^\pm_k(\chi u) = [B^\pm_k, \chi]u \in B^{s+1-m^\pm_k-1/p}_p(\Gamma^\pm)$, $g^\pm_k$ belongs to $B^{s+1-m^\pm_k-1/p}_p(\Gamma^\pm)$. Furthermore, (2.22) implies $\partial_\gamma g^\pm_k|\Gamma = 0$ and $\partial_\gamma^j g^\pm_k|\Gamma = 0$ for $j = 0, 1, \ldots, (s - \beta - 2/p) - m^\pm_k$. Consequently, $g^\pm_k$ and $\partial_\gamma g^\pm_k$ belong to $B^{s+1-m^\pm_k-1/p}_p(\Gamma^\pm)$. Then by Lemma 3.1 of [4] there exists $w \in V^{s}_{p,\beta}(D)$ satisfying $B^\pm_k w = g^\pm_k$ on $\Gamma^\pm$, $B^\pm_k \partial w/\partial z = \partial_\gamma g^\pm_k$ on $\Gamma^\pm$ and

$$\|w\|_{V^{s}_{p,\beta}(D)} + \left\|\frac{\partial w}{\partial z}\right\|_{V^{s}_{p,\beta}(D)} \leq c \sum_{k=1}^{\infty} \sum_{i+j=0}^{(s-\beta-2/p)} \|g^\pm_k\|_{B^{s+1-m^\pm_k-1/p}_p(\Gamma^\pm)} + \|\partial_\gamma g^\pm_k\|_{B^{s+1-m^\pm_k-1/p}_p(\Gamma^\pm)}.$$  

Therefore, $v = \tilde{v} - w$ satisfies the conditions of Lemma 11. Similarly the lemma can be proved for $s - \beta - 2/p$ an integer.  

**2.2. Regularity of solutions of elliptic boundary value problems.** We consider the boundary value problem

$$Lu = \sum_{|\alpha| \leq 2m} a_\alpha(x)D^\alpha u = f \quad \text{in } D,$$

$$B^\pm_k u = \sum_{|\alpha| \leq m^\pm_k} b^\pm_{k,\alpha}(x)D^\alpha u = g^\pm_k \quad \text{on } \Gamma^\pm,$$

$(k = 1, \ldots, m)$ where $L$ is an elliptic differential operator with smooth coefficients and $\{B^\pm_k\}$, $\{B^\pm_k\}$ are normal systems of boundary operators with smooth coefficients which cover the operator $L$ on $\Gamma^+$ and $\Gamma^-$, respectively.

The following regularity assertion has been proved in [4] (see also [9]).
Lemma 14. Let \( u \in V_{p,\beta}^{2m}(D) \) be a solution of the boundary value problem (2.23) where \( f \in V_{p,\beta+\delta}^s(D) \), \( g_k^\pm \in B_{p,\beta+\delta}^{s+2m-\frac{m-k-1}{p}}(\Gamma^\pm) \). Assume that the support of \( u \) is compact. Then \( u \in V_{p,\beta+\delta}^{s+2m}(D) \).

By using Theorems 5 and 6 this assertion can be extended to the spaces \( W_{p,\beta}^s \).

Theorem 8. Let \( u \in W_{p,\beta}^{2m}(D) \) (\( \beta > -2/p \)) be a solution of the boundary value problem (2.23) with compact support. If \( f \in W_{p,\beta+\delta}^s(D) \) and \( g^\pm_k \in B_{p,\beta+\delta}^{s+2m-\frac{m-k-1}{p}}(\Gamma^\pm) \) then \( u \in W_{p,\beta+\delta}^{s+2m}(D) \).

Proof. By Theorem 5 the function \( u \in W_{p,\beta}^{2m}(D) \) is the sum of a function

\[
   u_1 = \sum_{i+j=0}^{2m-1} (K\hat{u}_{ij}) \frac{y_1^{i}y_d^{j}}{i!j!} \in W_{p,\beta+\delta}^{s+2m}(D) \subset W_{p,\beta}^{2m}(D)
\]

(\( \hat{u}_{ij} \in W_{p,\beta+1/p}^{2m-1-j}(\mathbb{R}_+ \times \mathbb{R}^{n-2}) \)) and a function \( u_2 \in V_{p,\beta}^{2m}(D) \). Obviously,

\[
   Lu_2 = f - Lu_1 \in W_{p,\beta+\delta}^s(D) \cap V_{p,\beta}^0(D) = W_{p,\beta+\delta}^s(D),
\]

\[
   B_{p,\beta+\delta}^{s+2m-\frac{m-k-1}{p}}(\Gamma^\pm) \ni B_{p,\beta+\delta}^{s+2m-\frac{m-k-1}{p}}(\Gamma^\pm).
\]

Therefore, from Lemma 14 it follows that \( u_2 \in W_{p,\beta+\delta}^{s+2m}(D) \) and \( u = u_1 + u_2 \in W_{p,\beta+\delta}^{s+2m}(D) \).

References

