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AN OPTIMAL CONTROL PROBLEM FOR A FOURTH-ORDER VARIATIONAL INEQUALITY

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An optimal control problem is considered where the state of the system is described by a variational inequality for the operator $w \to \varepsilon\Delta^2 w - \varphi(\|\nabla w\|^2)\Delta w$. A set of nonnegative functions φ is used as a control region. The problem is shown to have a solution for every fixed $\varepsilon > 0$. Moreover, the solvability of the limit optimal control problem corresponding to $\varepsilon = 0$ is proved. A compactness property of the solutions of the optimal control problems for $\varepsilon > 0$ and their relation with the limit problem are established. This type of operator arises in the theory of nonlinear plates, and the choice of a most suitable function φ is of interest for applications [2]. The problem of control of the function w has been studied in [4] for the operator under consideration, and some statements of this work will be used. Nonstationary problems with analogous operators were analyzed in [6, 7]. Some general results on control of second-order variational inequalities can be found in [1]. The first section of this paper deals with the control problem for our fourth-order operator, the second considers a second-order operator, and the third studies the relationship between the solutions of the two problems.

I. Fourth-order operator. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\Omega$; let $H^s(\Omega)$ be the Sobolev space of functions having s generalized derivatives square summable in Ω. The closure of the smooth compactly supported functions in Ω in the $H^s(\Omega)$ norm is denoted by $H_0^s(\Omega)$. Let $\psi \in H^2(\Omega)$ be a given function, $\psi|_{\partial\Omega} < 0$. We define a convex and closed set in $H^{2,0}(\Omega) \equiv H^2(\Omega) \cap H_0^1(\Omega)$ as follows:

$$K_2 = \{w \in H^{2,0}(\Omega) | w(x) \geq \psi(x), x \in \Omega\}.$$

Consider the variational inequality

$$(1)\qquad w \in K_2, \quad \varepsilon(\Delta w, \Delta\overline{w} - \Delta w) + \varphi(\|\nabla w\|^2)(\nabla w, \nabla\overline{w} - \nabla w) \geq (f, \overline{w} - w) \quad \forall \overline{w} \in K_2.$$

Here $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$. Assume that $f \in L^2(\Omega)$. Let Φ be a convex and closed subset of $H^1(0,\infty)$ consisting of nonnegative functions. The cost functional is

$$E_\varepsilon(\varphi) = \|w(\varphi) - w_0\| + \|\varphi\|_1, \quad \varphi \in \Phi\,.$$

Here $w(\varphi)$ is the solution of the variational inequality (1) corresponding to φ (some conditions on φ ensuring the existence and uniqueness of solutions to (1) are given below); $w_0 \in L^2(\Omega)$ is a prescribed element; $\|\cdot\|_s$ is the norm in $H^s(\Omega)$ or in $H^s(0,\infty)$, $\|\cdot\|_0 \equiv \|\cdot\|$. The optimal control problem is to find $\varphi \in \Phi$ that

$$E_\varepsilon(\varphi) \le E_\varepsilon(\overline{\varphi}) \quad \forall \overline{\varphi} \in \Phi\,. \tag{2}$$

At this stage $\varepsilon > 0$ is assumed to be fixed. The dependence of the solutions on ε will be discussed later.

First, we present a well-known statement without proof.

LEMMA 1. *Let $\varphi \in \Phi$ and suppose $\sqrt{s}\varphi(s)$ is a nondecreasing function of s. Then the operator $w \to -\varphi(\|\nabla w\|^2)\Delta w$ is monotone from $H^1_0(\Omega)$ into its dual.*

This lemma is a particular case of a statement proved in [3]. Note that $H^1(0,\infty)$ functions are continuous in $[0,\infty)$ (see [5]).

Assume that for each $\varphi \in \Phi$, the function $\sqrt{s}\varphi(s)$ is nondecreasing. Set

$$\Pi^\varphi_\varepsilon(w) = \frac{\varepsilon}{2}\|\Delta w\|^2 + \frac{1}{2}\int\limits_0^{\|\nabla w\|^2} \varphi(s)\,ds - (f,w)\,,$$

which allows inequality (1) to be written as follows:

$$w \in K_2, \quad \partial\Pi^\varphi_\varepsilon(w)(\overline{w} - w) \ge 0 \quad \forall \overline{w} \in K_2\,.$$

Here $\partial\Pi^\varphi_\varepsilon(w)$ is the derivative of the functional Π^φ_ε at the point w. Observe that, according to Lemma 1, the operator $w \to \partial\Pi^\varphi_\varepsilon(w)$ is monotonous from $H^{2,0}(\Omega)$ into its dual, and therefore, the variational inequality (1) is equivalent to the problem of minimization of $\Pi^\varphi_\varepsilon(w)$ on K_2. It follows that, for every $\varphi \in \Phi$, (1) has a unique solution. This is a consequence of the coercivity and lower semicontinuity of Π^φ_ε on $H^{2,0}(\Omega)$.

