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COEFFICIENTS OF THE SINGULARITIES ON DOMAINS WITH CONICAL POINTS

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Abstract. As a model for elliptic boundary value problems, we consider the Dirichlet problem for an elliptic operator. Solutions have singular expansions near the conical points of the domain. We give formulas for the coefficients in these expansions.

1. Introduction. We consider bounded n-dimensional domains with conical points, as Kondrat'ev in [4]. For simplicity, we suppose that there is only one conical point and that it is located at 0. We denote by Ω our domain and we assume that its boundary is C^∞ outside 0 and that it coincides with a cone Γ in a neighborhood of 0. We denote by x the cartesian coordinates in $\mathbb{R}^n$ and by (r,θ) the spherical coordinates. The spherical section of Γ is denoted by G:

$$\Gamma \cap S^{n-1} =: G\,.$$

We are interested in the Dirichlet boundary value problem for an elliptic operator $P(x;D_x)$ of order $2m$. We assume that the coefficients of this operator are $C^\infty(\overline{\Omega}\setminus 0)$. We have to sharpen this assumption. We will consider three cases $(\mathcal{C}1)$, $(\mathcal{C}2)$ and $(\mathcal{C}3)$, each of them being more general than the previous one:

- $(\mathcal{C}1)$: P is homogeneous with constant coefficients; then there exists an operator $\mathcal{L}$ with $C^\infty(\overline{G})$ coefficients such that

$$P(D_x) = r^{-2m}\mathcal{L}(\theta; r\partial_r, \partial_\theta)\,.$$

- $(\mathcal{C}2)$: P has $C^\infty(\overline{\Omega})$ coefficients; then, if L denotes the principal part of

$P(0; D_x)$, then L satisfies the assumption of $(\mathcal{C}1)$ and the difference

$$R(x; D_x) := P(x; D_x) - L(D_x)$$

is a remainder.

- $(\mathcal{C}3)$: there exists an operator $\mathcal{L}$ with $C^\infty(\overline{G})$ coefficients such that the difference

$$R(x; D_x) := P(x; D_x) - r^{-2m}\mathcal{L}(\theta; r\partial_r, \partial_\theta)$$

is a remainder in a sense we are going to explain.

The Coulomb operator $-\Delta + 1/r$ satisfies the assumptions of $(\mathcal{C}3)$. To explain what we mean by remainder, we need some weighted Sobolev spaces.

As usual, $\overset{\circ}{H}{}^m(\Omega)$ denotes the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$ and $H^{-m}(\Omega)$ is its dual space. For any positive integer k and any real β, $H^k_\beta(\Omega)$ is defined as

$$H^k_\beta(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid r^{\beta-k+|\alpha|} D^\alpha_x u \in L^2(\Omega)\ \forall \alpha, |\alpha| \le k\}\,.$$

We also define $H^s_\beta(\Omega)$ for any positive real s in a natural way (cf. for instance appendix A in [1]), and for any negative s by duality.

For any $s > 0$ and any β, the operators P and L in the case $(\mathcal{C}2)$, and $r^{-2m}\,\mathcal{L}$ in the cases $(\mathcal{C}1)$ and $(\mathcal{C}3)$ are continuous $H^{s+m}_\beta(\Omega) \to H^{s-m}_\beta(\Omega)$. Moreover, in the case $(\mathcal{C}2)$, the remainder R is continuous $H^{s+m}_\beta(\Omega) \to H^{s-m}_{\beta-1}(\Omega)$.

