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TOWARDS THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION

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1. Introduction. The Cauchy problem for elliptic equations occurs in the study of many practical problems. For example, in investigations of a gravitational (electric, magnetic) field it is often necessary to determine the potential of the field in a domain outside the mass (charge, current) creating the field, from the values of the potential in a part of this domain. A related problem is that of extending an analytic vector field in $\mathbb{R}^3$ or a curve in $\mathbb{R}^2$ to a harmonic vector field outside with determination of its singularities. Such problems play a role, e.g., in the construction of magnetohydrostatic equilibria [24, 25, 54]. The solution of this problem is equivalent to the solution of a Cauchy problem for the Laplace equation that is ill-posed in the sense of Hadamard [3, 27, 46, 81]. For an introduction to the literature of such problems the reader is referred to [3, 27, 34, 46, 64, 81].

The solution of the Cauchy problem for the Laplace equation will exist only if strong compatibility or smoothness conditions are imposed on the initial data. It was Hadamard who showed that unless a certain compatibility relation holds among the Cauchy data no global solution can exist. He further showed that even if the data are such that a classical solution exists, this solution will not depend continuously on the data [26, 27].

In this paper we consider the following problems A, B and C:

PROBLEM A. *Find a function $u(y,x)$ which satisfies*

$$u_{yy}(y,x) + \Delta u(y,x) = 0, \quad (y,x) \in \mathbb{R}_+^{N+1} = \{(y,x) \mid y > 0,\ x \in \mathbb{R}^N\}, \tag{1.1}$$

$$u(0,x) = f(x)\,, \qquad x \in \mathbb{R}^N\,, \tag{1.2}$$

$$u_y(0,x) = g(x)\,, \qquad x \in \mathbb{R}^N\,. \tag{1.3}$$

PROBLEM B. *Find a function $u(y,x)$ which satisfies the equations*

$$u_{yy}(y,x) + u_{xx}(y,x) = 0,\ (y,x) \in D_Y = \{(y,x) \mid 0 < y, -\pi < x < \pi\}, \tag{1.4}$$

$$u(0,x) = f(x)\,, \qquad u_y(0,x) = g(x)\,, \qquad -\pi < x < \pi\,. \tag{1.5}$$

PROBLEM C. *Find a function $u(y,x)$ which satisfies*

$$u_{yy}(y,x) + \Delta u(y,x) = 0\,, \qquad y \in (0,Y)\,,\ x \in (-\pi,\pi)^N =: \Omega\,, \tag{1.6}$$

$$u(0,x) = f(x)\,, \qquad u_y(0,x) = g(x)\,, \qquad x \in (-\pi,\pi)^N\,, \tag{1.7}$$

$$u(y,x)\mid_{\partial\Omega} = 0\,, \qquad y \in (0,Y)\,. \tag{1.8}$$

These problems are well known to be ill-posed in the sense of Hadamard, and many attempts, many investigations from various aspects, as existence–uniqueness theorems [1, 5, 13–15, 17, 19, 20, 26, 27, 29–31, 37, 39–42, 47, 48, 51, 52, 57–59, 61, 62–67, 69, 73–79, 83, 94–99, 100, 101, 104, 106], stability estimates [3, 4, 7, 8, 21, 33, 35, 38, 40, 43–45, 48, 53, 55], regularization, least square methods [21, 28, 32, 48, 56, 81, 92, ...] for such problems have been discussed.

We shall use the theory of pseudodifferential operators with real analytic symbols (ΨDOAS) [18, 84–89] and the approximation theory of functions to regularize the above problems. We observe that every function in $L_p(\mathbb{R}^N)$ can be approximated by a sequence of entire functions of exponential type on $\mathbb{C}^N$, the restrictions of which to $\mathbb{R}^N$ belong to $L_p(\mathbb{R}^N)$ [60] (class $\mathfrak{M}_{\nu p}$), and in these classes our problems are well-posed. Therefore, we should first consider our problems with data in $\mathfrak{M}_{\nu p}$ and then approximate "improper" data in L_p by $\mathfrak{M}_{\nu p}$ functions and suggest the solutions with these approximation data as a regularization. We call this method the mollification method. Tran Duc Van in his joint work with other authors [87] has used this idea to approximate the solution of the well-posed Cauchy problem for the wave equation. Dinh Nho Hào in his joint work with Gorenflo [16] has observed that this method with a modification still works for many ill-posed problems. We hope to present an abstract version of this method in a later work.

In Section II we give a short survey on the Cauchy problem for elliptic equations. Section III is devoted to Problems A, B and C. Numerical experiments will be described in a succeeding work.

