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ESTIMATES OF SOLUTIONS TO LINEAR ELLIPTIC SYSTEMS AND EQUATIONS

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Dedicated to the memory of V. S. Vinogradov

Whenever nonlinear problems have to be solved through approximation methods by solving related linear problems a priori estimates are very useful. In the following this kind of estimates are presented for a variety of equations related to generalized first order Beltrami systems in the plane and for second order elliptic equations in $\mathbb{R}^m$. Different types of boundary value problems are considered. For Beltrami systems these are the Riemann–Hilbert, the Riemann and the Poincaré problem, while for elliptic equations the Dirichlet problem as well as entire solutions are involved.

1. Introduction. The simplest first order elliptic system in the plane is the Cauchy–Riemann system, in complex form given as

$$w_{\bar z} = 0\,, \quad w = u + iv\,, \quad z = x + iy\,, \quad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1}$$

Its solutions are the analytic functions, the real and imaginary part of which are harmonic, i.e. solutions to the second order elliptic equation

$$\Delta u = 0\,, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,. \tag{2}$$

Both equations are very simple and the theories of their solutions are very rich and beautiful. They are special cases of more complicated equations, namely the generalized Beltrami equation

$$w_{\bar z} + q_1 w_z + q_2 \overline{w_z} + aw + b\overline{w} + c = 0 \tag{3}$$

and the general second order equation

$$\sum_{j,k=1}^{2} a_{jk} u_{x_j x_k} + \sum_{j=1}^{2} b_j u_{x_j} + cu + d = 0\,, \qquad x_1 = x\,,\ x_2 = y\,, \tag{4}$$

respectively. The ellipticity condition for (3) is

$$|q_1(z)| + |q_2(z)| \le q_0 < 1\,, \tag{5}$$

while (4) is elliptic if (a_{jk}) is a symmetric matrix and for some positive λ

$$\lambda \sum_{j=1}^{2} \xi_j^2 \le \sum_{j,k=1}^{2} a_{jk} \xi_j \xi_k \quad \text{ for all } (\xi_1, \xi_2)\,. \tag{6}$$

Rather than considering (4) under (6) with two variables the corresponding equation with m independent variables will be studied.

Several special cases of (3) play an important role in complex analysis. If all the coefficients are zero except q_1 then (3) is the Beltrami equation, the solutions of which are known as quasiconformal mappings. Their theory became very important for complex analysis and has an interesting counterpart in $\mathbb{R}^m$. For $q_1 = q_2 = c = 0$ equation (3) is the basic equation for generalized analytic functions [51] and for pseudoanalytic functions [22], depending on the assumptions on the coefficients a and b. (3) was generalized in different directions. Without going into details we mention the works of Douglis [37], Gilbert–Hile [42, 43] and others, e.g. [6–9], [41], Bojarski [31], Delanghe and co-workers [34], Xu [61].

A variety of boundary value problems were studied in the past for analytic functions. They originated from suggestions by Riemann, and some were first solved by Hilbert and his student Haseman (see e.g. [3]) Hilbert even considered (3) with $q_1 = q_2 = c = 0$. Russian mathematicians, especially Gakhov [39] and Muskhelishvili [48] developed the theory of boundary value problems for analytic functions cosiderably. Mainly I. N. Vekua and his students, e.g. B. Bojarski, A. Dzhuraev, E. Obolashvili and V. S. Vinogradov, and W. Haak and his group, G. Hellwig, J. Jaenicke, W. Wendland and others extended these problems to more general systems (see [38, 44, 47, 51, 56, 59]). In the following three basic boundary value problems for (3) will be presented: the Riemann–Hilbert, the Poincaré and the Riemann boundary value problem. The last problem is sometimes also called the problem of linear conjugacy. There is an immense literature on this subject (see e.g. [2, 3, 44, 50, 51, 56, 59]) which cannot be quoted here.

The same is true of course for second order elliptic equations, where the literature is even more extensive. But there are a lot of well known books on this subject from which here mainly [40] is used. For second order equations two problems are considered here: the Dirichlet problem for bounded domains and entire solutions, i.e. solutions on the whole space $\mathbb{R}^m$ under special growth conditions.

How these estimates for linear problems can be used to treat related nonlinear equations as well as nonlinear boundary conditions can be found e.g. in [11, 16, 18, 21]. The method used is a combination of the Schauder imbedding method with the Newton approximation method. This procedure was in this context first applied by Wendland [57] to a quasilinear elliptic first order system in the plane.

2. Riemann–Hilbert problem for generalized Beltrami systems. Let for simplicity D be a bounded simply connected domain in the complex plane $\mathbb{C}$ with continuously differentiable boundary ∂D. For the case of multiply connected domains see [56]. For given Hölder continuous functions $\log\lambda$ and φ on the boundary ∂D, where $\lambda(z) \neq 0$, we look for a solution to

$$w_{\bar z} + q_1 w_z + q_2 \overline{w_z} + aw + b\overline{w} + c = 0\,, \tag{3}$$

where

$$|q_1(z)| + |q_2(z)| \le q_0 < 1 \tag{5}$$

and $a, b, c \in L_p(\overline{D})$ for $2 < p$, satisfying the boundary condition

$$\mathrm{Re}\{\overline{\lambda(z)}w(z)\} = \varphi(z)\,, \qquad z \in \partial D\,. \tag{7}$$

The solutions depend on the index

$$n = \frac{1}{2\pi i} \int_{\partial D} d\log\lambda(z) \tag{8}$$

of the problem, which by continuity is an integer number. As is known from analytic functions theory the homogeneous problem (7) with $\varphi = 0$ for nonnegative index has n linearly independent solutions. The general solution becomes unique by imposing side conditions. One possibility is to prescribe

$$\begin{gathered} \int_{\partial D} \mathrm{Im}\{\overline{\lambda(z)}w(z)\}\,|dz| = \kappa\,, \qquad \kappa \in \mathbb{R}\,, \\ w(z_k) = a_k\,, \qquad z_k \in D\,, \qquad a_k \in \mathbb{C}\,, \qquad 1 \le k \le n\,, \end{gathered} \tag{9}$$

for given points z_k and values κ, a_k; another is to fix for given points z_k on ∂D and $b_k \in \mathbb{R}$

$$\mathrm{Im}\{\overline{\lambda(z_k)}w(z_k)\} = b_k\,, \qquad 0 \le k \le 2n\,. \tag{10}$$

For negative index φ has to satisfy some conditions in order that (7) be solvable. In the case where D is the unit disc $\mathbb{D}$ these conditions are handled by replacing φ by $\varphi + h$ where h has the form

$$h(z) = \sum_{k=n+1}^{-n-1} h_k z^k\,, \qquad h_{-k} = \overline{h_k}\ (|k| \le -n-1)\,, \tag{11}$$

with coefficients to be determined properly so that the mentioned conditions are satisfied. Hence, in this modified problem besides the solution w the coefficients h_k, $0 \le k \le -n-1$, have to be found (see [51, 54, 56]). For more general domains D, the z on the right hand side of (11) has to be replaced by a conformal mapping from D onto $\mathbb{D}$.