Now, the assumption in the case $(\mathcal{C}3)$ is that there exists $\delta \in\,]0, 1]$ such that for any $s \ge 0$ and β the remainder R is continuous

$$H^{s+m}_\beta(\Omega) \to H^{s-m}_{\beta-\delta}(\Omega)\,. \tag{1.1}$$

If a_α denote the coefficients of R:

$$R(x; D_x) = \sum_{|\alpha| \le 2m} a_\alpha\, D^\alpha_x\,,$$

the assumption (1.1) holds if

$$\forall \gamma \in \mathbb{N}^n, \qquad D^\gamma_x a_\alpha \text{ is } \mathcal{O}(r^{|\alpha|-2m+\delta-|\gamma|})\,.$$

With the above assumptions, we are interested in the structure of any solution u of the Dirichlet problem

$$u \in \overset{\circ}{H}{}^m(\Omega),\ Pu \in H^{s-m}_\beta(\Omega), \qquad \text{with } s > 0 \text{ and } s - \beta > 0\,. \tag{1.2}$$

Since $s > 0$ and $s - \beta > 0$, $H^{s-m}_\beta(\Omega)$ is (compactly) embedded in $H^{-m}(\Omega)$. So Pu has in a sense more regularity than u. Of course, it is possible to consider more general situations than (1.2), i.e. to assume that u belongs to some weighted space. The solution of this problem would be essentially the same as for (1.2). We chose (1.2), because it is the natural framework when P is strongly elliptic.

The assumptions of the case $(\mathcal{C}2)$ are those of Kondrat'ev [4]. We also took these assumptions in our earlier works [2] and [3]. The assumptions of the case $(\mathcal{C}3)$ were introduced by Maz'ya and Plamenevskiĭ [7]. Kondrat'ev proved the existence of an expansion of the solutions of (1.2) in the form of a sum $\sum_i c_i S_i$ where the S_i only depend on Ω and P and the c_i are some coefficients. Maz'ya and

Plamenevskiĭ in [6] and [7] gave formulas for these coefficients; we also studied them in [2] and [3] in a different framework. What we give here extends in a certain sense [7] and [2].

To end this section, let us state a Fredholm theorem. Such a result is related to asymptotics of solutions: a solution of (1.2) can be split into a singular part (asymptotics) and a regular part (remainder) when the assumptions of the following theorem hold.

We need some notations. We denote by P^s_β the operator

$$P^s_\beta : \mathring{H}^m_{\beta-s}(\Omega) \cap H^{s+m}_\beta(\Omega) \to H^{s-m}_\beta(\Omega), \qquad u \mapsto Pu\,,$$

where $\mathring{H}^m_\gamma(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^m_\gamma(\Omega)$. We will simply denote by P_β the operator P^0_β.

$P^*_{-\beta}$ denotes the adjoint of P_β. It acts

$$P^*_{-\beta} : \mathring{H}^m_{-\beta}(\Omega) \to H^{-m}_{-\beta}(\Omega)\,.$$

For any $\lambda \in \mathbb{C}$, $\mathcal{L}(\lambda)$ is the operator

$$\mathcal{L}(\theta; \lambda, \partial_\theta) : \mathring{H}^m(G) \to H^{-m}(G)\,.$$

$\mathcal{L}(\lambda)$ is one-to-one except when λ belongs to a countable set in $\mathbb{C}$, which can be called the *spectrum* of $\mathcal{L}$ and is denoted by $\mathrm{Sp}(\mathcal{L})$.

THEOREM 1.1. *In the case* ($\mathcal{C}3$), *assume that* $s \ge 0$, $\beta \in \mathbb{R}$, $s - \beta \ge 0$ *and that*

$$\forall \lambda \text{ such that } \operatorname{Re}\lambda = s + m - \beta - n/2, \qquad \lambda \notin \mathrm{Sp}(\mathcal{L})\,.$$

Then P^s_β *is a Fredholm operator.*

2. The model problem. In this section, we will only study the case ($\mathcal{C}1$), when the operator is homogeneous with constant coefficients. We recall that then $P = L$.

For each $\lambda \in \mathrm{Sp}(\mathcal{L})$, the space

$$\mathcal{Z}^\lambda := \Big\{ u \;\Big|\; u = r^\lambda \sum_q \operatorname{Log}^q r\, u_q(\theta),\ u_q \in \mathring{H}^m(G),\ Lu = 0 \Big\}$$

does not reduce to 0: all functions of the form $r^\lambda u_0$ where $u_0 \in \operatorname{Ker}\mathcal{L}(\lambda)$ are in $\mathcal{Z}^\lambda$. Let σ^λ_ν, for $\nu = 1, \dots, N^\lambda$, denote a basis of this space.