THEOREM 1. *Suppose Φ satisfies the above conditions. Then the optimal control problem* (2) *has a solution.*

Proof. Choose a minimizing sequence $\varphi_n \in \Phi$. Then $\{\varphi_n\}$ is bounded in $H^1(0,\infty)$. Passing to a subsequence if necessary, we may assume that $\varphi_n \to \varphi$ weakly in $H^1(0,\infty)$. The problem

$$w_n \in K_2, \quad \partial\Pi^{\varphi_n}_\varepsilon(w_n)(\overline{w} - w_n) \ge 0 \quad \forall \overline{w} \in K_2 \tag{3}$$

has a solution for every n. By fixing $\overline{w} \in K_2$, we may deduce from (3) that

$$\Pi^{\varphi_n}_\varepsilon(w_n) \le \Pi^{\varphi_n}_\varepsilon(\overline{w}) \le c$$

with a constant c independent of n. Since $\varphi_n \geq 0$ we get

$$\|\Delta w_n\|^2 \leq c\,.$$

Recall that so far ε is considered to be fixed. The obtained estimate means that $\{w_n\}$ is bounded in $H^{2,0}(\Omega)$. Passing to a subsequence if necessary, we can assume that $w_n \to w$ weakly in $H^{2,0}(\Omega)$ and strongly in $H_0^1(\Omega)$. Let, moreover, $\|\nabla w_n\|^2 \leq \alpha$. Then, in addition, we may assume that $\varphi_n \to \varphi$ uniformly in $[0,\alpha]$. The latter follows from the compactness of the imbedding of $H^1(0,\alpha)$ in $C[0,\alpha]$. Now we can pass to the limit in (3) using the above-mentioned convergence. Indeed,

$$\varphi_n(\|\nabla w_n\|^2) \to \varphi(\|\nabla w\|^2)\,, \qquad \liminf \|\Delta w_n\|^2 \geq \|\Delta w\|^2\,.$$

Therefore, the limit function w satisfies

$$w \in K_2, \quad \partial \Pi_\varepsilon^\varphi(w)(\overline{w} - w) \geq 0 \quad \forall \overline{w} \in K_2\,, \tag{4}$$

and hence $w = w(\varphi)$. The lower semicontinuity of the norm gives

$$\inf_{\overline{\varphi} \in \Phi} E_\varepsilon(\overline{\varphi}) = \liminf_{n\to\infty} E_\varepsilon(\varphi_n) \geq E_\varepsilon(\varphi) \geq \inf_{\overline{\varphi} \in \Phi} E_\varepsilon(\overline{\varphi})\,.$$

This means that φ minimizes E_ε on Φ. The proof is complete.

2. Second-order operator. Let us introduce a convex and closed set in $H_0^1(\Omega)$ by

$$K_1 = \{w \in H_0^1(\Omega) \mid w(x) \geq \psi(x),\ x \in \Omega\}$$

and consider the variational inequality

$$w \in K_1, \quad \varphi(\|\nabla w\|^2)(\nabla w, \nabla \overline{w} - \nabla w) \geq (f, \overline{w} - w) \quad \forall \overline{w} \in K_1\,. \tag{5}$$

We assume that $\sqrt{s}\varphi(s)$ is strictly increasing for each $\varphi \in \Phi$. Moreover, we assume $\sqrt{s}\varphi(s) \to \infty$ as $s \to \infty$, uniformly in $\varphi \in \Phi$. Then for each fixed $\varphi \in \Phi$ there exists a unique solution of (5) (see [4]). The problem of minimization of the functional Π_0^φ on K_1 is equivalent to the variational inequality (5), analogously to (1).

