PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 27
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1992

ON A CLASS OF NONLINEAR ELLIPTIC EQUATIONS

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1. Introduction. Let Ω be a bounded domain of $\mathbb{R}^n$ with boundary Γ, $n \geq 1$. The goal of this note is to summarize results regarding existence and number of solutions of the equation

$$\begin{cases} \Delta\varphi - |\nabla\varphi|^q + \lambda\varphi^p = 0 & \text{in } \Omega\,, \\ \varphi > 0 \quad \text{in } \Omega\,, \quad \varphi = 0 & \text{on } \Gamma\,. \end{cases} \tag{1}$$

$|\ |$ denotes the Euclidean norm in $\mathbb{R}^n$, $\lambda > 0$, $p, q > 1$.

This equation was introduced in [CW_1] in connection with the evolution problem

$$\begin{cases} u_t = \Delta u - |\nabla u|^q + |u|^{p-1}u \quad \text{in } \Omega \times \mathbb{R}^+\,, \\ u(x,0) = \varphi(x) \quad \text{in } \Omega\,, \\ u(x,t) = 0 \quad \text{on } \Gamma \times \mathbb{R}^+\,. \end{cases} \tag{2}$$

More precisely, the following was proved in [CW_1]:

THEOREM 1. *Let u be a solution to* (2), *and let φ be a smooth function satisfying*

$$\varphi = 0 \quad \textit{on } \Gamma\,, \quad \varphi \geq 0 \quad \textit{in } \Omega\,, \tag{i}$$

$$\Delta\varphi - |\nabla\varphi|^q + \varphi^p = 0 \quad \textit{on } \Gamma\,, \tag{ii}$$

$$\Delta\varphi - |\nabla\varphi|^q + \varphi^p \geq 0 \quad \textit{in } \Omega\,, \tag{iii}$$

$$E(\varphi) = \frac{1}{2}\int_\Omega |\nabla\varphi(x)|^2\,dx - \frac{1}{p+1}\int_\Omega |\varphi(x)|^{p+1}\,dx \leq 0\,, \tag{iv}$$

and either

$$q < \frac{2p}{p+1} \ \textit{and } |\varphi|_{p+1} \textit{ large enough}, \quad \textit{or} \tag{v}$$

$$q = \frac{2p}{p+1} \ \textit{and } p \textit{ large enough.} \tag{vi}$$

Then u blows up in finite time. ($| \ |_r$ *denotes the usual* $L^r(\Omega)$*-norm.*)

Proof. It is enough to use the fact that $E(u(\cdot,t)) \le E(\varphi) \le 0$ to show that $F(t) = |u(\cdot,t)|_2^2$ satisfies a differential inequality that implies blow up. We refer to [CW_1] for details.

Next, to complete the example of blow up we need to construct a solution to (i)–(iv). To accomplish this one can remark that if φ satisfies (1) and $\lambda \le 2/(p+1)$ then (i)–(iv) hold.

Other proofs of blow up involve also (1) (see [F]).

It should be noted that, roughly speaking, one can assert that blow up occurs if and only if $q < p$ (see for instance [Q], [KP], [AW]).

We turn now to the study of (1).

2. The radial case.

In this section we assume that $\Omega = B(0,R)$ where $B(0,R)$ denotes the ball of center 0 and radius R in $\mathbb{R}^n$.

THEOREM 2. *Assume that* $\Omega = B(0,R)$. *Then any solution to* (1) *is radially symmetric.*

Proof. It is enough to adapt the arguments of [GNN].

In polar coordinates (1) becomes (for simplicity we keep the notation $\varphi = \varphi(r)$ for the solution):

$$\begin{cases} \varphi'' + \dfrac{n-1}{r}\varphi' - |\varphi'|^q + \lambda\varphi^p = 0 \quad \text{on } (-R,R)\,, \\ \varphi > 0 \quad \text{on } (-R,R)\,, \quad \varphi(\pm R) = 0\,. \end{cases} \tag{3}$$

This leads naturally to study for $a > 0$ the ordinary differential equation

$$\begin{cases} \varphi'' + \dfrac{n-1}{r}\varphi' - |\varphi'|^q + \lambda\varphi^p = 0 \quad \text{on } r > 0\,, \\ \varphi(0) = a\,, \quad \varphi'(0) = 0\,. \end{cases} \tag{4}$$

More precisely, if φ vanishes and if $z(a)$ denotes the first zero of φ then the solution to (4) will provide a solution to (3) on $(0, z(a))$. The complete solution will be obtained by symmetrization. We will assume $z(a) = +\infty$ when φ does not vanish.