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2. A short survey

2.1. *Existence and uniqueness theorems.* The first result on the Cauchy problem for the Laplace equation is that by Hadamard [26]. He proved, for a particular case of Problem B, the following

THEOREM 2.1. *A necessary and sufficient condition for the existence of a function* $u(y,x)$ *which satisfies* $u_{yy}+u_{xx}=0$ *in a neighborhood of the interval* $-\pi<x<\pi$ *of the* x*-axis, where* $y>0$ *and* $u(0,x)=f(x)$, $u_y(0,x)=g(x)$, $-\pi<x<\pi$, *is that*

$$H(x)=f(x)-\frac{1}{\pi}\int_{-\pi}^{\pi} g(\xi)\ln|x-\xi|\,d\xi \tag{2.1}$$

be an analytic function of x *for* $-\pi<x<\pi$. *Here* f *is of class* C^1 *and* g *is of class* C^0 *in* $(-\pi,\pi)$.

An analogous result for Problem C has been obtained by Payne and Sather in [67]. Furthermore, these authors have generalized the result of Hadamard to a class of degenerate elliptic equations.

THEOREM 2.2. *A necessary and sufficient condition for the existence of a function* $u(y,x)$ *which satisfies* $u_{yy}+y^\alpha u_{xx}=0$ $(\alpha>0)$ *in a neighborhood of the interval* $(-\pi,\pi)$ *of the* x*-axis, where* $y>0$, *and* $u(0,x)=f(x)$, $u_y(0,x)=g(x)$, $-\pi<x<\pi$, *is that*

$$H_\alpha(x)=f(x)+m_\beta\int_{-\pi}^{\pi}\frac{g(\xi)}{|x-\xi|^{1-\beta}}\,d\xi \qquad \left(\beta=\frac{2}{\alpha+2}\right) \tag{2.2}$$

be an analytic function of x *for* $-\pi<x<\pi$. *Here* f *is of class* C^1 *and* g *is of class* C^0 *in* $(-\pi,\pi)$, *and*

$$m_\beta=\frac{\beta^{1-\beta}\Gamma(\frac12(3-\beta))}{\sqrt{\pi}(1-\beta)\Gamma(\frac12(2-\beta))}\,. \tag{2.3}$$

An interesting result on existence and uniqueness for Problem A has been given by B. H. Li and Y. Q. Li [49]:

THEOREM 2.3. *Let* $f,g\in\mathcal{D}'(\mathbb{R}^N)$ *and let* (1.2), (1.3) *be satisfied in the sense of* $\mathcal{D}'$. *Then Problem* A *is solvable if and only if*

$$\mathcal{D}'\text{-}\lim_{y\searrow 0}\bar f_y(y,x)-g \tag{2.4}$$

is an entire function, where $\bar f$ *is a harmonic function such that*

$$\mathcal{D}'\text{-}\lim_{y\searrow 0}\bar f(y,x)=f(x)\,.$$

In [85], as a direct consequence of the result of Tran Duc Van in his recent work on ΨDOAS [84], we have also obtained an existence and uniqueness theorem for Problem A. Before outlining this result we need some definitions.

Let $x\in\mathbb{R}^N_x$, $N\ge1$, and $\xi\in\mathbb{R}^N_\xi$ be real variables, $D^\alpha=D_1^{\alpha_1}\ldots D_N^{\alpha_N}$, $D_j=-i\partial/\partial x_j$, $j=1,\ldots,N$, $\alpha=(\alpha_1,\ldots,\alpha_N)$, $|\alpha|=\alpha_1+\ldots+\alpha_N$.

DEFINITION 1. *The space* $W^{+\infty}(\mathbb{R}^N_x)$ *is the set of functions* $f:\mathbb{R}^N_x\to\mathbb{C}$, *satisfying the following conditions*: f *admits analytic continuation to an entire* *function on* $\mathbb{C}^N$ *and for each* $\varepsilon > 0$ *there exist constants* $r < \infty$ *and* C_ε, *possibly depending on* f, *such that*

$$|f(x+iy)| \le C_\varepsilon \exp(r|y| + \varepsilon|x|)\,, \qquad x+iy \in \mathbb{C}^N\,.$$

We list here some classes of functions which belong to $W^{+\infty}(\mathbb{R}^N_x)$: all functions $f \in L_2(\mathbb{R}^N_x)$ the support of whose Fourier transform $\widehat{f}(\xi)$ is compact (band-limited functions) [18], all functions in $\mathfrak{M}_{\nu p}$, $1 \le p \le +\infty$, $\nu < \infty$ [60], all functions in $W^{+\infty}(\mathbb{R}^N)$ [88, 89]. From the Paley–Wiener theorem it follows that $f \in W^{+\infty}(\mathbb{R}^N_x)$ if and only if its analytic continuation $f(z)$ is the Fourier–Laplace transform of an analytic functional with compact support.

