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L_p-THEORY OF BOUNDARY VALUE PROBLEMS FOR SOBOLEV TYPE EQUATIONS

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1. Introduction. In this paper we present our results on boundary value problems for linear equations not solved with respect to the time derivative of highest order

$$L(x,D_t,D_x)u = L_0(x,D_x)D_t^l u + \sum_{k=0}^{l-1} L_{l-k}(x,D_x)D_t^k u = f(t,x)\,. \tag{1}$$

Many problems of hydrodynamics lead to equations of this type. Let us consider some examples.

1. One of the first equations of type (1) was considered by C. G. Rossby [19] in 1939. It has the form

$$\varDelta D_t u + \beta D_{x_2} u = 0, \qquad n = 2\,. \tag{2}$$

It arose in the study of motion of some type of ocean waves. Now it is called in the literature the equation for Rossby waves ($\varDelta$ is the Laplacian in x).

2. S. L. Sobolev's equation [22] considered in the study of small oscillations of a rotating ideal fluid is

$$\varDelta D_t^2 u + \omega^2 D_{x_3}^2 u = f(t,x), \qquad n = 3 \tag{3}$$

($\omega/2$ is the angular velocity). S. L. Sobolev studied the Cauchy problem and the first and second boundary value problems for this equation and also formulated some new problems of mathematical physics. It was the first deep study of equations not solved for the highest derivative with respect to time. This is why now (3) is called the Sobolev equation and (1) is called an equation of Sobolev type.

3. The following equation was obtained for the problem of small oscillations of a rotating viscous fluid:

$$\varDelta D_t^2 u - 2\nu\varDelta^2 D_t u + \nu^2\varDelta^3 u + \omega^2 D_{x_3}^2 u = f(t,x), \qquad n = 3, \tag{4}$$

where $\nu > 0$ is the coefficient of viscosity (see, for example, [14], [17]).

4. Studying oscillations of a stratified ideal fluid leads to the equation

$$\Delta D_t^2 u + N^2(D_{x_1}^2 + D_{x_2}^2)u = 0, \qquad n = 3\,, \tag{5}$$

where N is the Väisälä–Brunt frequency. (5) is called the equation of internal waves [12], [16].

5. In the 1960s the equation

$$(\eta\Delta - 1)D_t u + \kappa\Delta u = f(t,x), \qquad n = 3\,, \tag{6}$$

was studied by G. I. Barenblatt, J. P. Zheltov and I. N. Kochina [1]. It describes the seepage of homogeneous liquids in fissure rocks.

In the 60s equation (6), for $n = 1$, also appeared in other physical papers, not connected with seepage problems (see, for example, [2], [3]).

Appearance of equations of type (1) in many physical applications stimulated the interest of mathematicians in them. Since the fifties, the study of equations of Sobolev type has gone in different directions. In particular, the qualitative behaviour of solutions of some boundary value problems has been investigated together with spectral problems. Many papers were devoted to construction of a general theory of boundary value problems for those equations.

In the literature, the most popular problems of type (1) are Sobolev's equation (3) and the equation for internal waves (5). Many papers by S. L. Sobolev, R. A. Aleksandryan, T. I. Zelenyak and V. N. Maslennikova were devoted to the qualitative properties of solutions of (3). Since the 70s different properties of solutions of (5) have been investigated in papers by S. A. Gabov, S. Ya. Sekerzh-Zen'kovich, A. G. Sveshnikov and others.

M. I. Vishik's, S. A. Galpern's, A. A. Dezin's, A. L. Pavlov's, Ya. A. Dubinskiĭ's, B. K. Romanko's, J. Lagnese's, T. W. Ting's, G. I. Eskin's, A. G. Kostyuchenko's, R. E. Showalter's and other papers were devoted to construction of a general theory of boundary value problems for equations of Sobolev type (see, for example, the bibliography in [24]). However, most of the papers consider the case when the symbol $L_0(x, i\xi)$ of the operator $L_0(x, D_x)$ does not vanish for $\xi \in \mathbb{R}_n$. (Of the above equations only (6) satisfies this condition.) In this case for some classes of equations of Sobolev type a theory analogous to the theory of boundary value problems for hyperbolic and parabolic partial differential equations can be constructed (see, for example, [15], [18], [20], [21]). But there is no analogous theory in the case when $L_0(x, i\xi)$ may be zero at $\xi \in \mathbb{R}_n$. S. A. Galpern [13] first observed this fact when constructing the L_2-theory of the Cauchy problem. This aspect for mixed problems was studied in detail in [24].