Now consider the optimal control problem with the same cost functional:

$$E_0(\varphi) = \|w(\varphi) - w_0\| + \|\varphi\|_1\,,$$

where $w(\varphi)$ is the solution of (5). An element $\varphi \in \Phi$ is to be found so that

$$E_0(\varphi) \leq E_0(\overline{\varphi}) \quad \forall \overline{\varphi} \in \Phi\,. \tag{6}$$

THEOREM 2. *Under the above conditions on Φ, the optimal control problem* (6) *has a solution.*

Proof. Let $\varphi_n \in \Phi$ be a minimizing sequence. Without loss of generality, we may assume that $\varphi_n \to \varphi$ weakly in $H^1(0,\infty)$. The variational inequality

$$w_n \in K_1, \quad \varphi_n(\|\nabla w_n\|^2)(\nabla w_n, \nabla \overline{w} - \nabla w_n) \geq (f, \overline{w} - w_n) \quad \forall \overline{w} \in K_1 \tag{7}$$

has a solution for every n. An equivalent form of (7) is

$$w_n \in K_1, \quad \Pi_0^{\varphi_n}(w_n) \leq \Pi_0^{\varphi_n}(\overline{w}) \quad \forall \overline{w} \in K_1\,. \tag{8}$$

Let us show that $\Pi_0^{\varphi_n}(w_n)$ is coercive uniformly in $\varphi \in \Phi$. Indeed, we have

$$\Pi_0^{\varphi}(w) - \Pi_0^{\varphi}(0) = \int_0^1 \partial \Pi_0^{\varphi}(sw)(w)\, ds\,.$$

Therefore,

$$\Pi_0^{\varphi}(w) = \int_0^{1/2} (\partial \Pi_0^{\varphi}(sw) - \partial \Pi_0^{\varphi}(0))(w)\, ds$$
$$+ \frac{1}{2}\partial \Pi_0^{\varphi}(0)(w) + \int_{1/2}^1 \partial \Pi_0^{\varphi}(sw)(w)\, ds\,.$$

According to Lemma 1, the first term of the right-hand side is non-negative; the second is equal to $-\frac{1}{2}(f,w)$, and the third is $\partial \Pi_0^{\varphi}(\bar{s}w)(w)$, $\bar{s} \in [1/2, 1]$. Consequently,

$$\Pi_0^{\varphi}(w) \ge \tfrac{1}{2}\partial \Pi_0^{\varphi}(\bar{s}w)(w) - \tfrac{1}{2}(f,w)$$
$$\ge \tfrac{1}{2}\|\nabla w\|(\varphi(\|\nabla w\|^2 \bar{s}^2)\|\bar{s}\nabla w\| - c) \to \infty$$

as $\|\nabla w\| \to \infty$, uniformly in $\varphi \in \Phi$. Fixing $\overline{w}$ in (8), we may assume that $\Pi_0^{\varphi_n}(w_n) \le c$ with a constant c independent of n. By the coercivity of Π_0^{φ}, we conclude that there exists a constant c independent of n such that

$$\|\nabla w_n\|^2 \le c\,.$$

As previously, we can assume additionally that $\varphi_n \to \varphi$ strongly in $C[0,\alpha]$. Let also $w_n \to w$ weakly in $H_0^1(\Omega)$. Note that $w \in K_1$. Now we wish to pass to the limit in (7). Inequality (8), equivalent to (7), takes the form

$$\frac{1}{2}\int_0^{\|\nabla w_n\|^2} \varphi_n(s)\, ds - (f, w_n) \le \frac{1}{2}\int_0^{\|\nabla \overline{w}_n\|^2} \varphi_n(s)\, ds - (f, \overline{w})\,.$$

At the same time, by the above considerations,

$$\liminf \int_0^{\|\nabla w_n\|^2} \varphi_n(s)\, ds \ge \int_0^{\|\nabla w\|^2} \varphi(s)\, ds\,.$$

Thus, after passing to the lower limit in both sides of (8), we obtain

$$\Pi_0^{\varphi}(w) \le \Pi_0^{\varphi}(\overline{w}) \quad \forall \overline{w} \in K_1\,,$$

which is equivalent to

$$w \in K_1, \quad \varphi(\|\nabla w\|^2)(\nabla w, \nabla \overline{w} - \nabla w) \ge (f, \overline{w} - w) \quad \forall \overline{w} \in K_1\,.$$

This means that $w = w(\varphi)$. The proof is completed as in Theorem 1.