Definition 2. *A sequence of functions* $f_n \in W^{+\infty}(\mathbb{R}^N_x)$ *is said to converge to* $f \in W^{+\infty}(\mathbb{R}^N_x)$ *if and only if for each* $\varepsilon > 0$ *there exists a constant* $r < \infty$ *such that* (*with* $z = x+iy$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}^N$)

$$\sup_{z\in\mathbb{C}^N} |f_n(z) - f(z)| \exp(-r|y| - \varepsilon|x|) \to 0\,, \qquad n \to \infty\,.$$

Let the entire function $A(\xi)$, $\xi \in \mathbb{R}^N_\xi$, be expanded in the Taylor series

$$A(\xi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \xi^\alpha\,, \qquad a_\alpha = (iD)^\alpha A(0)/\alpha!\,.$$

Denote by $W^{-\infty}(\mathbb{R}^N_x)$ the space of all continuous linear functionals defined on $W^{+\infty}(\mathbb{R}^N_x)$. Let $h \in W^{-\infty}(\mathbb{R}^N_x)$ and

$$\langle A(D)h, \varphi\rangle := \langle h, A(-D)\varphi\rangle\,, \qquad \forall\, \varphi \in W^{+\infty}(\mathbb{R}^N_x)\,.$$

Theorem 2.4. *Let* $f, g \in W^{+\infty}(\mathbb{R}^N_x)$ $(W^{-\infty}(\mathbb{R}^N_x))$. *Then there exists a unique solution of Problem* A *in* $C^2(\mathbb{R}^1_+; W^{+\infty}(\mathbb{R}^N_x))$ $(C^2(\mathbb{R}^N_+; W^{-\infty}(\mathbb{R}^N_x)))$, *and*

$$u(y,x) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n} \Delta^n f(x)}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{2n-1} \Delta^{n-1} g(x)}{(2n-1)!}\,. \tag{2.5}$$

For any $y > 0$, *the right side of* (2.5) *converges in the sense of* $W^{+\infty}(\mathbb{R}^N_x)$ $(W^{-\infty}(\mathbb{R}^N_x))$.

Medeiros in [52] has given another existence and uniqueness theorem for Problem C.

Uniqueness results have received quite extensive treatment [1, 5, 9, 13–15, 20, 26, 29, 30, 31, 37, 39, 40–47, 49, 51–53, 57–59, 61–67, 69–79, 85, 94–99, 100–102, 106]. We note here again important works by Carleman [9], Hörmander [30, 31], Lavrent'ev [43, 44], Lavrent'ev, Romanov and Shishat-skiĭ [46], Müller [59], Heinz [29], Cordes [15], Landis [40–42] and by Payne and his collaborators [37, 62–67]. Uniqueness results for elliptic equations with multiple characteristics were considered in [94–99, 106].

2.2. *Stable methods for solutions.* The first results concerning stable methods to solve the Cauchy problems for the Laplace equation are contained in the works of Lavrent'ev [43, 45] and Pucci [70, 71] (see also John [35]). These authors established stability estimates and proposed some stable methods. For further works see [51, 53]. Other approaches have been tried, such as: establishing stability estimates [3, 4, 7, 8, 21, 22, 33–35, 46, 53], regularization by various methods, e.g., least squares, Carleman estimates, stable summation of Fourier series, finding a Carleman function, Tikhonov regularization ... [6, 7, 20, 21, 22, 28, 55, 56, 81, 90–93, 104, ...]. For a function-theoretic method see [11, 12].

3. A mollification method

3.1. *Problem* A. Suppose that instead of the exact data f and g (which are supposed to belong to $L_p(\mathbb{R}^N)$, $1 \le p \le \infty$) we only have the measured data f_ε and g_ε such that

$$\|f_\varepsilon - f\| \le \varepsilon\,, \qquad \|g_\varepsilon - g\| \le \varepsilon\,. \tag{3.1}$$

We introduce *mollification operators* M_ν which map a function $f \in L_p$ to a function f_ν in $\mathfrak{M}_{\nu p}$, and we require that $f_\nu \to f$ in L_p as $\nu = (\nu_1, \ldots, \nu_N) \to \infty$.

For simplicity of presentation, we first describe results only for the case $N = 1$.

We need some notations. Let $\Delta_h f = \Delta_h f(x) = f(x+h) - f(x)$, where h is any real number. Then

$$\Delta_h^k f(x) = \Delta_h \Delta_h^{k-1} f(x) = \sum_{l=0}^{k} (-1)^{l+k} \binom{k}{l} f(x+lh)\,, \qquad k \in \{1, 2, \ldots\},$$
$$\Delta_h^0 = f\,.$$

Let

$$\omega^k(\delta) = \omega^k(f,\delta)_p = \sup_{|h|\le\delta} \|\Delta_h^k f(\cdot)\|_p\,, \qquad \omega(\delta) = \omega(f,\delta)_p = \omega^1(f,\delta)\,.$$

$\omega^k(\delta)$ is called the modulus of continuity of order k of the function f in the L_p metric. It is well known that if $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, then $\lim_{\delta\to 0} \omega(\delta) = 0$. For $p = \infty$ this property does not hold in general. However, it is satisfied in a trivial way if f is uniformly continuous on every compact subset of $\mathbb{R}$.