In the next section we give some results on the L_p-theory of the Cauchy problem and mixed problems in a quarter-space for two classes of equations (1) in the case when the symbol $L_0(x, i\xi)$ degenerates at $\xi = 0$. These results reflect a considerable difference between the theory of well-posedness for boundary value problems and the corresponding results for classical equations. The solvability of a boundary value problem depends not only on the smoothness of its data but on some additional requirements, such as orthogonality conditions for $f(t,x)$.

We apply a construction of approximate solutions [24] for the boundary value problems. This method uses a special regularization of functions given by S. V. Uspenskiĭ [23]:

$$F(x)=\lim_{h\to 0}(2\pi)^{-k}\int\limits_{h}^{h^{-1}} v^{-|\alpha|-1}\int\limits_{\mathbb{R}_k}\int\limits_{\mathbb{R}_k}\exp\left(i\frac{x-y}{v^{\alpha}}\right)G(\xi)F(y)\,d\xi\,dy\,dv\,,$$

$$G(\xi)=2m\langle\xi\rangle^{2m}\exp(-\langle\xi\rangle^{2m}),\qquad \langle\xi\rangle^2=\sum_{i=1}^{k}\xi_i^{2/\alpha_i}\,.$$

Our method is applicable to boundary value problems for some class of linear systems not of Cauchy–Kovalevsky type [4], [5], [11] and for some others. In particular, it is applicable to boundary value problems for quasielliptic equations in a half-space [24], [6], [7]. Some results can be used in the theory of hyperbolic equations. For example, [8] establishes an interesting connection between the L_p-theory of the Cauchy problem for a certain hyperbolic system of the dynamics of a stratified fluid and the L_p-theory of the Cauchy problem for certain equations of Sobolev type. An analogous connection exists for mixed problems.

2. The Cauchy problem. In this section we consider the following Cauchy problem for two classes of equations of Sobolev type:

$$\begin{cases} L(x,D_t,D_x)u=f(t,x), & t>0,\ x\in\mathbb{R}_n\,,\\ D_t^k u|_{t=0}=0, & k=0,\dots,l-1\,. \end{cases}\tag{7}$$

We formulate some conditions on the differential operators $L(x,D_t,D_x)$.

1) $L(x,D_t,D_x)$ has the form

$$\begin{aligned} L(x,D_t,D_x)&=\widetilde{L}_1(x,D_t,D_x)+\widetilde{L}_2(x,D_x)\\ &=\Big(L_0(x,D_x)D_t^l+\sum_{k=0}^{l-1}L_{l-k}(x,D_x)D_t^k\Big)+\widetilde{L}_2(x,D_x)\,, \end{aligned}$$

where the symbol $\widetilde{L}_1(x,i\eta,i\xi)$ of the operator $\widetilde{L}_1(x,D_t,D_x)$ is homogeneous with respect to the vector $\overrightarrow{\alpha}=(\alpha_0,\alpha_1,\dots,\alpha_n)=(\alpha_0,\alpha)$, $\alpha_0\ge 0$ and $1/\alpha_i$ are natural numbers, i.e.

$$\widetilde{L}_1(x,c^{\alpha_0}i\eta,c^{\alpha}i\xi)=c\widetilde{L}_1(x,i\eta,i\xi),\qquad c\ge 0\,.$$

The operator $\widetilde{L}_2(x,D_x)$ has the form

$$\widetilde{L}_2(x,D_x)=\sum_{1-\alpha_0 l\le\beta\alpha<1}\alpha_\beta(x)D_x^{\beta}\,.$$

2) $L_0(x,D_x)$ is quasielliptic, i.e. $L_0(x,i\xi)=0$ for $\xi\in\mathbb{R}_n$ if and only if $\xi=0$.

3) $L(x, D_t, D_x)$ has variable coefficients which are smooth and constant outside a certain compact set $K \subset \mathbb{R}_n$.

The first class we consider contains the equations which are defined by operators $L(x, D_t, D_x)$ with $\alpha_0 = 0$. The second class contains the equations defined by operators $L(x, D_t, D_x)$ with $\alpha_0 > 0$.

We assume supplementary conditions for the second class of equations:

4) $\widetilde{L}_1(x, \tau, i\xi) \neq 0$, $\operatorname{Re} \tau \geq 0$, $\xi \in \mathbb{R}_n \setminus \{0\}$, $|\tau| + |\xi| \neq 0$.

We now give some examples of equations for which 1)–4) are satisfied.