THEOREM 2.1. *In the case* ($\mathcal{C}1$), *assume the same hypotheses about* s *and* β *as in Theorem* 1.1. *Let* η *be a cut-off function which is equal to* 1 *in a neighborhood of* 0 *and has its support in another neighborhood of* 0 *where* Ω *coincides with the cone* Γ. *Assume that* $u \in \mathring{H}^m(\Omega)$ *is such that* $Pu \in H^{s-m}_\beta(\Omega)$. *Then there exist* *coefficients* c_ν^λ *such that*

$$u - \sum_{\substack{\lambda\in\mathrm{Sp}(\mathcal{L})\\ m-n/2<\mathrm{Re}\,\lambda<s-\beta+m-n/2}} \sum_{\nu=1}^{N^\lambda} c_\nu^\lambda \eta\sigma_\nu^\lambda \in H_\beta^{s+m}(\Omega)\,.$$

If $\operatorname{Ker} P_0 \subseteq H_\beta^{s+m}(\Omega)$, then the coefficients c_ν^λ only depend on Pu. This is the reason for the introduction of the following assumption.

(2.1) If $u \in \overset{\circ}{H}{}^m(\Omega)$ is such that $Pu = 0$ then $u \in H_{\beta-s}^m(\Omega)$.

If (2.1) holds, as a consequence of a well-known regularity result for corner problems, such an element of the kernel belongs to any space $H_{\beta-s+t}^{m+t}(\Omega)$ for any $t \ge 0$ (see Theorem 2.9 of [2] for instance).

Now, we are going to construct dual singular functions. We start with a result from [6].

Lemma 2.2. *For all* $\lambda \in \mathrm{Sp}(\mathcal{L})$, *there exists a basis* τ_ν^λ, *for* $\nu = 1, \ldots, N^\lambda$, *of the space*

$$\widetilde{\mathcal{Z}}^\lambda := \Big\{u \;\Big|\; u = r^{-\bar\lambda+2m-n} \sum_q \mathrm{Log}^q\, r v_q(\theta),\ v_q \in \overset{\circ}{H}{}^m(G),\ L^* u = 0\Big\}$$

such that $\forall \mu, \mu' \in \mathrm{Sp}(\mathcal{L})$, $\forall \nu, \nu'$

$$\int_\Omega L(\eta\sigma_\nu^\mu)\overline{\tau_{\nu'}^{\mu'}} = \delta_{\mu\mu'}\delta_{\nu\nu'}\,.$$

See [6] and [2] for more details.

We set $T_\nu^\lambda := \eta\tau_\nu^\lambda$. We have

$$\forall\lambda,\ m-n/2 < \mathrm{Re}\,\lambda < s-\beta+m-n/2\,,$$
$$T_\nu^\lambda \notin H_0^m(\Omega) \text{ and } T_\nu^\lambda \in \overset{\circ}{H}{}^m_{s-\beta}(\Omega).$$

Due to assumption (2.1), there exists $Y_\nu^\lambda \in \overset{\circ}{H}{}^m(\Omega)$ such that $P^* Y_\nu^\lambda = P^* T_\nu^\lambda$. We set

$$K_\nu^\lambda := T_\nu^\lambda - Y_\nu^\lambda\,.$$

Theorem 2.3. *In the case* ($\mathcal{C}1$) *and with the hypothesis* (2.1), *assume the same hypotheses about s and β as in Theorems* 1.1 *and* 2.1. *Then*

$$c_\nu^\lambda = \int_\Omega Pu\overline{K_\nu^\lambda}\,dx\,.$$

When $P = \Delta$ and when Γ is a plane sector with opening ω, the σ_ν^λ are the functions $r^{k\pi/\omega}\sin(k\pi\theta/\omega)$ for $k \in \mathbb{N}^*$ and the τ_ν^λ are the functions $-(k\pi)^{-1} \times r^{-k\pi/\omega}\sin(k\pi\theta/\omega)$.