Suppose that $K(\xi)$ is a nonnegative even function of one variable, of exponential type 1, satisfying the condition

$$\int_{-\infty}^{\infty} K(t)\,dt = 1\,, \tag{3.2}$$

and K is chosen so that the integral

$$\int_{-\infty}^{\infty} K(\xi)|\xi|^l\,d\xi\,, \tag{3.3}$$

where l is a fixed natural number, is finite. For K we may choose a function of the form

$$\mu\left(\frac{\sin(\xi/\lambda)}{\xi}\right)^{\lambda}, \tag{3.4}$$

where $\lambda \geq l+2$ is an even number and μ is a positive constant such that (3.2) holds.

For $f \in L_p(\mathbb{R})$ and $\nu \in \mathbb{R}^+$ the function

$$f_\nu(x) = M_\nu f(x) = \int\limits_{-\infty}^{\infty} K_\nu(\xi - x) f(\xi)\, d\xi\,, \tag{3.5}$$

where

$$K_\nu(\xi) = \sum_{j=1}^{l} (-1)^{j-1} \binom{l}{j} \frac{\nu}{j} K\left(\frac{\xi\nu}{j}\right) \tag{3.6}$$

is well defined. Since K is an entire function of one variable, of exponential type 1, f_ν is an entire function of spherical type ν, lying in $L_p(\mathbb{R})$ ([60], p. 186). Now we have the following

LEMMA 1 ([60], §5.2). *If f has derivatives of order m lying in L_p, $k = l - m$, $0 \leq m \leq l$, then f_ν is an entire function of spherical type ν and*

$$\|f_\nu - f\|_p \leq \frac{c(m)}{\nu^m} \omega^k(f^{(m)}, 1/\nu)_p\,, \tag{3.7}$$

where

$$c(m) = \int\limits_{-\infty}^{\infty} K(|\xi|)|\xi|^m (1 + |\xi|)^k\, d\xi\,. \tag{3.8}$$

If s is a nonnegative integer with $s \leq m$ then

$$\|f_\nu^{(s)} - f^{(s)}\|_p \leq \frac{c}{\nu^{m-s}} \omega^k(f^{(s)}, 1/\nu)_p\,, \qquad \nu > 0\,, \tag{3.9}$$

where

$$c = \int\limits_{0}^{\infty} K(\xi)(1 + |\xi|)^l\, d\xi\,. \tag{3.10}$$

Now instead of (1.1)–(1.3) with the measured data f_ε and g_ε we consider the *mollified problem*: Find $u_\nu^\varepsilon(y, x)$ such that

$$\frac{\partial^2 u_\nu^\varepsilon(y,x)}{\partial y^2} + \frac{\partial^2 u_\nu^\varepsilon(y,x)}{\partial x^2} = 0\,, \qquad (y,x) \in \mathbb{R}^+ \times \mathbb{R}\,, \tag{3.11}$$

$$u_\nu^\varepsilon(0,x) = f_\nu^\varepsilon(x)\,, \qquad x \in \mathbb{R}\,, \tag{3.12}$$

$$\frac{\partial u_\nu^\varepsilon(0,x)}{\partial y} = g_\nu^\varepsilon(x)\,, \qquad x \in \mathbb{R}\,. \tag{3.13}$$

THEOREM 3.1. *Problem* (3.11)–(3.13) *is well-posed in* $C^2(\mathbb{R}; W^{+\infty}(\mathbb{R}_x^N))$ *and*

$$u_\nu^\varepsilon(y,x) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n} (f_\nu^\varepsilon)^{(2n)}(x)}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1} y^{2n-1} (g_\nu^\varepsilon)^{(2n-1)}(x)}{(2n-1)!}\,. \tag{3.14}$$

Furthermore, *for every* m, $0 \le m \le l$, $k = l - m$,

$$\begin{aligned}\|u_\nu^\varepsilon(y,\cdot) - u(y,\cdot)\|_p &\\ &\le \frac{c(m)}{\nu^m}\omega^k\left(\frac{\partial^m u(y,\cdot)}{\partial x^m}, \frac{1}{\nu}\right)_p + c_l\varepsilon\left[\cosh(y\nu) + \frac{\sinh(y\nu)}{\nu}\right],\end{aligned} \tag{3.15}$$

where $c_l \le 2^l - 1$ *does not depend on* ν.