EXAMPLE 1. The Sobolev equation (3) and the equation for internal waves (5) are equations of the first class. The respective differential operators have the homogeneity vector $\overrightarrow{\alpha} = (0, 1/2, \ldots, 1/2)$.

EXAMPLE 2. Consider a pseudo-parabolic partial differential equation

$$L_0(x, D_x) D_t u + L_1(x, D_x) u = f(t, x)\,,$$

where $L_0(x, D_x)$ and $L_1(x, D_x)$ are homogeneous elliptic operators. Let $\operatorname{ord} L_0 = 2m$, $\operatorname{ord} L_1 = 2k$ and $m \leq k$; then $\overrightarrow{\alpha} = ((k-m)/k, 1/2k, \ldots, 1/2k)$ is the homogeneity vector. If $m = k$, then this is an equation of the first class. If $m < k$ and $L_0(x, i\xi) > 0$, $L_1(x, i\xi) > 0$, $\xi \in \mathbb{R}_n \setminus \{0\}$, it is an equation of the second class.

EXAMPLE 3. The equation for small oscillations of a rotating viscous fluid (4) is an equation of the second class. In this case we can write $\widetilde{L}_1(x, D_t, D_x) = \Delta(D_t - \nu\Delta)^2$, $\widetilde{L}_2(x, D_x) = \omega^2 D_{x_3}^2$, $\overrightarrow{\alpha} = (1/3, 1/6, 1/6, 1/6)$.

We now define a certain function space.

Let $r = (r_0, r_1, \ldots, r_n)$, $0 \leq \sigma \leq 1$, $\gamma > 0$, $G \subseteq \mathbb{R}_n$.

We denote by $W^r_{p,\sigma,\gamma}(\mathbb{R}^+_1 \times G)$ the space of locally integrable functions $u(t, x)$ in $\mathbb{R}^+_1 \times G$ which have generalized derivatives $D_t^{r_0} u(t, x)$, $D_{x_i}^{r_i} u(t, x)$, $i = 1, \ldots, n$, and finite norm

$$\begin{aligned} \|u(t,x), W^r_{p,\sigma,\gamma}(\mathbb{R}^+_1 \times G)\| = {} & \|e^{-\gamma t}(1 + \langle x\rangle)^{-\sigma} u(t,x), L_p(\mathbb{R}^+_1 \times G)\| \\ & + \|e^{-\gamma t}(1 + \langle x\rangle)^{-\sigma} D_t^{r_0} u(t,x), L_p(\mathbb{R}^+_1 \times G)\| \\ & + \sum_{k=1}^{n} \|e^{-\gamma t} D_{x_k}^{r_k} u(t,x), L_p(\mathbb{R}^+_1 \times G)\|\,, \end{aligned}$$

where $\langle x\rangle^2 = \sum_{i=1}^n x_i^{2/\alpha_i}$.

If $\sigma = 0$, then $W^r_{p,\sigma,\gamma}(\mathbb{R}^+_1 \times G)$ is denoted by $W^r_{p,\gamma}(\mathbb{R}^+_1 \times G)$, i.e. $W^r_{p,\gamma}(\mathbb{R}^+_1 \times G)$ is a Sobolev space with weight $e^{-\gamma t}$.

Let $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha_{\min} = \min(\alpha_1, \ldots, \alpha_n)$, $p' = p/(p-1)$. We define vectors $s = (s_0, s_1, \ldots, s_n)$ and $r = (r_0, r_1, \ldots, r_n)$ by

$$s_0 = \begin{cases} 1/\alpha_0 - l & \text{for } \alpha_0 > 0, \\ l & \text{for } \alpha_0 = 0, \end{cases} \qquad s_j = \begin{cases} 0 & \text{for } \alpha_0 > 0, \\ 1/\alpha_j & \text{for } \alpha_0 = 0, \end{cases}$$

$$r_0 = s_0 + l, \qquad r_j = s_j + 1/\alpha_j, \qquad j = 1, \ldots, n.$$

For simplicity we henceforth assume that $f(t,x) = 0$, $x \notin K \subset \mathbb{R}_n$, where K is a compact set.