3. The general problem. Now, we will work in the framework of the general case ($\mathcal{C}3$). All the above results can be extended in a certain sense to the case

$(\mathcal{C}3)$. We are going to introduce auxiliary functions. In the case $(\mathcal{C}2)$, the structure of these functions is more precisely known.

First, we construct elements of the kernel of P^* in the same way by subtracting a corrective function Y_ν^λ from T_ν^λ. The difference lies in the construction of Y_ν^λ. They cannot be found in $\mathring{H}^m(\Omega)$ in general but in a larger space.

PROPOSITION 3.1. *In the case* $(\mathcal{C}3)$ *and with the hypothesis* (2.1), *let* $\lambda \in \mathrm{Sp}(\mathcal{L})$ *such that* $m - n/2 < \mathrm{Re}\,\lambda$. *Set* $\gamma(\lambda) := \mathrm{Re}\,\lambda - m + n/2$. *Then* $\forall \varepsilon > 0$, *we have*

$$T_\nu^\lambda \in H^m_{\gamma(\lambda)+\varepsilon}(\Omega) \quad \textit{and} \quad T_\nu^\lambda \notin H^m_{\gamma(\lambda)}(\Omega)\,. \tag{3.1}$$

Let $\delta' = \min\{\delta, \gamma(\lambda)\}$, *where* δ *was introduced in* (1.1). *Then there exists* Y_ν^λ *which satisfies the homogeneous Dirichlet conditions and such that*

$$P^* T_\nu^\lambda = P^* Y_\nu^\lambda \quad \textit{and} \quad \forall \varepsilon > 0,\ Y_\nu^\lambda \in H^m_{\gamma(\lambda)-\delta'+\varepsilon}(\Omega)\,.$$

Set $K_\nu^\lambda := T_\nu^\lambda - Y_\nu^\lambda$. *The* K_ν^λ *for* $\lambda \in \mathrm{Sp}(\mathcal{L})$, $m - n/2 < \mathrm{Re}\,\lambda < s - \beta + m - n/2$ *and for* $\nu = 1, \ldots, N^\lambda$ *form a basis of* $\mathrm{Ker}\,P^*_{s-\beta}/\,\mathrm{Ker}\,P^*_0$.

Remark 3.2. In the case $(\mathcal{C}2)$, the Y_ν^λ can be constructed as a sum of terms

$$T^\lambda_{\nu,j} = \eta r^{-\overline{\lambda}+2m-n+j} \sum \mathrm{Log}^q\, r v^\lambda_{\nu,j,q}(\theta)$$

with $1 \le j \le \mathrm{Re}\,\lambda - m + n/2$ and of an element $X_\nu^\lambda \in \mathring{H}^m(\Omega)$. In the case $(\mathcal{C}1)$, the $T^\lambda_{\nu,j}$ are zero and $Y_\nu^\lambda = X_\nu^\lambda$ (see §4 of [2]).

Proof. *First step.* Let us prove the existence of Y_ν^λ. By construction, $L^* T_\nu^\lambda = 0$ in a neighborhood of 0; as a consequence of the assumption (1.1), $P^* T_\nu^\lambda \in H^{-m}_{\gamma(\lambda)+\varepsilon-\delta'}(\Omega)$. We want to prove that

$$P^* T_\nu^\lambda \in \mathrm{Rg}\, P^*_{\gamma(\lambda)+\varepsilon-\delta'}\,. \tag{3.2}$$

But the regularity of T_ν^λ yields $P^* T_\nu^\lambda \in \mathrm{Rg}\, P^*_{\gamma(\lambda)+\varepsilon}$. We choose ε small enough such that the ranges of $P^*_{\gamma(\lambda)+\varepsilon}$ and of $P^*_{\gamma(\lambda)+\varepsilon-\delta'}$ are closed. We have

$$\mathrm{Rg}\, P^*_{\gamma(\lambda)+\varepsilon} = (\mathrm{Ker}\, P_{-\gamma(\lambda)-\varepsilon})^\perp, \qquad \mathrm{Rg}\, P^*_{\gamma(\lambda)+\varepsilon-\delta'} = (\mathrm{Ker}\, P_{-\gamma(\lambda)-\varepsilon+\delta'})^\perp.$$

The hypothesis (2.1) yields that $\mathrm{Ker}\, P_{-\gamma(\lambda)-\varepsilon} = \mathrm{Ker}\, P_{-\gamma(\lambda)-\varepsilon+\delta'}$. So, we have obtained (3.2).