P r o o f. Formula (3.14) is a consequence of Theorem 2.4. We have

$$\begin{aligned} u_\nu^\varepsilon(y,x) - u(y,x) = {}& \cos(yd/dx) f_\nu^\varepsilon + [\sin(yd/dx)/(d/dx)] g_\nu^\varepsilon \\ & - \cos(yd/dx) f + [\sin(yd/dx)/(d/dx)] g \\ ={}& \{\cos(yd/dx) f_\nu^\varepsilon - \cos(yd/dx) f_\nu \\ & + [\sin(yd/dx)/(d/dx)] g_\nu^\varepsilon - [\sin(yd/dx)/(d/dx)] g_\nu\} \\ & + \{\cos(yd/dx) f_\nu - \cos(yd/dx) f \\ & + [\sin(yd/dx)/(d/dx)] g_\nu - [\sin(yd/dx)/(d/dx)] g\} \\ =:{}& \Sigma_1 + \Sigma_2\,. \end{aligned}$$

From the fact that $f_\nu, g_\nu \in \mathfrak{M}_{\nu p}$ and (3.5) it can be verified that

$$\begin{aligned} &\cos(yd/dx) f_\nu + [\sin(yd/dx)/(d/dx)] g_\nu \\ &\qquad = M_\nu(\cos(yd/dx) f + [\sin(yd/dx)/(d/dx)] g) = M_\nu u(y,x)\,. \end{aligned}$$

Thus,

$$\|\Sigma_2\|_p = \|M_\nu(u(y,\cdot)) - u(y,\cdot)\|_p \le \frac{c(m)}{\nu^m}\omega^k\left(\frac{\partial^m u(y,\cdot)}{\partial x^m}, \frac{1}{\nu}\right)_p.$$

On the other hand, since f_ν^ε, f_ν, g_ν^ε, g_ν belong to $\mathfrak{M}_{\nu p}$, we obtain from the Bernstein–Nikol'skiĭ inequalities ([60], p. 115)

$$\|\Sigma_1\|_p \le \cosh(y\nu)\|f_\nu^\varepsilon - f_\nu\|_p + \frac{\sinh(y\nu)}{\nu}\|g_\nu^\varepsilon - g_\nu\|_p\,.$$

But

$$\begin{aligned} \|f_\nu^\varepsilon - f_\nu\|_p &= \Big\| \int_{-\infty}^{\infty} K_\nu(\xi - x)(f^\varepsilon(x) - f(x))\,dx \Big\|_p \\ &\le \int_{-\infty}^{\infty} |K_\nu(\xi)|\,d\xi\,\|f^\varepsilon - f\|_p \le c_l\varepsilon \end{aligned}$$

where $c_l \le 2^l - 1$ does not depend on ν ([60], p. 190). Analogously,

$$\|g_\nu^\varepsilon - g_\nu\|_p \le c_l\varepsilon\,.$$

Thus,

$$\|\Sigma_1\|_p \le c_l\varepsilon\left[\cosh(y\nu) + \frac{\sinh(y\nu)}{\nu}\right],$$

and (3.15) is proved.

THEOREM 3.2. *Let*

$$\|\partial^l u(y,\cdot)/\partial x^l\|_p \le M \quad \text{for } y \in [0,Y]\,, \tag{3.16}$$

where m is a nonnegative integer. Then, with $m = l-1$,

$$\|u^\varepsilon_\nu(y,\cdot) - u(y,\cdot)\|_p \le \frac{c(m)M}{\nu^l} + 2c_l\varepsilon e^{y\nu}\,. \tag{3.17}$$

Furthermore, for $y > 0$ let

$$\Omega(\varepsilon) = \inf_{\nu>0}\left\{\frac{c(m)M}{\nu^l} + 2c_l\varepsilon e^{y\nu}\right\}. \tag{3.18}$$

Then

$$\Omega(\varepsilon) = c(m)M\left(\frac{1}{y}\ln\frac{lc(m)M}{2yc_l\varepsilon\left(\frac{1}{y}\ln\frac{lc(m)M}{2yc_l\varepsilon}\right)^{(1/l)+l}} + o(1)\right)^{-1/l} \quad \text{as } \varepsilon \to 0. \tag{3.19}$$

The infimum on the right hand side of (3.18) *is attained if ν is the solution $\nu(\varepsilon)$ of the equation*

$$\varepsilon = \frac{c(m)Ml}{2yc_l}\frac{e^{-y\nu^2}}{\nu^{l+2}}\,, \tag{3.20}$$

which can be written in the form

$$\nu(\varepsilon) = \left(\frac{1}{y}\ln\frac{lc(m)M}{2yc_l\varepsilon\left(\frac{1}{y}\ln\frac{lc(m)M}{2yc_l\varepsilon}\right)^{(1/l)+l}} + o(1)\right)^{-1/l} \quad \text{as } \varepsilon \to 0\,. \tag{3.21}$$

Proof. The estimate (3.17) is a direct consequence of (3.15) and (3.16). To prove (3.19)–(3.21) we need the following result:

LEMMA 2 ([16]). *Let*

$$\Omega_c(\varepsilon) = \inf_{\delta>0}(\delta + c(\delta)\varepsilon)\,. \tag{3.22}$$

If $c(\delta) = c_0\exp(s\delta^{-\eta})$, $\eta > 0$, then

$$\Omega_c(\varepsilon) = \left(\frac{1}{s}\ln\frac{1}{sc_0\eta\varepsilon\left(\frac{1}{s}\ln\frac{1}{sc_0\eta\varepsilon}\right)^{(\eta+1)/\eta}} + o(1)\right)^{-\eta} \quad \text{as } \varepsilon \to 0\,. \tag{3.23}$$

The infimum is attained if δ is the solution $\delta(\varepsilon)$ of the equation

$$\varepsilon = \frac{1}{c_0 s\eta}\delta^{\eta+1}e^{-s\delta^{-\eta}}\,, \tag{3.24}$$

which can be written in the form

$$\delta(\varepsilon) = \left(\frac{1}{s} \ln \frac{1}{sc_0\eta\varepsilon\left(\dfrac{1}{s}\ln\dfrac{1}{sc_0\eta\varepsilon}\right)^{(\eta+1)/\eta}} + o(1)\right)^{-\eta} \qquad as\ \varepsilon \to 0\,. \tag{3.25}$$

Our estimates (3.19)–(3.21) now follow directly from this lemma if we take $c_0 = c_l/c(m)M, s = y, \eta = 2/l$.

When $N > 1$, we can also approximate the measured data f^ε and g^ε by functions of classes $\mathfrak{M}_{\nu p}$, $\nu = (\nu_1, \ldots, \nu_N)$, by the above process, but since the description is somewhat lengthy and no new idea is offered, we do not write it down here.

3.2. *Problem* B. Instead of the exact data f and g in (1.5), let there be given measured data f^ε and g^ε such that

$$\|f - f^\varepsilon\|_{L_p(-\pi,\pi)} \le \varepsilon\,, \qquad \|g - g^\varepsilon\|_{L_p(-\pi,\pi)} \le \varepsilon\,. \tag{3.26}$$

The following lemma helps us to treat this problem.

LEMMA 3. *If f and g can be extended to entire functions in $\mathbb{R}$ of exponential type, then the solution of Problem* B *can be written in the form*

$$u(y,x) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n} f^{(2n)}(x)}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{2n-1} g^{(2n-1)}(x)}{(2n-1)!}\,. \tag{3.27}$$

Furthermore, this series converges in the sense of W^∞, for fixed $y \in (0, Y)$, uniformly in x for $|x| < \pi$.

Proof. Denote the extensions of f and g to all of $\mathbb{R}$ by the same notations f and g, respectively. Now we consider Problem A with these data. By Theorem 2.4 the solution of Problem A exists uniquely in $W^\infty(\mathbb{R}_x)$ and

$$u(y,x) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n} f^{(2n)}(x)}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{2n-1} g^{(2n-1)}(x)}{(2n-1)!}\,.$$

For fixed $y \in (0, Y)$ this series converges uniformly in every compact subset of $\mathbb{R}$.

It is clear that the function $u(y, x)$ satisfies (1.4)–(1.5). On the other hand, since $g(x)$ is an entire function of exponential type, so is $\int_{-\pi}^{\pi} g(\xi) \ln|x-\xi|\, d\xi$ ([60], §3.6). Thus $f(x) - \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\xi)\ln|x-\xi|\,d\xi$ is also an entire function, or Problem B has a unique solution (Theorem 2.1). The lemma is proved.

Now for $\nu \in \mathbb{R}^+$ we define

$$\begin{aligned} f_\nu^\varepsilon(x) &= M_\nu^* f^\varepsilon(x) = \int_{-\pi}^{\pi} K_\nu(\xi - x) f^\varepsilon(\xi)\, d\xi\,,\\ g_\nu^\varepsilon(x) &= M_\nu^* g^\varepsilon(x) = \int_{-\pi}^{\pi} K_\nu(\xi - x) g^\varepsilon(\xi)\, d\xi\,. \end{aligned} \tag{3.28}$$

By Lemma 1, f_ν^ε and g_ν^ε are elements of $\mathfrak{M}_{\nu p}$. Therefore, from Lemma 3,

$$u_\nu^\varepsilon(y,x) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n} (f_\nu^\varepsilon)^{(2n)}(x)}{(2n)!} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} y^{2n-1} (g_\nu^\varepsilon)^{(2n-1)}(x)}{(2n-1)!} \tag{3.29}$$

is the unique solution of Problem B with the data f_ν^ε instead of f, and g_ν^ε instead of g. We hope that u_ν^ε can serve as an approximation to $u(y,x)$. Indeed, the following theorems will show how well $u_\nu^\varepsilon(y,x)$ approximates $u(y,x)$.