Let us assume that we are in the subcritical case, i.e. that

$$p < \frac{n+2}{n-2} \quad \text{if } n \ge 3\,, \qquad \text{no restriction if } n < 3\,. \tag{5}$$

Under this assumption we have:

THEOREM 3. (i) *If* $q < 2p/(p+1)$ *then for any* $R, \lambda > 0$ *there exists a solution to* (3); *moreover*, *this solution is unique when* $n = 1$.

(ii) *If* $q = 2p/(p+1)$ *then if a solution to* (3) *exists for some* R *a solution exists for any* R.

(a) *If* $n = 1$, $\lambda \le (2p)^p/(p+1)^{2p+1}$ *then* (3) *has no solution.*

(b) *If* $n = 1$, $\lambda > (2p)^p/(p+1)^{2p+1}$ *then* (3) *has a unique solution.*

(c) *If* $n \geq 2$, $\lambda \leq (2p)^p/(p+1)^{2p+1}$ *then* (3) *has no solution.*
(d) *If* $n \geq 2$, *there exists* λ^* *such that for* $\lambda > \lambda^*$, (3) *has a solution.*

(iii) *If* $q > 2p/(p+1)$ *then there exists a number* $R(\lambda)$ *such that*
(a) *for any* $R \geq R(\lambda)$ *the problem* (3) *has at least one solution*,
(b) *for any* $R < R(\lambda)$ *the problem* (3) *has no solution*,
(c) *for any* $R > R(\lambda), q > p$ *the problem* (3) *has at least two solutions.*

Proof. Most of the proofs of the above assertions are based on a careful analysis of the properties of φ, solution to (4). We are going to restrict ourselves to the last assertion of the theorem which is maybe the more fascinating.

First we claim that

$$\varphi'(r) < 0 \quad \text{when} \quad \varphi(r) > 0\,.$$

Letting $r \to 0$ in the first equation of (4) we get $n\varphi''(0) = -\lambda a^p < 0$. Hence since φ is smooth and $\varphi'(0)=0$, $\varphi'<0$ around 0. Denote by r_0 the first point in the set $\{r > 0 : \varphi(r) > 0\}$ where $\varphi'(r_0) = 0$. Then $\varphi''(r_0) = -\lambda\varphi(r_0)^p < 0$. Hence, φ' is decreasing around r_0 and by definition of r_0 one cannot have $\varphi'(r_0) = 0$. This completes the proof of our assertion.

Next we have

$$H(r) = \frac{\varphi'^2}{2} + \frac{\lambda}{p+1}\varphi^{p+1} \quad \text{is decreasing when} \quad \varphi(r) > 0\,. \tag{6}$$

It is enough to multiply the equation (4) by φ' to get

$$H'(r) = [\varphi'' + \lambda\varphi^p]\varphi' = \left[-\frac{n-1}{r}\varphi' + |\varphi'|^q\right]\varphi' < 0$$

and the result follows.

We now show that

$$\sqrt{\frac{p+1}{2\lambda}}\,a^{-(p-1)/2} \leq z(a)\,. \tag{7}$$

We can assume without loss of generality that $z(a) < +\infty$. Then on $(0, z(a))$ one has by (6)

$$\frac{1}{2}\varphi'^2 \leq H(r) \leq H(0) = \frac{\lambda}{p+1}a^{p+1}\,,$$

hence

$$|\varphi'| \leq \sqrt{\frac{2\lambda}{p+1}}\,a^{(p+1)/2}\,.$$

Integrating between 0 and $z(a)$ we get

$$a = \Big|\int_0^{z(a)} \varphi'(s)\,ds\Big| \leq z(a)\sqrt{\frac{2\lambda}{p+1}}\,a^{(p+1)/2}$$

and (7) follows.

In the same spirit one has

$$\left(\frac{p+1}{\lambda}\right)^{1/q} a^{1-p/q} \le z(a)\,. \tag{8}$$

This is a slightly sharper estimate than the one contained in [CW$_i$] and the proof we give here is different.