Second step. Let us prove that the K_ν^λ are linearly independent modulo $\mathring{H}^m(\Omega)$. Suppose that there exist nonzero coefficients c_ν^λ such that $\sum c_\nu^\lambda K_\nu^\lambda \in \mathring{H}^m(\Omega)$. Let ξ be the largest real part of the λ which are associated with a nonzero coefficient. Since the whole sum belongs to $\mathring{H}^m(\Omega)$, we deduce by construction of the K_ν^λ that

$$\exists \varrho > 0, \qquad \sum_{\mathrm{Re}\,\lambda = \xi} c_\nu^\lambda T_\nu^\lambda \in H^m_{\xi-m+n/2-\varrho}(\Omega)\,.$$

The form of the T_ν^λ (cf. (3.1)) allows us to show that the coefficients in the above sum are all zero. We have obtained a contradiction.

Third step. Let γ be $s-\beta$ and let n_γ be the cardinal of the set

$$\{K_\nu^\lambda \mid m-n/2<\operatorname{Re}\lambda<s-\beta+m-n/2 \text{ and } \nu=1,\dots,N^\lambda\}\,.$$

We have to show that the $\dim\operatorname{Ker}P_\gamma^* = \dim\operatorname{Ker}P_0^* + n_\gamma$. We rely on an index calculus. Choose $\gamma_0,\dots,\gamma_J$ such that

$$\begin{gathered}
0\le\gamma_0\le\dots\le\gamma_J=\gamma\,,\\
\forall j=1,\dots,J,\qquad \gamma_j-\gamma_{j-1}\le\delta\,,\\
\operatorname{Sp}(\mathcal{L})\cap\{\lambda\in\mathbb{C}\mid m-n/2<\operatorname{Re}\lambda<m-n/2+\gamma_0\}=\emptyset\,,\\
\forall j=1,\dots,J,\qquad \operatorname{Sp}(\mathcal{L})\cap\{\lambda\in\mathbb{C}\mid \operatorname{Re}\lambda=m-n/2+\gamma_j\}=\emptyset.
\end{gathered}\tag{3.3}$$

For each $j=1,\dots,J$, the functions $u\in\mathring{H}^m_{-\gamma_{j-1}}(\Omega)$ such that $Lu\in H^{-m}_{-\gamma_j}(\Omega)$ can be written as a sum of a regular part in $H^m_{-\gamma_j}(\Omega)$ and a singular part which is a combination of the $\eta\sigma_\nu^\lambda$ with $\lambda\in\operatorname{Sp}(\mathcal{L})$ and $m-n/2+\gamma_{j-1}<\operatorname{Re}\lambda<m-n/2+\gamma_j$. Due to (1.1), the same holds for the operator P. Applying the result of appendix B of [1] for each pair $(P_{-\gamma_{j-1}},P_{-\gamma_j})$ and summing over $j=1,\dots,J$, we get

$$\operatorname{Ind}P_{-\gamma_0}-\operatorname{Ind}P_{-\gamma}=n_\gamma\,.$$

As a consequence of the assumption (2.1), $\operatorname{Ker}P_{-\gamma_0}=\operatorname{Ker}P_{-\gamma}$. Then

$$\operatorname{Codim}\operatorname{Rg}P_{-\gamma}-\operatorname{Codim}\operatorname{Rg}P_{-\gamma_0}=n_\gamma\,.$$

So for the adjoints, we get

$$\dim\operatorname{Ker}P_\gamma^*-\dim\operatorname{Ker}P_{\gamma_0}^*=n_\gamma\,.$$