Basic tools in the theory of generalized Beltrami systems in the plane are two integral operators introduced by I. N. Vekua in his treatment of generalized analytic functions [51],

$$T\varphi(z) := -\frac{1}{\pi}\int_D \varphi(\zeta)\frac{d\zeta}{\zeta - z}\,, \quad \Pi\varphi(z) := -\frac{1}{\pi}\int_D \varphi(\zeta)\frac{d\xi\, d\eta}{(\zeta - z)^2} \quad (\zeta = \xi + i\eta)\,.$$

An a priori estimate for solutions to equation (3) was first given in [28] by B. Bojarski in the case of a Dirichlet problem. By reductio ad absurdum he proved the following result (see Theorem 4.7 in [28]).

THEOREM 0. *Let w be a solution to equation* (3) *under the above assumptions. Then w can be represented in the form*

$$w(z) = f(\chi(z))e^{\varphi(z)} + w_0(z)\,.$$

Here f is an analytic function in $\chi(D)$ while χ is a homeomorphism of the equation

$$w_{\bar z} + q_0(z)w_z = 0\,, \qquad q_0(z) := \begin{cases} q_1(z) + q_2(z)\dfrac{\overline{w_z(z)}}{w_z(z)} & \text{if } w_z(z) \neq 0, \\ q_1(z) + q_2(z) & \text{if } w_z(z) = 0, \end{cases}$$

mapping the z-plane onto the χ-plane, φ is a continuous function on the plane, holomorphic outside the domain D and vanishing at infinity, $\chi, \varphi \in W_p^1(\mathbb{C})$, $2<p$. Moreover, $\|\varphi_z\|_p$, $\|\varphi_{\bar z}\|_p$, $\|\chi_z\|_p$, $\|\chi_{\bar z}\|_p$ are bounded by a constant depending only on the domain D, on q_0 and $\|a\|_p$, $\|b\|_p$. In particular, χ and φ are Hölder continuous and the Hölder exponent depends only on the same quantities. The function $w_0 \in W_p^1(\mathbb{C})$ is a particular solution of the inhomogeneous equation (3), *which is Hölder continuous on the entire plane $\mathbb{C}$, analytic outside D and asymptotically* 1 *at infinity.*

In the special case of the Beltrami equation (3) with $q_2 = a = b = c = 0$ this result is contained already in [25]. Later V. S. Vinogradov [52, 53] applied Bojarski's procedure to treat the general case of a Riemann–Hilbert boundary condition (see also Bojarski's thesis [27]) in both cases of nonnegative as well as negative index. In [54] he extended the result to quasilinear equations, which were extensively studied by Bojarski [30]. Bojarski [32] and his student Iwaniec [33, 46] considered nonlinear first order equations, too. While Bojarski and Vinogradov gave an indirect proof of the a priori estimate later Wendland [57, 58] and Begehr and Hsiao [13, 16] developed a direct one. This method was at the same time used in the case of the problem of linear conjugacy (see Begehr and Hile [11, 12]) and for the Poincaré problem (see Begehr and Wen [18]).

There are estimates for classical as well as for weak solutions. The latter are more important because they serve to solve nonlinear problems under more general conditions.

THEOREM 1. *Let D be a bounded domain the boundary of which has a Hölder* *continuously varying tangent, let* $a, b \in C^\alpha(\overline{D})$ *with*

$$\|a\|_\alpha + \|b\|_\alpha \le K\,,$$

and $q_1, q_2 \in C^{1+\alpha}(\overline{D})$ *with*

$$\|q_1\|_0 + \|q_2\|_0 \le q_0 < 1\,, \quad \|q_{1z}\|_\alpha + \|q_{1\bar z}\|_\alpha + \|q_{2z}\|_\alpha + \|q_{2\bar z}\|_\alpha \le M\,.$$

Then there exist constants γ_ν $(1 \le \nu \le 4)$ *depending on* D, z_k $(1 \le k \le n)$, λ, α, q, K *and* M *but not on* q_1, q_2, a, b, φ, κ, a_k $(1 \le k \le n)$ *such that for any* $w \in C^{1+\alpha}(\overline{D})$ *satisfying* (7), (9) *with* $\log\lambda, \varphi \in C^{1+\alpha}(\partial D)$ *the estimate*

$$\begin{aligned}(12)\qquad \|w\|_{1+\alpha} \le \gamma_1\|\varphi\|_{1+\alpha} + \gamma_2|\kappa| + \gamma_3\sum_{k=1}^{n}|a_k| \\ + \gamma_4\|w_{\bar z} - q_1 w_z - q_2\overline{w_z} - aw - b\overline{w}\|_\alpha\end{aligned}$$

holds. In the case of (10) *instead of* (9) *the two middle terms are replaced by* $\gamma_2\sum_{k=0}^{2n}|b_k|$. *For* $n < 0$ *the estimate* (12) *is also valid, but then these middle terms do not occur.*

For a direct proof see e.g. [15, 17, 19, 59]. In connection with weak solutions a direct proof of the corresponding a priori estimate requires restrictive conditions on the constant q_0, i.e. on the ellipticity of (3) (see [16]). But the estimate holds without this restriction (compare [4, 30, 56]).

Theorem 2. *Let* D *be a bounded domain with continuously differentiable boundary, let* $a, b \in L_p(\overline{D})$ *with*

$$\|a\|_p + \|b\|_p \le K \qquad (2 < p)$$

and let q_1, q_2 *be measurable and satisfying* (5). *Then there exist constants* γ_ν $(1 \le \nu \le 4)$ *depending on* α, p, q_0, λ, z_k $(1 \le k \le n)$, K *but not on* q_1, q_2, a, b, φ, κ, a_k $(1 \le k \le n)$ *such that for any* $w \in W_p^1(\overline{D})$ *satisfying* (7), (9) *with* $\log\lambda, \varphi \in C^\alpha(\partial D)$, $1 < 2\alpha < 2$, *the estimate*

$$\begin{aligned}(13)\qquad &\|w\|_0 + \|w_z\|_p + \|w_{\bar z}\|_p \\ &\le \gamma_1\|\varphi\|_\alpha + \gamma_2|\kappa| + \gamma_3\sum_{k=1}^{n}|a_k| + \gamma_4\|w_{\bar z} - q_1 w_z - q_2\overline{w_z} - aw - b\overline{w}\|_p\end{aligned}$$

holds.