THEOREM 3.3. *For every m, $0 \le m \le l$, $k = l - m$, we have*

$$\|u_\nu^\varepsilon(y,\cdot) - u(y,\cdot)\|_{L_p(-\pi,\pi)} \le \frac{c(m)}{\nu^m}\omega^k\left(\frac{\partial^m u(y,\cdot)}{\partial x^m}, \frac{1}{\nu}\right)_p + c_l\varepsilon\left[\cosh(y\nu) + \frac{\sinh(y\nu)}{\nu}\right],$$

where c_l does not depend on ν.

THEOREM 3.4. *Let*

$$\|\partial^l u(y,\cdot)/\partial y^l\|_{L_p(-\pi,\pi)} \le M \quad \textit{for any } y > 0\,,$$

where l is a positive integer. Then the estimates (3.17)–(3.21) *are valid, with the norm in* (3.17) *being that of $L_p(-\pi,\pi)$.*

The proofs are similar to those of Theorems 3.1 and 3.2.

3.3. *Problem* C. We can repeat, of course, the method 3.2 for Problem C. This means that when the data are not precisely given, we approximate them by entire functions of exponential type, and therefore all the above results remain valid. But since the data, and so the solution, of Problem C are periodic, we can approximate the data by trigonometric polynomials.

Let $\tau_\nu(t)$ $(\nu = 0,1,2,\ldots)$ be a trigonometric polynomial of order not higher than ν, having the following properties:

$$\int_{-\pi}^{\pi} \tau_\nu(t)\,dt = 1\,, \tag{3.30}$$

$$\int_{-\pi}^{\pi} |\tau_\nu(t)|\,dt \le c \quad (\nu = 1,2,\ldots)\,, \tag{3.31}$$

where c is a constant not depending on ν. Obviously

$$\tau_0(t) = 1/(2\pi)\,. \tag{3.32}$$

For $\nu > 0$ the polynomials τ_ν are defined nonuniquely. One may obtain such polynomials, for example, by means of the formula

$$d_\nu(t) = \frac{1}{c_\nu}\left(\frac{\sin(\lambda t/2)}{\sin(t/2)}\right)^{2\sigma}, \tag{3.33}$$

where σ is a positive integer not depending on λ, $\lambda \in \mathbb{N}^+$ (then $\nu = (\lambda-1)\sigma$). Here

$$c_\nu = \int_{-\pi}^{\pi} \left(\frac{\sin(\lambda t/2)}{\sin(t/2)}\right)^{2\sigma} dt \sim \lambda^{2\sigma-1}, \qquad \lambda = 1, 2, \ldots \tag{3.34}$$

([60], p. 87). Define

$$K_\nu^*(t) := \sum_{k=1}^{l} \frac{(-1)^{k-1}\binom{l}{k}}{k} \sum_{s=0}^{k-1} \tau_\nu\left(\frac{t+2s\pi}{k}\right). \tag{3.35}$$

Then $K_\nu^*(t)$ is a trigonometric polynomial of order not higher than ν. Furthermore, if

$$\tau_\nu(t) = \sum_{\lambda=-\nu}^{\nu} a_\lambda e^{i\lambda t} \qquad (\overline{a}_\lambda = \overline{a}_{-\lambda}), \tag{3.36}$$

then

$$\sum_{s=0}^{k-1} \tau_\nu\left(\frac{t+2s\pi}{k}\right) = \sum_{\lambda=-\nu}^{\nu} a_\lambda \sum_{s=0}^{k-1} e^{i\lambda(t+2s\pi)/k}, \tag{3.37}$$

and therefore

$$K_\nu^*(t) = \sum_{k=1}^{l} \frac{(-1)^{k-1}\binom{l}{k}}{k} \sum_{s=0}^{k-1} \Big[\sum_{\lambda=-\nu}^{\nu} a_\lambda \sum_{s=0}^{k-1} e^{i\lambda(t+2s\pi)/k}\Big] = \sum_{\lambda=-\nu}^{\nu} k_\lambda^* e^{i\lambda t}. \tag{3.38}$$

Now let

$$f_\nu(x) = \int_{-\pi}^{\pi} K_\nu^*(\xi) f(x+\xi)\, d\xi. \tag{3.39}$$

Then $f_\nu(t)$ is a trigonometric polynomial of order not higher than ν. Furthermore, as an analogue to Lemma 1 we have