We end the proof by noting that the construction of γ_0 implies $\operatorname{Ker}P_{\gamma_0}^*=\operatorname{Ker}P_0^*$. ■

We are now going to construct the singularities, i.e. a basis of functions belonging to $\mathring{H}^m(\Omega)$, which are not in $H^{s+m}_\beta(\Omega)$ and such that $Pu\in H^{s-m}_\beta(\Omega)$. In the case $(\mathcal{C}1)$, such a basis is formed by the $\eta\sigma_\nu^\lambda$ (cf. Theorem 2.1). Such a result extends to the case $(\mathcal{C}3)$ only if $s-\beta\le\delta$. Let us state that with $s=0$:

LEMMA 3.3. *In the case* $(\mathcal{C}3)$, *let* τ *and* τ' *be such that* $0<\tau'-\tau\le\delta$. *Assume that*

$$\operatorname{Sp}(\mathcal{L})\cap\{\lambda\in\mathbb{C}\mid\operatorname{Re}\lambda=m-n/2+\tau\}=\emptyset$$

and that $u\in\mathring{H}^m_{\tau'}(\Omega)$ *is such that* $Pu\in H^{-m}_\tau(\Omega)$. *Then there exist coefficients* c_ν^λ *such that*

$$u-\sum_{\substack{\lambda\in\operatorname{Sp}(\mathcal{L})\\ m-n/2-\tau'<\operatorname{Re}\lambda<m-n/2-\tau}}\ \sum_{\nu=1}^{N^\lambda}c_\nu^\lambda\eta\sigma_\nu^\lambda\in H^m_\tau(\Omega)\,.$$

As we already explained in the above proof, this is a simple consequence of the assumptions of (1.1) and of the corresponding result for L which is known [4].

In the case $(\mathcal{C}2)$, when $s-\beta>1$, $P(\eta\sigma_\nu^\lambda)$ does not belong to $H^{s-m}_\beta(\Omega)$ in general but there exist

$$\sigma_{\nu,j}^\lambda=r^{\lambda+j}\sum\operatorname{Log}^q r\,u_{\nu,j,q}^\lambda(\theta)$$

where $u^\lambda_{\nu,j,q} \in \mathring{H}^m(G)$ and such that

$$P\Big[\eta\Big(\sigma^\lambda_\nu + \sum_{1\le j\le s-\beta+m-n/2} \sigma^\lambda_{\nu,j}\Big)\Big] \in H^{s-m}_\beta(\Omega)$$

(see §4.B of [2]).

In the general case ($\mathcal{C}3$), we have another construction, which is less explicit, as in the previous Proposition 3.1.

PROPOSITION 3.4. *In the case* ($\mathcal{C}3$) *and with the hypothesis* (2.1), *let* $\lambda \in \mathrm{Sp}(\mathcal{L})$ *such that* $m - n/2 < \mathrm{Re}\,\lambda$. *With the notation of Proposition* 3.1 *for all* $\varepsilon > 0$, *we have*

$$\eta\sigma^\lambda_\nu \in H^m_{-\gamma(\lambda)+\varepsilon}(\Omega) \quad \textit{and} \quad \eta\sigma^\lambda_\nu \notin H^m_{-\gamma(\lambda)}(\Omega)\,. \tag{3.4}$$

Then there exists Z^λ_ν *which satisfies the homogeneous Dirichlet conditions and such that*

$$P(\eta\sigma^\lambda_\nu - Z^\lambda_\nu) \in C^\infty_0(\overline{\Omega}\setminus 0) \quad \textit{and} \quad \forall\varepsilon>0,\ Z^\lambda_\nu \in H^m_{-\gamma(\lambda)-\delta+\varepsilon}(\Omega)\,.$$

Set $S^\lambda_\nu := \eta\,\sigma^\lambda_\nu - Z^\lambda_\nu$ *and* $F^\lambda_\nu := PS^\lambda_\nu$. *The* F^λ_ν *for* λ, ν *satisfying*

$$\lambda \in \mathrm{Sp}(\mathcal{L}),\ m - n/2 < \mathrm{Re}\,\lambda < s - \beta + m - n/2 \ \textit{and}\ \nu = 1,\dots,N^\lambda \tag{3.5}$$

form a basis of $(H^{s-m}_\beta(\Omega) \cap \mathrm{Rg}\,P_0)/\,\mathrm{Rg}\,P^s_\beta$.