Here the same remarks as for Theorem 1 with respect to (9) and to negative index apply. The proof for negative n which is not included in [4, 15, 17] may be given similarly to [19, 56].

Replacing indirect proofs of a priori estimates by direct ones is no hairsplitting because a constructive proof gives some information on the size of the constants γ_ν, while an indirect proof only shows their existence. But for numerical procedures knowledge of the magnitude of the constants is important. Existence and uniqueness results for Riemann–Hilbert problems for quasilinear equations of type

(3) and nonlinear equations are for example given in [3, 13–16, 30, 32, 33, 46, 49, 54, 57–59].

3. Discontinuous Poincaré problem for generalized Beltrami systems. The difference between the Poincaré boundary condition and the Riemann–Hilbert problem is that in (7), w is replaced by w_z, i.e.

$$\operatorname{Re}\{\overline{\lambda(z)}w_z(z)\} = \varphi(z)\,, \qquad z \in \partial D\,. \tag{14}$$

Instead of assuming Hölder continuity here λ and φ are allowed to have discontinuities of first kind. While for analytic functions there is no difference in principle between (7) and (14) the discontinuous Riemann–Hilbert problem needs some extra treatment (see e.g. [18, 19, 47, 48, 56]). Problem (3), (14) is reducible to this problem by setting

$$U := w_z$$

so that after some calculations by differentiating (3)

$$U_{\bar z} + \mu_1 U_z + \mu_2 \overline{U_z} + AU + B\overline{U} + H = 0 \tag{15}$$

with

$$H = \nu_1 w_{\bar z} + \nu_2 \overline{w_{\bar z}} + Cw + D\overline{w} + E$$

and proper coefficients (see [19], p. 312), in particular,

$$|\mu_1(z)| + |\mu_2(z)| \le q_0 < 1$$

with q_0 from (5). Condition (14) means the Riemann–Hilbert condition for U. The connection between w and U is given by

$$w(z) = w(z_0) + \int_{z_0}^{z} \{U(\zeta)\,d\zeta - [q_1 U + q_2\overline{U} + aw + b\overline{w} + c](\zeta)\,d\overline{\zeta}\}\,. \tag{16}$$

The integral involved is path-independent because of (3) and the fact that D is simply connected.

The coefficients λ and φ in (14) are assumed to have at most a finite number of discontinuities on ∂D. Let $c_1, \ldots, c_m$ be those points ordered in accordance with the orientation on ∂D and denote the open arcs between them by Γ_μ, $1 \le \mu \le m$, so that $\partial D \setminus \{c_1, \ldots, c_m\} = \bigcup_{\mu=1}^{m} \Gamma_\mu$. Then

$$\lambda, \varphi_0 \in C^\beta(\overline{\Gamma_\mu})\,, \quad 1 \le \mu \le m\,, \quad 0 < \beta < 1\,, \quad |\lambda| = 1\,,$$
$$\lambda(c_\mu - 0) = e^{i\theta_\mu}\,\lambda(c_\mu + 0)\,, \quad \varphi_\mu := \theta_\mu/\pi - k_\mu\,,$$
$$k_\mu := [\theta_\mu/\pi] + I_\mu\,,$$

where $I_\mu \in \{0, 1\}$ such that $|\varphi_\mu| < 1$,

$$\varphi(z) = \varphi_0(z)\prod_{\mu=1}^{m}|z - c_\mu|^{-\beta_\mu}\,, \quad 0 \le \beta_\mu\,, \quad \beta_\mu + \varphi_\mu < 1\,.$$

For this discontinuous problem

$$\kappa := \frac{1}{2}\sum_{\mu=1}^{m} k_\mu$$

is called the index. If the set of discontinuities is empty $\kappa = n$ is an integer. But in general only 2κ is an integer. For vanishing or positive κ the side conditions (9) or (10) with κ instead of n serve to determine the solution of (3), (14) uniquely if, moreover, at some $z_0 \in \overline{D}$ the value $w(z_0)$ is fixed. But in order to solve (16) with U known for w some smallness assumptions on the coefficients a and b have to be imposed [18, 19]. Modification of (14) for negative κ in order to handle the conditions for the problem to be solvable again is done by replacing φ by $\varphi + h$ where in case $D = \mathbb{D}$

$$h(z) := \begin{cases} \displaystyle\sum_{k=\kappa+1}^{-\kappa-1} h_k z^k \,, \ h_{-k} = \overline{h_k} \ (|k| \le -\kappa - 1) & \text{if } -\kappa \in \mathbb{N}, \\ \displaystyle\sum_{k=[\kappa]+2}^{-[\kappa]} h_k z^{k-1/2} \,, \ h_{1-k} = \overline{h_k} \ (|k| \le [-\kappa]) & \text{if } \frac{1}{2} - \kappa \in \mathbb{N}, \end{cases} \tag{17}$$

with unknown coefficients h_k to be determined properly. As before this function has to be modified for general D. Using Muskhelishvili's theory of singular integrals in [19] via direct estimation rather than by reductio ad absurdum an a priori estimate for solutions to the modified problem (3), (14), (10) (with $n = \kappa$) is proved.

THEOREM 3. *Let D be a bounded domain with continuously differentiable boundary, let*

$$|q_1(z)| + |q_2(z)| \le q_0 < 1 \,, \quad \|q_{1z}\|_p + \|q_{2z}\|_p \le K \quad (2 < p) \,,$$
$$|a(z)| + |b(z)| \le \varepsilon K \,, \quad \|a_z\|_p + \|b_z\|_p \le \varepsilon K \,, \quad 2\varepsilon K < 1 \,,$$

and suppose λ and φ have the properties described above. Then under some more technical restrictions there exist nonnegative constants γ_μ depending on α, φ_μ, β_μ $(1 \le \mu \le m)$, and M depending on p, q_0, α, β, λ, ε, K, c_μ, φ_μ, β_μ $(1 \le \mu \le m)$, z_k $(0 \le k \le 2n)$ but not on w, q_1, q_2, a, b, c, φ, b_k $(0 \le k \le 2n)$ such that

$$\|w\|_\alpha + \Big\| \prod_{\mu=1}^{m} (z - c_\mu)^{\gamma_\mu} w_z \Big\|_0 + \Big\| \prod_{\mu=1}^{m} (z - c_\mu)^{\gamma_\mu} w_{\bar z} \Big\|_0 \le M \Big\{ \|\varphi_0\|_\beta + \sum_{k=0}^{2\kappa} |b_k| + \|c\|_0 + \|c_z\|_p \Big\} \tag{18}$$

for any solution to (3), (14), (10) (*with w_z instead of w*). *Here for negative index κ neither the side condition* (10) *for w_z nor the second term on the right hand side of* (18) *occur.*

Related nonlinear problems are discussed in [18, 19].