THEOREM 1. *Suppose equation* (1) *has constant coefficients. Assume* $f(t,x) \in W^s_{p,\gamma}(\mathbb{R}^+_{n+1})$ *and* $D^k_t f|_{t=0} = 0$, $k = 0, \ldots, s_0 - 1$. *If* $|\alpha|/p' + l\alpha_0 > 1$, *then there exists* $\gamma_0 > 0$ *such that the Cauchy problem* (7) *has a unique solution* $u(t,x) \in W^r_{p,\gamma}(\mathbb{R}^+_{n+1})$ *provided* $\gamma \geq \gamma_0$. *Moreover*,

$$\|u, W^r_{p,\gamma}\| \leq c\|f, W^s_{p,\gamma}\|, \tag{8}$$

where $c > 0$ *is a constant depending on* γ_0 *and* $\operatorname{diam} K$.

COROLLARY. *Suppose* (1) *has variable coefficients but* $L_0(x, D_x)$ *has constant coefficients. Suppose that* $f(t,x)$ *satisfies the assumptions of the theorem and* $|\alpha|/p' + l\alpha_0 > 1$. *Then the Cauchy problem* (7) *has a unique solution* $u(t,x) \in W^r_{p,\gamma}(\mathbb{R}^+_{n+1})$ *provided* $\gamma \geq \gamma_0$, *where* $\gamma_0 > 0$ *is sufficiently large. The solution satisfies the estimate* (8).

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied and* $|\alpha|/p' + l\alpha_0 \leq 1$. *Suppose that*

$$\int_{\mathbb{R}_n} x^\beta f(t,x)\,dx = 0, \qquad |\beta| = 0, \ldots, N-1, \tag{9}$$

where $|\alpha|/p' + l\alpha_0 + N\alpha_{\min} > 1 \geq |\alpha|/p' + l\alpha_0 + (N-1)\alpha_{\min}$. *Then there exists* $\gamma_0 > 0$ *such that the Cauchy problem* (7) *is well-posed in the weighted Sobolev spaces* $W^r_{p,\gamma}(\mathbb{R}^+_{n+1})$, $\gamma > \gamma_0$.

THEOREM 3. *Let the assumptions of Theorem* 1 *be satisfied and* $|\alpha| + l\alpha_0 > 1$. *Then there exists* $\gamma_0 > 0$ *such that the Cauchy problem* (7) *has a unique solution* $u(t,x) \in W^r_{p,\sigma,\gamma}(\mathbb{R}^+_{n+1})$ *provided* $\gamma \geq \gamma_0$ *and* $|\alpha|/p > \sigma > 1 - |\alpha|/p' - l\alpha_0$. *The solution satisfies the estimate*

$$\|u, W^r_{p,\sigma,\gamma}\| \leq c\|f, W^s_{p,\gamma}\|, \tag{10}$$

where $c > 0$ *is a constant depending on* γ_0 *and* $\operatorname{diam} K$.

COROLLARY. *Let the assumptions of the Corollary to Theorem* 1 *be satisfied and* $|\alpha| + l\alpha_0 > 1$. *Then the Cauchy problem* (7) *is well-posed in the spaces* $W^r_{p,\sigma,\gamma}(\mathbb{R}^+_{n+1})$, $\gamma > \gamma_0$, $|\alpha|/p > \sigma > 1 - |\alpha|/p' - l\alpha_0$, *where* $\gamma_0 > 0$ *is sufficiently large.*

We illustrate these statements by the example of the Cauchy problem for the equation of small oscillations of a rotating fluid:

$$\Delta D^2_t u - 2\nu\Delta^2 D_t u + \nu^2\Delta^3 u + \omega^2 D^2_{x_3} u = f(t,x), \qquad n = 3,$$
$$u|_{t=0} = 0, \qquad D_t u|_{t=0} = 0,$$

with $\nu \geq 0$.

We recall that this equation belongs to the first class if $\nu = 0$, and to the second class if $\nu > 0$.

If $p > 3$, then this problem has a unique solution $u \in W^r_{p,\gamma}$ for any $f \in C_0^\infty$. This is the Corollary to Theorem 1. However, for $p \le 3$ it is not difficult to show that the Cauchy problem, generally speaking, is unsolvable in $W^r_{p,\gamma}$ [9]. The problem is well-posed for $3/2 < p \le 3$ if $\int_{\mathbb{R}_3} f(t,x)\,dx = 0$, and for $1 < p \le 3/2$ if $\int_{\mathbb{R}_3} f(t,x)\,dx = \int_{\mathbb{R}_3} x_j f(t,x)\,dx = 0$, $j = 1,2,3$. This follows from Theorem 2. Since $|\alpha|+l\alpha_0 > 1$ we may apply Theorem 3, which gives the well-posedness of the Cauchy problem in the weighted Sobolev spaces $W^r_{p,\sigma,\gamma}$ for $p > 1$, $\sigma_1 < \sigma < \sigma_2$.