LEMMA 4 ([60, §5.3]). *Suppose $f \in L_p(-\pi,\pi)$ and f has a generalized derivative $f^{(m)}$. Furthermore, suppose that the even nonnegative trigonometric polynomials $\tau_\nu(t)$ of order ν satisfy along with the condition* (3.30) *the further condition*

$$\int_0^{\pi} \tau_\nu(t) t^m\, dt \le \frac{a_m}{(\nu+1)^m}, \tag{3.40}$$

where the constant a_m does not depend on $\nu = 0, 1, 2, \ldots$ Then the function $f_\nu(t)$ defined by (3.39) *approximates f in the metric of $L_p(-\pi,\pi)$ with the following estimate*:

$$\|f - f_\nu\|_{L_p(-\pi,\pi)} \le b_m \frac{\omega_p^k(f^{(m)}, \frac{\pi}{\nu+1})}{(\nu+1)^m} \qquad \text{for } \nu = 0, 1, 2, \ldots, \tag{3.41}$$

where $b_m = 2(\pi^m + 2a_l/\pi)$, $k = l - m$.

R e m a r k. The function $d_\nu(t)$ in (3.33) with $2\sigma - m \geq 3$ satisfies the condition of Lemma 4.

Now, suppose that instead of the exact f and g in (1.7), we only have the measured f^ε and g^ε such that

$$\|f - f^\varepsilon\|_{L_p(-\pi,\pi)} \leq \varepsilon\,, \qquad \|g - g^\varepsilon\|_{L_p(-\pi,\pi)} \leq \varepsilon\,, \tag{3.42}$$

$$f^\varepsilon(-\pi) = f^\varepsilon(\pi) = g^\varepsilon(-\pi) = g^\varepsilon(\pi) = 0\,. \tag{3.43}$$

For a positive integer ν, let

$$f_\nu^\varepsilon(x) = \int_{-\pi}^{\pi} K_\nu^* f^\varepsilon(x+\xi)\,d\xi\,, \tag{3.44}$$

$$g_\nu^\varepsilon(x) = \int_{-\pi}^{\pi} K_\nu^* g^\varepsilon(x+\xi)\,d\xi\,. \tag{3.45}$$

It is not hard to see that

$$u_\nu^\varepsilon(y,x) = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n} (f_\nu^\varepsilon)^{(2n)}(x)}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1} y^{2n-1} (g_\nu^\varepsilon)^{(2n-1)}(x)}{(2n-1)!} \tag{3.46}$$

is the unique solution of Problem C with the data f_ν^ε and g_ν^ε instead of f and g, respectively. On the other hand, from (3.38) and (3.44), (3.45) we have

$$f_\nu^\varepsilon(x) = \sum_{\lambda=-\nu}^{\nu} (f_\nu^\varepsilon)_\lambda e^{i\lambda x}\,, \qquad g_\nu^\varepsilon(x) = \sum_{\lambda=-\nu}^{\nu} (g_\nu^\varepsilon)_\lambda e^{i\lambda x}\,,$$

where

$$(f_\nu^\varepsilon)_\lambda = k_{-\lambda}^* \int_{-\pi}^{\pi} f^\varepsilon(x) e^{-i\lambda x}\,dx\,, \tag{3.47}$$

$$(g_\nu^\varepsilon)_\lambda = k_{-\lambda}^* \int_{-\pi}^{\pi} g^\varepsilon(x) e^{-i\lambda x}\,dx\,. \tag{3.48}$$

Thus,

$$u_\nu^\varepsilon(y,x) = \sum_{\lambda=-\nu}^{\nu} (f_\nu^\varepsilon)_\lambda \cosh(\lambda y) e^{i\lambda x} + \sum_{\lambda=-\nu}^{\nu} (g_\nu^\varepsilon)_\lambda \frac{\sinh(\lambda y)}{\lambda} e^{i\lambda x} \tag{3.49}$$

is a trigonometric polynomial of order not higher than ν.

The same estimates as in Theorems 3.1, 3.2 can be established, but we do not write them down here again.

R e m a r k. Although our estimates for the regularizing solutions are not of Hölder continuity type, they are *stronger than any logarithmic continuity*. An important feature is that our estimates are uniform for y with $0 < y \leq Y$. This has not been established in earlier works (see, e.g, [38], [43], [46], ...). In [6] Cannon and DuChateau have given a direct process for solving Problem B.

Their estimates are also not of Hölder continuity type, but are stronger than any logarithmic continuity. They are not valid for all $y \in (0, Y]$, as ours are, but only for $y \in (0, Y/3]$.

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Added in proof. Recently, there have been some new papers on the Cauchy problem for elliptic equations:

1. Dinh Nho Hào, *A mollification method for ill-posed problems*, preprint A-92-35, FB Mathematik, FU Berlin.
2. A. V. Fursikov, *The Cauchy problem for a second-order elliptic equation in a conditionally well-posed formulation*, Trans. Moscow Math. Soc. 52 (1990), 139–176.
3. M. V. Klibanov and F. Santosa, *A computational quasi-reversibility method for Cauchy problems for Laplace's equation*, SIAM J. Appl. Math. 5 (1991), 1653–1675.