Integrating between 0 and $z(a)$ and using Hölder's inequality we get

$$a = \Big|\int_0^{z(a)} \varphi'(s)\,ds\Big| \le \Big(\int_0^{z(a)} |\varphi'(s)|^{q+1}\,ds\Big)^{1/(q+1)} z(a)^{1-1/(q+1)}\,. \tag{9}$$

Next from the first equation of (4) we deduce after multiplication by $\varphi' < 0$

$$\varphi''\varphi' + |\varphi'|^{q+1} + \lambda\varphi^p\varphi' = -\frac{n-1}{r}\varphi'^2 < 0 \quad \text{ on } (0, z(a))\,.$$

Integrating between 0 and $z(a)$ we get

$$\frac{\varphi'(z(a))^2}{2} + \int_0^{z(a)} |\varphi'(s)|^{q+1}\,ds - \frac{\lambda}{p+1}a^{p+1} < 0$$

from which it follows that

$$\int_0^{z(a)} |\varphi'(s)|^{q+1}\,ds < \frac{\lambda}{p+1}a^{p+1}\,.$$

Combining this inequality and (9) yields (8).

From (7) and (8) it results that

$$z(a) \ge \operatorname{Max}\bigg(\sqrt{\frac{p+1}{2\lambda}}\,a^{-(p-1)/2}, \left(\frac{p+1}{\lambda}\right)^{1/q} a^{1-p/q}\bigg)\,.$$

If we are in the case $p < q$ then

$$\lim_{a\to 0} z(a) = +\infty\,, \qquad \lim_{a\to+\infty} z(a) = +\infty\,. \tag{10}$$

So, we see that the function $z(a)$, which is continuous, is bounded from below by a positive constant. Set

$$R_\lambda = \inf_{a>0} z(a)\,.$$

Clearly for $R < R_\lambda$ there is no a such that $z(a) = R$ and (4) has no solution. If $R > R_\lambda$, by (10), there are at least two a such that $z(a) = R$ and (4) has at least two solutions. This completes the proof of the assertions (iii)(b) and (c) of the theorem in the case $q > p$. The proof of (iii)(b) in the case where $2p/(p+1) < q < p$ is much more involved and we refer the reader to [CV] or [V] for details.

The interested reader will find a proof of the other assertions in [CW$_1$] or [CW$_2$] except for (ii)(d) which is in [V] and has been obtained independently by J. Hulshof and F. B. Weissler (cf. [W]).

Remark. A consequence of (ii)(c) is that for λ small enough the problem

$$\begin{cases} \Delta\varphi - |\nabla\varphi|^{2p/(p+1)} + \lambda\varphi^p = 0 \quad \text{in } \mathbb{R}^n\,, \\ \varphi > 0 \quad \text{in } \mathbb{R}^n\,, \quad \lim_{|x|\to+\infty}\varphi(x) = 0\,, \end{cases}$$

admits a continuum of radially symmetric solutions and also of course since the problem is invariant by translations, continua of nonsymmetric solutions (see [P] for this kind of problems).

3. The general case. We would like to conclude this note showing that some of the results obtained for a ball extend to the general case. We will restrict ourselves to the following very simple result contained in [V], referring the reader to [CV] and [V] for more.

THEOREM 4. *Assume that* $p = q$. *Then if*

$$\lambda \le p\,\mathrm{diam}(\Omega)^{-p} \tag{11}$$

where $\mathrm{diam}(\Omega)$ *denotes the diameter of* Ω *then* (1) *has no solution.*

Proof. If φ is a solution to (1), by the strong maximum principle one has $\partial\varphi/\partial n < 0$ on Γ where n denotes the unit outward normal to Γ. Hence, integrating the first equation of (1) over Ω we get

$$\int_\Omega |\nabla\varphi(x)|^p - \lambda\varphi(x)^p\,dx = \int_\Omega \Delta\varphi(x)\,dx = \int_\Gamma \frac{\partial\varphi(x)}{\partial n}\,d\sigma(x) < 0\,,$$

which reads also

$$\int_\Omega |\nabla\varphi(x)|^p\,dx < \lambda \int_\Omega \varphi(x)^p\,dx\,.$$

Using the Poincaré Inequality

$$\int_\Omega \varphi(x)^p\,dx \le \frac{1}{p}(\mathrm{diam}(\Omega))^p \int_\Omega |\nabla\varphi(x)|^p\,dx$$

we obtain

$$\int_\Omega |\nabla\varphi(x)|^p\,dx < \lambda \int_\Omega \varphi(x)^p\,dx \le \frac{\lambda}{p}(\mathrm{diam}(\Omega))^p \int_\Omega |\nabla\varphi(x)|^p\,dx\,.$$

This leads to a contradiction if (11) holds.

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