To finish this section we formulate a statement which shows that the orthogonality conditions (9) are close to being necessary conditions for the solvability of the Cauchy problem (7) in the spaces $W^r_{p,\gamma}$.

THEOREM 4. *Let $f(t,x) \in C_0^\infty(\mathbb{R}^+_{n+1})$ and $\alpha_1 = \ldots = \alpha_n$. If the Cauchy problem* (7) *is well-posed in the spaces $W^r_{p,\gamma}$ for $p \le 2$ then the condition* (9) *holds.*

The proofs of the theorems for $\alpha_0 > 0$ are given in [4], [9]. The case $\alpha_0 = 0$ is proved analogously.

3. Initial boundary value problems. In this section we consider the following initial boundary value problems in the quadrant $\mathbb{R}^{++}_{n+1} = \{t > 0, x_n > 0, x' \in \mathbb{R}_{n-1}\}$ for two classes of equations (1):

$$
(11) \qquad \begin{cases} L(x,D_t,D_x)u = f(t,x), \quad t > 0, x \in \mathbb{R}^+_n, \\ B_j(D_t,D_x)u|_{x_n=0} = 0, \quad j = 1,\ldots,\mu, \\ D_t^k u|_{t=0} = 0, \quad k = 0,\ldots,l-1. \end{cases}
$$

We now define some conditions on the differential operators $B_j(D_t,D_x)$.

First note that from the conditions on the operator $L(x,D_t,D_x)$ there exists $\gamma_1 > 0$ such that for $\operatorname{Re}\tau \ge \gamma_1$, $\xi' \in \mathbb{R}_{n-1} \setminus \{0\}$ the equation $L(x,\tau,i\xi',i\lambda) = 0$ has no real roots. Let

$$
M^+(x;\tau,\xi',\lambda) = \prod_{k=1}^{\mu} (\lambda - \lambda_k^+(x;\tau,\xi'))\,,
$$

where we assume $\lambda_k^+(x;\tau,\xi')$, $k = 1,\ldots,\mu$, are all the roots with positive imaginary part.

I. Assume that the number of the boundary operators at $x_n = 0$ is equal to μ.

II. If we consider the mixed problem for equations of the first class ($\alpha_0 = 0$) then the operators B_j have the form

$$
B_j(D_t,D_x) = b_j(D_t)D_{x_n}^{m_j} + \sum_{k=0}^{m_j-1} B_{j,k}(D_t,D_{x'})D_{x_n}^k\,,
$$

and for the second class ($\alpha_0 > 0$) the operators B_j have the form

$$B_j(D_t, D_x) = D_{x_n}^{m_j} + \sum_{k=0}^{m_j-1} B_{j,k}(D_{x'})D_{x_n}^k .$$

Now assume only that the symbols $B_j(i\eta, i\xi)$ are homogeneous with respect to the vector $\overrightarrow{\alpha} = (\alpha_0, \alpha)$, i.e.

$$B_j(c^{\alpha_0} i\eta, c^{\alpha} i\xi) = c^{\beta_j} B_j(i\eta, i\xi), \quad c > 0 ,$$

where $0 \le \beta_j < 1$, $j = 1, \dots, \mu$.

III. We suppose that the Lopatinskiĭ condition holds. This means that $B_j(i\eta, i\xi)$, $j = 1, \dots, \mu$, are linearly independent modulo $M^+(x;\tau,\xi',\lambda)$ for $x \in \mathbb{R}_n^+$, $\operatorname{Re}\tau \ge \gamma_1$, $\xi' \in \mathbb{R}_{n-1} \setminus \{0\}$, i.e. $\det b_{j,k}(x;\tau,\xi') \neq 0$, where $b_{j,k}(x;\tau,\xi')$ is defined by

$$\sum_{k=1}^{\mu} b_{j,k}(x;\tau,\xi')(i\lambda)^{k-1} \equiv B_j(i\tau, i\xi', i\lambda) \; (\operatorname{mod} M^+(x;\tau,\xi',\lambda)) .$$

As an example, we discuss some mixed problems, namely the initial boundary value problems in a quadrant of the space $\{t > 0, x_k > 0\}$ for the Sobolev equation (3), the equation of internal waves (5) and the equation of small oscillations of a viscous rotating fluid (4). For simplicity, we restrict ourselves to $x_k = x_3$.