Proof. *First step.* As a consequence of Proposition 3.1, for any $u \in \mathring{H}^m(\Omega)$ such that $Pu \in H^{-m}_{\beta-s}(\Omega)$, the following equivalence holds:

$$u \in H^m_{\beta-s}(\Omega) \Leftrightarrow \forall\lambda,\nu \text{ as in (3.5)},\ \langle Pu,\, K^\lambda_\nu\rangle = 0\,.$$

Due to a classical regularity result for corner problems (see for instance the statement given in [2], p. 33), if $u \in \mathring{H}^m_{\beta-s}(\Omega)$ satisfies $Pu \in H^{s-m}_\beta(\Omega)$, then $u \in H^{s+m}_\beta(\Omega)$. Thus we only have to consider the above equivalence.

Since the K^λ_ν are *functions* as well as all elements of the kernel of any operator P^*_τ, there exist $\widetilde{F}^\lambda_\nu \in C^\infty_0(\overline{\Omega}\setminus 0) \cap \mathrm{Rg}\,P_0$ such that

$$\forall\lambda,\nu \text{ and } \lambda',\nu' \text{ as in (3.5)}, \quad \langle \widetilde{F}^\lambda_\nu, K^{\lambda'}_{\nu'}\rangle = \delta_{\lambda,\lambda'}\delta_{\nu,\nu'}\,.$$

Let $\widetilde{S}^\lambda_\nu \in \mathring{H}^m(\Omega)$ be such that $P\widetilde{S}^\lambda_\nu = \widetilde{F}^\lambda_\nu$. We now have to construct the S^λ_ν satisfying the assertions of Proposition 3.4 as linear combinations of the $\widetilde{S}^\lambda_\nu$.

Second step. We use again the γ_j satisfying (3.3) we have introduced in the previous proof. Applying Lemma 3.3 for $\tau = -\gamma_1$ and $\tau' = -\gamma_0$, we deduce that the $\widetilde{S}^\lambda_\nu$ for $m - n/2 + \gamma_0 < \mathrm{Re}\,\lambda < m - n/2 + \gamma_1$ generate

$$\{\eta\,\sigma^\lambda_\nu \mid m - n/2 + \gamma_0 < \mathrm{Re}\,\lambda < m - n/2 + \gamma_1\}$$

modulo $\mathring{H}^m_{-\gamma_1}(\Omega)$. Thus, for any such λ, there exist $Z^\lambda_\nu \in \mathring{H}^m_{-\gamma_1}(\Omega)$ such that $\eta\sigma^\lambda_\nu - Z^\lambda_\nu$ is a linear combination of the $\widetilde{S}^{\lambda'}_{\nu'}$. So, the functions S^λ_ν are constructed for $m - n/2 + \gamma_0 < \mathrm{Re}\,\lambda < m - n/2 + \gamma_1$.

For the next step, corresponding to the weights $-\gamma_1$ and $-\gamma_2$, we use the same arguments where we replace the $\widetilde{S}_\nu^\lambda$ for $m-n/2+\gamma_1<\operatorname{Re}\lambda<m-n/2+\gamma_2$ by the functions

$$\widetilde{S}_\nu^\lambda - \sum_{m-n/2+\gamma_0<\operatorname{Re}\lambda'<m-n/2+\gamma_1} d_{\nu'}^{\lambda'}\, S_{\nu'}^{\lambda'}$$

where, according to Lemma 3.3, the coefficients $d_{\nu'}^{\lambda'}$ are chosen such that all the above functions belong to $H_{-\gamma_1}^m(\Omega)$.

Step by step, we reach $\gamma_J=\gamma$ and our S_ν^λ are independent and their number is n_γ, which is what we need. ∎

Now it is not too difficult to deduce from the two previous propositions and from the Green formula the three following statements.