4. Riemann problem for generalized Beltrami systems. The main problem in solving the linear Riemann problem for (3) is to find proper entire solutions to (3), i.e. functions satisfying (3) in the whole plane. The reason is that proper analytic functions can be used to reduce the general problem to an equation (3) without any jump condition on Γ (see e.g. [10, 11, 39]). For, say, a simple, smooth arc Γ in $\mathbb{C}$ and a function Φ defined on both sides of Γ define

$$\Phi^{\pm}(\zeta) := \lim_{z\to\zeta} \Phi(z)\,,$$

where z not on Γ nontangentially tends to a point ζ of Γ different from the endpoints, approaching Γ from the right and from the left hand side, respectively.

Let, for simplicity, Γ be a single, simply closed, smooth curve in $\mathbb{C}$ rather than a finite set of such curves, mutually disjoint, and let G and g be Hölder continuous functions defined on Γ such that $|G(z)| = 1$ on Γ. Denote by D^+ and D^- the inner and outer domain of Γ, respectively. We look for a solution to (3) in $\mathbb{C}\setminus\Gamma$ satisfying

$$w^+ = Gw^- + g \quad \text{on } \Gamma \tag{19}$$

and e.g. vanishing at infinity. Let X be the analytic factorization of G, i.e. the analytic function behaving as z^{-n} at infinity and satisfying the homogeneous condition (19):

$$X^+ = GX^-\,.$$

This function is given by

$$X(z) := \begin{cases} \exp L(z)\,, & z\in D^+,\\ z^{-n}\exp L(z)\,, & z\in D^-,\end{cases}$$

where

$$n = \operatorname{ind} G = \frac{1}{2\pi i}\int_\Gamma d\log G(\zeta) \in \mathbb{Z}$$

is the index of (19). Moreover, let

$$\varphi(z) := \frac{1}{2\pi i}\int_\Gamma \frac{g(\zeta)}{X^+(\zeta)}\,\frac{d\zeta}{\zeta - z} \quad (z\notin\Gamma)\,.$$

This function is analytic and as the Plemelj–Sokhotskiĭ formula shows (see [39, 45, 48]), it satisfies

$$\varphi^+ = \varphi^- + \frac{g}{X^+} \quad \text{on } \Gamma\,.$$

Then, obviously,

$$\left(\frac{w}{X} - \varphi\right)^+ = \left(\frac{w}{X} - \varphi\right)^- \quad \text{on } \Gamma\,.$$

Hence, $\omega := w/X - \varphi$ is continuous on Γ and satisfies in $\mathbb{C}\setminus\Gamma$

$$\omega_{\bar z} + q_1\omega_z + q_2\overline{\omega_z} + \left(a + \frac{X'}{X}q_1\right)\omega + \left(b\frac{\overline{X}}{X} + \frac{\overline{X'}}{X}q_2\right)\overline{\omega} + \widetilde{c} = 0\,,$$

$$\widetilde{c} := \frac{c}{X} + q_1\varphi' + q_2\overline{\varphi'} + \left(a + \frac{X'}{X}\,q_1\right)\varphi + \left(b\frac{\overline{X}}{X} + \frac{\overline{X'}}{X}q_2\right)\overline{\varphi}\,.$$

The behaviour of ω at infinity is controlled by that of w and X.

In order to prove an a priori estimate for entire solutions to (3) this equation is stepwise reduced first to an inhomogeneous, then to a homogeneous Beltrami equation and finally to the Cauchy–Riemann equation (see [10, 11]). A proper space for solutions is the space of functions with derivatives in

$$L_{(p,p')} := L_p \cap L_{p'}\,, \qquad 1 \le p' < 2 < p < \infty\,,$$

and in $L_{p,2}$ for $2 < p < \infty$, respectively (see [51]).

THEOREM 4. *Besides* (5), *let for some* ε, $0 < \varepsilon < 1$,

$$|q_1(z)| + |q_2(z)| = O(|z|^{-\varepsilon}) \qquad \textit{as } z \to \infty\,,$$

and let $a, b \in L_{(p,p')} \cap L_{p,2}$ *for suitable* p, p' *satisfying in particular*

$$\frac{2}{1+\varepsilon} < p' < 2 < p < \frac{4}{2-\varepsilon}\,,$$

with

$$\|a\|_{(p,p')} + \|b\|_{(p,p')} \le K\,.$$

Let w *be an entire function vanishing at infinity and such that* $w_z, w_{\bar z} \in L_{(p,p')}$, $w_{\bar z} \in L_{p,2}$. *Then there exists a constant* M *depending on* K, ε, p, p' *and* q_0 *such that*

$$\|w\|_0 + \|w_z\|_{(p,p')} + \|w_{\bar z}\|_{(p,p')} \le M\|w_{\bar z} + q_1 w_z + q_2\overline{w_z} + aw + b\overline{w}\|_{(p,p')}\,. \tag{20}$$

Moreover, equation (3) under the above conditions together with $c \in L_{(p,p')} \cap L_{p,2}$ can be shown to be uniquely solvable by an entire function satisfying (20). In particular, (20) shows that the homogeneous problem, i.e. $c \equiv 0$, is only trivially solvable (see [11]). If instead of being an entire solution w satisfies

$$w^+ = w^- + g \qquad \text{on } \Gamma$$

with $g \in C^\alpha(\Gamma)$ then (20) has to be replaced by

$$\begin{aligned} \|w\|_0 + \|w_z\|_{(p,p')} &+ \|w_{\bar z}\|_{(p,p')} \\ &\le M\{\|g\|_\alpha + \|w_{\bar z} + q_1 w_z + q_2\overline{w_z} + aw + b\overline{w}\|_{(p,p')}\}\,. \end{aligned} \tag{21}$$

While here solutions in the weak sense are involved, classical ones may be considered, too. The disadvantage then is that the Π-operator (see [51], Chapter I, §8)

$$(\Pi f)(z) := -\frac{1}{\pi}\int_{\mathbb{C}} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2} \qquad (\zeta = \xi + i\eta)$$

for $f \in C^\alpha(\mathbb{C}) \cap L_p(\mathbb{C})$ may be estimated by

$$\|\Pi f\|_{p,\alpha} \le M(p,\alpha)\|f\|_{p,\alpha}\,,$$

where $\|f\|_{p,\alpha} := \|f\|_p + \|f\|_\alpha$ and $M(p,\alpha)$ is some fixed constant > 1, while for $f \in L_p$

$$\|\Pi f\|_p \le \Lambda_p \|f\|_p$$

with Λ_p continuous in p and $\Lambda_2 = 1$. Hence, an estimate of type (20) for classical solutions to (3) in $\mathbb{C}$ can be proven only under strong restrictions, namely q_0 has to be so small that $M(p,\alpha)q_0 < 1$. In order to get a better result, the constant $M(p,\alpha)$ has to be considered in more detail. In [10] for the case $q_1 = q_2 = 0$ the following result is derived.