As boundary condition for Sobolev's equation we require one relation at $x_3 = 0$. For the first initial boundary value problem the boundary operator has the form $B_1(D_t, D_x) = 1$ ($\beta_1 = 0$), but for the second initial boundary value problem $B_1(D_t, D_x) = D_t^2 D_{x_3} + \omega^2 D_{x_3}$ ($\beta_1 = 1/2$).

For the equation of internal waves we also require one boundary condition at $x_3 = 0$. For the first initial boundary value problem $B_1(D_t, D_x) = 1$ ($\beta_1 = 0$), but for the second initial boundary value problem $B_1(D_t, D_x) = D_t^2 D_{x_3}$ ($\beta_1 = 1/2$).

For the equation of small oscillations of a rotating viscous fluid we require three boundary conditions. In the case of the first initial boundary value problem the corresponding boundary operators have the form $B_j(D_t, D_x) = D_{x_3}^{j-1}$, $\beta_j = (j-1)/6$, $j = 1, 2, 3$.

Let $\alpha_{\min}$ and the vectors s, r be as defined in Section 2. For simplicity we henceforth assume that $f(t,x) \equiv 0$ for $x \notin K$, where $K \subset \mathbb{R}_n^+$ is compact.

THEOREM 5. *Suppose equation* (1) *has constant coefficients. Assume* $f(t,x) \in W_{p,\gamma}^s(\mathbb{R}_{n+1}^{++})$ *and* $D_t^k f|_{t=0} = 0$, $k = 0, \dots, s_0 - 1$. *If* $|\alpha|/p' + l\alpha_0 > 1$, *then there exists* $\gamma_0 > \gamma_1$ *such that the mixed problem* (11) *has a unique solution* $u(t,x) \in W_{p,\gamma}^r(\mathbb{R}_{n+1}^{++})$ *provided* $\gamma \ge \gamma_0$, *and for the solution the estimate* (8) *holds.*

COROLLARY. *Suppose* (1) *has variable coefficients but* $L_0(x, D_x)$ *has constant coefficients. Suppose that* $f(t,x)$ *satisfies the assumptions of the theorem and* $|\alpha|/p' + l\alpha_0 > 1$. *Then the initial boundary value problem* (11) *is well-posed in the weighted Sobolev spaces* $W_{p,\gamma}^r(\mathbb{R}_{n+1}^{++})$, $\gamma > \gamma_0$, *where* $\gamma_0 > \gamma_1$ *is sufficiently large.*

THEOREM 6. *Let the assumptions of Theorem* 5 *be satisfied and* $|\alpha|/p' + l\alpha_0 \leq 1$. *Suppose that*

$$\int_{\mathbb{R}_n^+} x^\beta f(t,x)\,dx = 0, \qquad |\beta| = 0,\ldots,N-1\,,$$

where $|\alpha|/p'+l\alpha_0+N\alpha_{\min}>1\geq|\alpha|/p'+l\alpha_0+(N-1)\alpha_{\min}$. *Then there exists* $\gamma_0 > \gamma_1$ *such that the mixed problem* (11) *has a unique solution* $u(t,x) \in W^r_{p,\gamma}(\mathbb{R}^{++}_{n+1})$ *provided* $\gamma \geq \gamma_0$. *The solution satisfies the estimate* (8).

THEOREM 7. *Let the assumptions of Theorem* 5 *be satisfied and* $|\alpha| + l\alpha_0 > 1$. *Then there exists* $\gamma_0 > \gamma_1$ *such that the initial boundary value problem* (11) *has a unique solution* $u(t,x) \in W^r_{p,\sigma,\gamma}(\mathbb{R}^{++}_{n+1})$ *provided* $\gamma \geq \gamma_0$ *and* $|\alpha|/p > \sigma > 1 - |\alpha|/p' - l\alpha_0$. *The solution satisfies the estimate* (10).

COROLLARY. *Let the assumptions of the Corollary to Theorem* 5 *be satisfied and* $|\alpha| + l\alpha_0 > 1$. *Then the mixed problem* (11) *is well-posed in the spaces* $W^r_{p,\sigma,\gamma}(\mathbb{R}^{++}_{n+1})$, $\gamma > \gamma_0$, $|\alpha|/p > \sigma > 1 - |\alpha|/p' - l\alpha_0$, *where* $\gamma_0 > \gamma_1$ *is sufficiently large.*

The statements of Theorems 5–7 for $\alpha_0 = 0$ strengthen the results of the author [10]. For $\alpha_0 > 0$ these results are new.

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