With the functions S_ν^λ we have just constructed, we have the extension of Theorem 2.1 to the case $(\mathcal{C}3)$.

THEOREM 3.5. *In the case* $(\mathcal{C}3)$ *and with the hypothesis* (2.1), *assume the same hypotheses about* s *and* β *as in Theorems* 1.1 *and* 2.1. *Assume that* $u\in \mathring{H}^m(\Omega)$ *is such that* $Pu\in H_\beta^{s-m}(\Omega)$. *Then there exist coefficients* c_ν^λ *such that*

$$u-\sum_{\substack{\lambda\in\operatorname{Sp}(\mathcal{L})\\ m-n/2<\operatorname{Re}\lambda<s-\beta+m-n/2}}\ \sum_{\nu=1}^{N^\lambda} c_\nu^\lambda S_\nu^\lambda \in H_\beta^{s+m}(\Omega)\,.$$

As a result of the previous constructions, we have some independent functions $\widetilde{S}_\nu^\lambda$ such that

$$\langle P\widetilde{S}_\nu^\lambda,\, K_{\nu'}^{\lambda'}\rangle=\delta_{\lambda,\lambda'}\,\delta_{\nu,\nu'}\,,$$

and the singularities S_ν^λ are a basis of the space generated by the $\widetilde{S}_\nu^\lambda$. Thus, we can show

LEMMA 3.6. *Under the assumptions of Theorem* 3.5, *there exists a basis* $\widetilde{K}_\nu^\lambda$ *of the space generated by the* K_ν^λ *for* $m-n/2<\operatorname{Re}\lambda<s-\beta+m-n/2$ *such that*

$$\langle PS_\nu^\lambda, \overline{\widetilde{K}_{\nu'}^{\lambda'}}\rangle=\delta_{\lambda,\lambda'}\delta_{\nu,\nu'}\,.$$

The $\widetilde{K}_\nu^\lambda$ *have the form*

$$\widetilde{K}_\nu^\lambda=\sum_{\operatorname{Re}\lambda'\geq\operatorname{Re}\lambda} d_{\nu'}^{\lambda'}\,K_{\nu'}^{\lambda'}\,.$$

With these new elements of the kernel of P^* we have

THEOREM 3.7. *Under the assumptions of Theorem* 3.5

$$c_\nu^\lambda=\int_\Omega Pu\overline{\widetilde{K}_\nu^\lambda}\,dx\,.$$

The above results have to be compared with the following statements of [7]: Corollaries 3.1 and 3.2, Theorems 3.3 and 3.4. Our hypothesis (2.1) is more general than the hypothesis of [7], which in our framework would correspond to

$$P \text{ is one-to-one } \mathring{H}^m(\Omega) \to H^{-m}(\Omega)\,.$$

The paper [5] gives similar expressions for the coefficients of the singularities in a different framework.

In the case $(\mathcal{C}2)$, under the extra assumption

$$\text{(3.6)}\qquad \forall\lambda,\lambda' \in \mathrm{Sp}(\mathcal{L}) \text{ such that } \operatorname{Re}\lambda, \operatorname{Re}\lambda' \in\,]m-n/2, s-\beta+m-n/2[\,, \quad \lambda-\lambda' \notin \mathbb{N}\setminus 0\,,$$

the $\widetilde{K}_\nu^\lambda$ and the K_ν^λ coincide and the formula for the coefficients is the same as in Theorem 2.3:

$$c_\nu^\lambda = \int_\Omega Pu\overline{K_\nu^\lambda}\,dx\,.$$

In the case $(\mathcal{C}2)$, it is natural to consider P as an operator acting between ordinary Sobolev spaces:

$$P : H^{s+m}(\Omega) \cap \mathring{H}^m(\Omega) \to H^{s-m}(\Omega)\,.$$

The above formulas have no longer any sense in general because Pu is not flat enough. In [2], we have proved formulas for the coefficients, where we subtract from Pu some function whose Taylor expansion at 0 is the same as the Taylor expansion of Pu in 0.

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