THEOREM 5. *For some* ε, $0 < \varepsilon < 1$, *let*

$$(1 + |z|^{1+\varepsilon})[|a(z)| + |b(z)|] \le K$$

and let $g \in C^\alpha(\Gamma)$ *for* $\alpha = (p-2)/p$, *where* $2 < p < 2/(1-\varepsilon)$. *Then there exists a constant* M *depending on* K, ε, p *and* Γ *such that for any function* w *with* $w_{\bar z} \in L_{p,2}(\mathbb{C})$ *satisfying the jump condition* $w^+ = w^- + g$ *on* Γ *and vanishing at infinity we have*

$$\|w^\pm\|_\alpha \le M[\|g\|_\alpha + \|w_{\bar z} + aw + b\overline{w}\|_{p,2}]\,, \tag{22}$$

where $\|w^\pm\|_\alpha := \|w\|_{\alpha,\overline{D^+}} + \|w\|_{\alpha,\overline{D^-}}$.

5. Second order elliptic equations. A priori estimates for solutions to the Dirichlet problem for elliptic equations of second order

$$Lu := \sum_{j,k=1}^{m} a_{jk} u_{x_j x_k} + \sum_{j=1}^{m} b_j u_{x_j} + cu = d \tag{23}$$

in a regular domain $D \in C^{2,\alpha}$ of $\mathbb{R}^m$ follow from Schauder estimates (see e.g. [40], 6.2). Let $a_{jk}, b_j, c, d \in C^\alpha(\overline{D})$, (a_{jk}) being symmetric and satisfying

$$\lambda \sum_{j=1}^{m} \xi_j^2 \le \sum_{j,k=1}^{m} a_{jk}\xi_j\xi_k$$

for some positive constant λ for all real $\xi = (\xi_1, \ldots, \xi_m)$ in D. Moreover, suppose

$$\|a\|_\alpha := \sum_{j,k=1}^{m} \|a_{jk}\|_\alpha\,,\ \|b\|_\alpha := \sum_{j=1}^{m} \|b_j\|_\alpha\,,\ \|c\|_\alpha \le \Lambda\,.$$

Then there exists a constant C depending on m, α, λ, Λ and D such that

$$\|u\|_{2,\alpha,D} \le C\{\|u\|_{0,D} + \|u\|_{2,\alpha,\partial D} + \|Lu\|_{\alpha,D}\} \tag{25}$$

(see e.g. [60]). If, moreover, $c \le 0$ on D the maximum principle is valid, giving

$$\|u\|_{0,D} \le K\{\|u\|_{0,\partial D} + \|Lu\|_{0,D}\}$$

for any $u \in C^{2,\alpha}(\overline{D})$, where K depends on λ, Λ and D. Hence, the following result holds (see [1]).

THEOREM 6. *Under the above assumptions there exists a constant M depending on m, α, λ, Λ and D such that for all $u \in C^{2,\alpha}(\overline{D})$*

$$\|u\|_{2,\alpha,D} \le M\{\|u\|_{2,\alpha,\partial D} + \|Lu\|_{\alpha,D}\}\,. \tag{26}$$

As in the case of the generalized Beltrami equation the Dirichlet condition has to be replaced by some growth condition if instead of a bounded domain the elliptic equation is considered in the whole space $\mathbb{R}^m$. Because an elliptic equation satisfying (24) locally may be reduced to the canonical form it is no restriction to assume L to behave asymptotically as the Laplace operator Δ at infinity. More precisely, in [12] it is assumed that for some positive ε

$$a_{jk}(x) - \delta_{jk} = O(|x|^{-\varepsilon})\,, \quad b_j(x) = O(|x|^{-1-\varepsilon})\,,$$
$$c(x) = O(|x|^{-2-\varepsilon})\,, \quad d(x) = O(|x|^{-2-\varepsilon}) \quad \text{as } |x| \to \infty\,.$$

A proper kind of Hölder norm in the space of certain entire functions satisfying some growth condition is

$$\|u\|_{\sigma,\alpha} := \|u\|_\sigma + \|u\|_{(\sigma,\alpha)}\,,$$

where

$$\|u\|_\sigma := \sup_{x\in\mathbb{R}^m} (1+|x|)^{-\sigma}|u(x)|\,,$$
$$\|u\|_{(\sigma,\alpha)} := \sup_{\substack{x,y\in\mathbb{R}^m \\ 0<2|x-y|\le 1+|x|}} (1+|x|)^{\alpha-\sigma}\frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

with some real σ and $0 < \alpha < 1$. Let ∇ denote the gradient operator and $\nabla_2 = (\partial^2/\partial x_j \partial x_k)$ the Hesse matrix rather than the Laplacian, $a = (a_{jk})$, $b = (b_j)$. Similarly to the case of a bounded domain, a Schauder estimate can be derived (see [12]).

THEOREM 7. *Let $u \in C^2(\mathbb{R}^m)$, $2 \le m$, with $\|u\|_\sigma$ finite for some real σ, let* (24) *hold in $\mathbb{R}^m$ and*

$$\|a\|_{0,\alpha}\,,\ \|b\|_{-1,\alpha}\,,\ \|c\|_{-2,\alpha} \le \Lambda\,.$$

Then there exists a constant C depending on m, α, σ, λ and Λ but not on u, a, b, c such that

$$\|u\|_{\sigma,1} + \|\nabla u\|_{\sigma-1,1} + \|\nabla_2 u\|_{\sigma-2,\alpha} \le C\{\|u\|_0 + \|Lu\|_{\sigma-2,\alpha}\}\,. \tag{27}$$

This estimate leads to a priori estimates. But there is a crucial difference between the cases $m = 2$ and $2 < m$. While for the latter the homogeneous equation

$$\Delta u + cu = 0$$

under the above decay condition on c at infinity only has the trivial solution in $\mathbb{R}^m$ vanishing at infinity, the situation for $m = 2$ is more involved. In that case, as for the Poisson equation, an entire solution in general grows as a multiple of

$\log|x|$. But solutions of this kind need not be unique, as is shown in [12] by an example. Hence, for $m = 2$ in the sequel it is assumed that $c \equiv 0$.

THEOREM 8. *Suppose there exist constants* $0 < \delta$, $0 < \alpha < 1$, $0 \le \Lambda$ *such that*

$$\|a - I\|_{-\delta,\alpha}\,,\ \ \|b\|_{-1-\delta,\alpha}\,,\ \ \|c\|_{-2-\delta,\alpha}\ \le \Lambda\,,$$

where $I = (\delta_{jk})$ *denotes the* $m \times m$ *identity matrix. Let* $c \le 0$ *and* $c \equiv 0$ *if* $m = 2$ *and* $u \in C^2(\mathbb{R}^m)$.

(a) *If for* $3 \le m$ *and* $2 < \tau < m$ *the function* u *has finite norm* $\|u\|_{2-\tau}$, *then*

$$\|u\|_{2-\tau,1} + \|\nabla u\|_{1-\tau,1} + \|\nabla_2 u\|_{-\tau,\alpha} \le M\|Lu\|_{-\tau,\alpha}\,, \tag{28}$$

where $M = M(m, \alpha, \delta, \tau, \lambda, \Lambda)$.

(b) *For* $m = 2$ *assume that* $u(x) - \gamma \log|x|$ *vanishes at infinity for some constant* γ. *Then for any* $\varepsilon > 0$

$$\|u\|_{\varepsilon,1} + \|\nabla u\|_{\varepsilon-1,1} + \|\nabla_2 u\|_{\varepsilon-2,\alpha} \le M\|Lu\|_{-\tau,\alpha}\,, \tag{29}$$

and also $|\gamma| \le M\|Lu\|_{-\tau,\alpha}$, *where* $M = M(\alpha, \delta, \tau, \lambda, \Lambda)$. *If, moreover,* $0 < \varepsilon < 1$, $\varepsilon \le \delta$, $2 + \varepsilon \le \tau$, *then*

$$\|\Delta u\|_{-2-\varepsilon,\alpha} \le M\|Lu\|_{-\tau,\alpha} \tag{30}$$

and for $1 \le |x|$

$$|u(x) - \gamma \log|x|| \le M\|Lu\|_{-\tau,\alpha}(1 + |x|)^{-\varepsilon}\ \log(1 + |x|)$$

with $M = M(\alpha, \delta, \tau, \varepsilon, \lambda, \Lambda)$.

This result in [12] is used to give an existence and uniqueness proof for entire solutions not only with the indicated behaviour at infinity but also for solutions which behave asymptotically as a harmonic polynomial to which in case $m = 2$ a term of the kind $\gamma \log|x|$ is added. These solutions are shown to exist under some stronger decay conditions on the coefficients than those formulated in Theorem 8. In order to study nonlinear equations the Hölder norms are not suitable as was mentioned earlier but proper L_p -estimates are not yet available.

6. Initial and boundary value problem for a composite type system. The simplest form of a general linear composite system of first order of three real equations in complex form is

$$\begin{aligned} w_{\bar z} + q_1 w_z + q_2 \overline{w_z} + a_1 w + a_2 \overline{w} + a_3 \omega + c = 0\,,\\ \omega_y + b_1 w + b_2 \overline{w} + b_3 \omega + d = 0\,, \end{aligned} \tag{31}$$

where the coefficients are complex except a_3, b_3 and d and such that $b_2 = \overline{b_1}$. The unknown function w is complex while ω is real.

Any second order elliptic equation in the plane can be reduced to such a system. For simplicity (31) is studied in the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$. Natural initial and boundary conditions are

$$\mathrm{Re}\{\overline{\lambda} w\} = \varphi \quad \text{on } \Gamma = \partial\mathbb{D}\,, \qquad \omega = \psi \quad \text{on } \gamma = \{|z| = 1, \mathrm{Im} z \le 0\}\,, \tag{32}$$

where $|\lambda(z)| = 1$, φ and ψ are real-valued. As in Section 2 the Riemann–Hilbert problem (7) is influenced by its index n, so is the case for problem (32). Hence for $n \geq 0$ the side conditions (10) are imposed in order to get unique solutions of problem (31), (32) while for its solvability for $n < 0$ the first condition in (32) is modified by replacing φ by $\varphi + h$ with h from (11).

THEOREM 9. *Let* $a_3 = b_1 = b_2 = 0$, *let* q_1, q_2 *satisfy* (5), *suppose that*

$$\|a_1\|_p + \|a_2\|_p \leq K \quad (2 < p)\,, \qquad \|b_3\|_{\alpha'} \leq K \quad (0 < \alpha' < 1)\,,$$

and let $\lambda, \varphi \in C^\alpha(\mathit{\Gamma})$ $(1/2 < \alpha < 1)$, $\psi \in C^{\alpha'}(\gamma)$, $c \in L_p(\overline{\mathbb{D}})$ *and* $d \in C^{\alpha'}(\overline{\mathbb{D}})$. *Then any solution* (w, ω) *of* (31), (32) *satisfies* $w \in W^1_{p_0}(\mathbb{D})$, $2 < p_0 \leq \min\{p, 1/(1-\alpha)\}$, $\omega \in C^1_y(\overline{\mathbb{D}})$ *and*

$$\|w\|_\beta + \|w_z\|_{p_0} + \|w_{\bar z}\|_{p_0} \leq M_1 \Big\{ \|\varphi\|_\alpha + \sum_{\mu=0}^{2n} |b_\mu| + \|c\|_p \Big\}\,, \tag{33}$$
$$\|\omega\|_\beta + \|\omega_y\|_0 \leq M_2 \{ \|\psi\|_{\alpha'} + \|d\|_{\alpha'} \}\,,$$

where $0 < \beta \leq \min\{\alpha, \alpha', 1 - 2/p_0\}$ *and* M_1 *and* M_2 *depend on* α, p, p_0, q_0, K, λ *and on* α' *and* K, *respectively.*

THEOREM 10. *Let the coefficients of problem* (31), (32) *satisfy the conditions of the preceding theorem but* $a_3 \in L_p(\overline{\mathbb{D}})$, $b_1, b_2 \in C^\beta(\overline{\mathbb{D}})$, $b_2 = \overline{b_1}$, *with*

$$2M_1M_2\|a_3\|_p\|b_1\|_\beta \leq k < 1\,.$$

Then any solution of (31), (32) *satisfies*

$$\begin{aligned} &\|w\|_\beta + \|w_z\|_{p_0} + \|w_{\bar z}\|_{p_0} + \|\omega\|_\beta + \|\omega_y\|_0 \\ &\qquad \leq M \Big\{ \|\varphi\|_\alpha + \|\psi\|_{\alpha'} + \sum_{\mu=0}^{2n} |b_\mu| + \|c\|_p + \|d\|_{\alpha'} \}\,, \end{aligned} \tag{34}$$

where M *depends on* α, α', β, p, p_0, q_0, k, K *and* λ.

In both estimates (33) and (34) the term $\sum_{\mu=0}^{2n} |b_\mu|$ has to be replaced by 0 if n is negative. These results are proved in [20].

7. Initial and boundary value problem for a pseudoparabolic equation. A pseudoparabolic equation related to the generalized Beltrami equation (3) is

$$\begin{aligned} &w_{t\bar z} + q_1 w_{tz} + q_2 \overline{w_{tz}} + a_1 w_t + a_2 \overline{w_t} \\ &\qquad + \widehat{b}_1 w_{\bar z} + \widehat{b}_2 \overline{w_{\bar z}} + \widehat{b}_3 w_z + \widehat{b}_4 \overline{w_z} + c_1 w + c_2 \overline{w} + f = 0\,. \end{aligned}$$

Here the coefficients are functions of $(t, z) \in I \times D$ where $I = [0, T]$ is a compact interval with some positive T and D is a simply connected domain in $\mathbb{C}$ with smooth boundary ∂D.

For any Banach space V the spaces of continuous and of continuously differentiable mappings, respectively, from I into V endowed with the norms

$$\|w\|_{0,V} := \sup_{t\in I} \|w(t)\|_V \,, \qquad \|w\|_{1,V} := \|w\|_{0,V} + \|w'\|_{0,V} \,,$$

are denoted by $C(I;V)$ and $C^1(I;V)$, respectively. They are Banach spaces themselves. Quite obviously Theorem 2 can be generalized to (see [5, 35])

THEOREM 11. *Let D be a C^1 domain, and let α and p be real numbers satisfying*

$$1 < 2\alpha < 2 < p < \frac{1}{1-\alpha} \,.$$

Let $\lambda \in C^\alpha(\partial D)$, $a, b, c \in C(I; L_p(\overline{D}))$, $g \in C^1(I; C^\alpha(\partial D))$, $a_k \in \partial D$, $b_k \in C^1(I)$ for $0 \le k \le 2n$, $d \in W_p^1(D)$. Then any solution $W \in C^1(I; W_p^1(D))$ of the problem

$$\begin{aligned} w_{t\bar z}(t,z) + a(t,z) w_t(t,z) + b(t,z)\overline{w_t(t,z)} &= c(t,z) \qquad \text{in } I \times D \,, \\ \operatorname{Re}\{\overline{\lambda(z)} w(t,z)\} &= g(t,z) + h(t,z) \qquad \text{in } I \times \partial D \,, \\ \operatorname{Im}\{\overline{\lambda(a_k)} w(t,a_k)\} &= b_k(t) \qquad (0 \le k \le 2n \,,\ 0 \le n) \qquad \text{in } I \,, \\ w(0,z) &= d\,(z) \qquad \text{in } D \end{aligned} \tag{35}$$

satisfies the a priori estimate

$$\begin{aligned} \|w\|_{1,W_p^1(D)} \le \gamma_1 \|g\|_{1,C^\alpha(\partial D)} + \gamma_2 \sum_{k=0}^{2n} \|b_k\|_{C^1(I)} \\ + \gamma_3 \|c\|_{0,L_p(\overline{D})} + \gamma_4 \|d\|_{W_p^1(D)} \,, \end{aligned} \tag{36}$$

where the constants $\gamma_1, \dots, \gamma_4$ depend on α, p, a_k, T, D, λ and on an upper bound for $\|a\|_{0,L_p(\overline{D})} + \|b\|_{0,L_p(\overline{D})}$. Here $h(t,z) = 0$ if $0 \le n$ and for $n < 0$

$$h(t,z) := \sum_{\nu=n+1}^{-n-1} h_\nu(t) \omega^\nu(z) \,, \qquad h_{-\nu}(t) = \overline{h_\nu(t)} \qquad (|\nu| \le -n-1) \,,$$

where ω is the conformal mapping from D onto the unit disc satisfying $\omega(0) = 0$, $\omega'(0) > 0$ and $h_\nu \in C^1(I)$ are to be determined properly, so that for $n < 0$, (35) is solvable. Moreover, the compatibility conditions

$$\begin{aligned} \operatorname{Re}\{\overline{\lambda(z)}\, d(z)\} &= g(0,z) + h(0,z) \qquad \text{on } \partial D \,, \\ \operatorname{Im}\{\overline{\lambda(a_k)}\, d(a_k)\} &= b_k(0) \qquad (0 \le k \le 2n \,,\ 0 \le n) \end{aligned} \tag{37}$$

are assumed to hold.

A similar a priori estimate can be obtained for the initial Riemann–Hilbert boundary problem for the more general linear pseudoparabolic equation

$$\begin{aligned} w_{t\bar z} + q_1 w_{tz} + q_2 \overline{w_{tz}} + a_1 w_t + a_2 \overline{w_t} \\ + \widehat{b}_1 w_{\bar z} + \widehat{b}_2 \overline{w_{\bar z}} + \widehat{b}_3 w_z + \widehat{b}_4 \overline{w_z} + c_1 w + c_2 \overline{w} + f = 0 \end{aligned}$$

under appropriate conditions (see [5, 35]). In this case the constants $\gamma_1, \ldots, \gamma_4$ depend on $\|a_1\|_{0,L_p(\overline{D})} + \|a_2\|_{0,L_p(\overline{D})}$, $\sum_{j=1}^4 \|\widehat{b}_j\|_{0,L_\infty(\overline{D})}$, $\|c_1\|_{0,L_p(\overline{D})} + \|c_2\|_{0,L_p(\overline{D})}$, too. The proof of (36) is based on

Lemma. *Under the assumptions of Theorem* 11 *but with* $g \in C(I; C^\alpha(\partial D))$ *any solution to the equation*

$$w_{\bar z}(t,z) + a(t,z)w(t,z) + b(t,z)\overline{w(t,z)} = c(t,z) \qquad \text{in } I \times D$$

satisfying the boundary and side conditions from (35) *but no initial condition fulfils for any* $t \in I$ *the estimate*

$$\|w(t,\cdot)\|_{W_p^1(D)} \le \gamma_1 \|g(t,\cdot)\|_{C^\alpha(\partial D)} + \gamma_2 \sum_{k=0}^{2m} |b_k(t)| + \|c(t,\cdot)\|_{L_p(\overline{D})} \,.$$

Here γ_1, γ_2, γ_3 *depend on* t, α, p, q_0, a_k, D, λ *and on an upper bound for* $\|a(t,\cdot)\|_{L_p(\overline{D})} + \|b(t,\cdot)\|_{L_p(\overline{D})}$.

8. Half-Dirichlet problems for first order elliptic equations on the unit ball in higher dimensions. The elements of the complex Clifford algebra $\mathbb{C}_m$ $(3 \le m)$ are represented in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{C}$, $A = \{\alpha_1, \ldots, \alpha_h\} \subset \{1, \ldots, m\}$, $1 \le \alpha_1 < \ldots < \alpha_h \le m$. The basis elements

$$e_A = e_{\alpha_1 \ldots \alpha_h} = e_{\alpha_1} \ldots e_{\alpha_h} \,, \qquad e_\emptyset = e_0$$

obey the multiplication rules

$$e_j e_k + e_k e_j = -2\delta_{jk} \,, \qquad 1 \le j,k \le m \,.$$

For more details see [34, 41]. Each element $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ is identified with the element $x = \sum_{j=1}^m x_j e_j$ of $\mathbb{C}_m$. The Dirac operator

$$\overline{\partial} := \sum_{j=1}^m e_j \frac{\partial}{\partial x_j}$$

is related to the Laplace operator in $\mathbb{R}^m$ by $\overline{\partial}^2 = -\Delta$. Its fundamental solution is

$$E(x) = -\frac{1}{\omega_m} \frac{x}{|x|^m} \,,$$

where $|x| := (\sum_{j=1}^m x_j^2)^{1/2}$ and ω_m is the area of the unit sphere in $\mathbb{R}^m$.

Let J_j denote the (real-) linear mappings from $\mathbb{C}_m$ into itself satisfying

$$J_j(e_j) = -e_j \,, \qquad J_j(e_k) = e_k \text{ for } j \ne k \qquad (1 \le j,k \le m) \,.$$

Every real-linear mapping from $\mathbb{C}_m$ into itself may be written in the form

$$\sum_A C_A J_A(a) \,, \qquad C_A \in \mathbb{C}_m \,,$$

where for $A = \{\alpha_1, \ldots, \alpha_h\}$, $0 \le \alpha_1 < \ldots < \alpha_h \le m$, $J_A := J_{\alpha_1} \ldots J_{\alpha_h}$, $J_\emptyset$ is the identity mapping and J_0 is given by

$$J_0\Big(\sum_A a_A e_A\Big) = \sum_A \overline{a_A} e_A \,.$$

In [36, 61] the following special boundary value problem in the unit ball B is studied. For given $F \in L_p(\overline{B}; \mathbb{C}_m)$ $(m < p)$, $C_A(x) \in C(\overline{B}; \mathbb{C}_m)$, $f \in C^\alpha(\partial B; \mathbb{C}_m)$ $(0 < \alpha < 1$ or $\alpha = (p-m)/p)$ find a $w \in C^\alpha(\overline{B}; \mathbb{C}_m)$ such that $\overline{\partial} w \in L_p(\overline{B}; \mathbb{C}_m)$ $(m < p)$ and

$$\begin{gathered} \overline{\partial} w(x) + \sum_A C_A(x) J_A w(x) = F(x) \quad \text{in } B\,, \\ (1 + ix)(w(x) - f(x)) = 0 \quad \text{on } \partial B\,. \end{gathered} \tag{38}$$

Since for $|x| = 1$ we have

$$\frac{1+ix}{2}\frac{1-ix}{2} = 0\,, \qquad \frac{1+ix}{2} + \frac{1-ix}{2} = 1\,,$$

this boundary condition is called a *half-Dirichlet condition.*

While the operator T_D given by

$$T_D \phi(x) := \frac{1}{\omega_m} \int_D \frac{y-x}{|y-x|^m} \phi(y)\, dy$$

on $L_p(\overline{D}; \mathbb{C}_m)$ for any domain D in $\mathbb{R}^m$ is useful for treating differential equations for the operator $\overline{\partial}$, in the case of $D = B$ the operator T_1 is more appropriate, where

$$T_1 \phi(x) := \frac{1}{\omega_m} \int_B \left\{ \frac{y-x}{|y-x|^m} - i \frac{1+xy}{|1+xy|^m} \right\} \phi(y)\, dy\,.$$

It obviously satisfies $(1+ix) T_1 \phi(x) = 0$ on ∂B. Moreover, it is a compact operator on $L_p(\overline{B}; \mathbb{C}_m)$ satisfying $\overline{\partial} T_1 \phi(x) = \phi(x)$ in B. Hence, by means of the operator T_1 problem (38) can be reduced to the half-Dirichlet problem for left regular functions in B. The solution to this problem is given by

$$K f(x) := -\frac{1}{\omega_n} \int_B \frac{y-x}{|y-x|^m}\, d\sigma_y (1 + iy)\, f(y)\,,$$

where the surface element $d\sigma_y$ is

$$d\sigma_y = \sum_{j=1}^m (-1)^{j-1} e_j\, dy_1 \wedge \ldots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \ldots \wedge dy_m$$

(see [41]). Problem (38) is then seen to be equivalent to the integral equation

$$v(x) + \sum_A C_A(x) J_A (T_1 v(x)) = F(x) + \sum_A C_A(x) J_A (K f(x))\,. \tag{39}$$

The relation of w and v is

$$w(x) = T_1 v(x) + Kf(x) .$$

For $C_A \in C(\overline{B}; \mathbb{C}_m)$, $A \subset \{0, 1, \ldots, m\}$ the operator $\sum_A C_A J_A(T_1)$ is a bounded linear operator on $L_p(\overline{B}; \mathbb{C}_m)$ satisfying

$$\Big\| \sum_A C_A J_A(T_1 v) \Big\|_p \le C \Big\| \sum_A C_A \Big\|_0 \|v\|_p .$$

Here C is a suitable real constant independent of the C_A and v. From the properties of the operators T_1 and K one can deduce the following a priori estimate for solutions to (38).

THEOREM 12. *Let $C_A \in C(\overline{B}; \mathbb{C}_m)$, $A \subset \{0, 1, \ldots, m\}$, and $\| \sum_A C_A \|_0 \le M$ with $CM < 1$. Let $F \in L_p(\overline{B}; \mathbb{C}_m)$, $f \in C^\alpha(\partial B; \mathbb{C}_m)$. Then any solution to problem* (38) *satisfies*

$$\|w\|_\alpha + \|\overline{\partial} w\|_p \le \gamma_1 \|f\|_\alpha + \gamma_2 \|F\|_p ,$$

where the constants γ_1 and γ_2 only depend on α, p, C, M.

This estimate can be used to solve nonlinear problems of the form

$$\begin{aligned} \overline{\partial} w &= F(x, w) \quad \text{in } B , \\ (1 + ix)(w - f(x, w)) &= 0 \quad \text{on } \partial B \end{aligned} \tag{40}$$

(see [21]). Similarly another half-Dirichlet problem

$$(1 - ix)(w - f(x, w)) = 0 \quad \text{on } \partial B$$

can be treated.

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