3. On the relationship between the solutions as $\varepsilon \to 0$. We assume the same conditions on $\varphi \in \Phi$ as in the previous section. We need the following statement on the approximation of a function satisfying a bound of the form $\overline{w} \geq \psi$ by a sequence of more smooth functions [4].

LEMMA 2. *For every* $\overline{w} \in K_1$ *there exists a sequence* $\overline{w}^n \in K_2$ *strongly converging to* $\overline{w}$ *in* $H_0^1(\Omega)$.

Let φ_ε be a solution of problem (2), and let $w(\varphi_\varepsilon)$ be the corresponding solution of the variational inequality (1). The relation between the solutions of the optimal control problems (2) and (6) is characterized by the following statement.

THEOREM 3. *Passing to subsequences if necessary, we have*

$$\begin{aligned} \varphi_\varepsilon &\to \varphi && \text{weakly in } H^1(0,\infty)\,, \\ w(\varphi_\varepsilon) &\to w && \text{weakly in } H_0^1(\Omega)\,, \\ E_\varepsilon(\varphi_\varepsilon) &\to E_0(\varphi)\,. \end{aligned}$$

Here φ *is a solution of problem* (6), *and* w *is the solution of* (5) *corresponding to* φ.

Proof. Let $\varphi \in \Phi$ be any fixed element. Then for every ε

$$E_\varepsilon(\varphi_\varepsilon) \leq E_\varepsilon(\varphi)\,. \tag{9}$$

Let us show that the solutions $w(\varphi) \equiv w_\varepsilon(\varphi)$ of the variational inequality (1) corresponding to φ have H^1 norms bounded uniformly in ε. This means, in particular, the boundedness of the right-hand side of (9). The variational inequality

$$w_\varepsilon(\varphi) \in K_2, \quad \partial \Pi_\varepsilon^\varphi(w_\varepsilon(\varphi))(\overline{w} - w_\varepsilon(\varphi)) \geq 0 \quad \forall \overline{w} \in K_2\,,$$

is equivalent to

$$\Pi_\varepsilon^\varphi(w_\varepsilon(\varphi)) \leq \Pi_\varepsilon^\varphi(\overline{w}) \quad \forall \overline{w} \in K_2\,.$$

Hence, for all ε,

$$\frac{\varepsilon}{2}\|\Delta w_\varepsilon(\varphi)\|^2 + \frac{1}{2}\int\limits_0^{\|\nabla w_\varepsilon(\varphi)\|^2} \varphi(s)\,ds - (f, w_\varepsilon(\varphi)) \leq c\,,$$

and thus

$$\|\nabla w_\varepsilon(\varphi)\|^2 \leq \alpha$$

with α independent of ε

Therefore, $E_\varepsilon(\varphi)$ is bounded uniformly in ε, and then from (9) it follows that

$$\|\varphi_\varepsilon\|_1 \leq c\,.$$

Passing to a subsequence if necessary, we may assume that $\varphi_\varepsilon \to \varphi$ weakly in $H^1(0,\infty)$. Then from the inequality

$$w_\varepsilon(\varphi_\varepsilon) \in K_2\,, \quad \partial \Pi_\varepsilon^{\varphi_\varepsilon}(w_\varepsilon(\varphi_\varepsilon))(\overline{w} - w_\varepsilon(\varphi_\varepsilon)) \geq 0 \quad \forall \overline{w} \in K_2 \tag{10}$$

we get an estimate for $w_\varepsilon(\varphi_\varepsilon)$. Indeed, (10) is equivalent to

$$\Pi_\varepsilon^{\varphi_\varepsilon}(w_\varepsilon(\varphi_\varepsilon)) \leq \Pi_\varepsilon^{\varphi_\varepsilon}(\overline{w}) \quad \forall \overline{w} \in K_2\,. \tag{11}$$

So

$$\varepsilon\|\Delta w_\varepsilon(\varphi_\varepsilon)\|^2 + \|\nabla w_\varepsilon(\varphi_\varepsilon)\| \le \sqrt{\alpha}$$

with some constant α independent of ε. Taking a subsequence if necessary we may assume that, as $\varepsilon \to 0$,

$$\begin{aligned} w_\varepsilon(\varphi_\varepsilon) &\to w \quad \text{weakly in } H_0^1(\Omega),\ w \in K_1\,, \\ \varepsilon w_\varepsilon(\varphi_\varepsilon) &\to 0 \quad \text{weakly in } H^{2,0}(\Omega)\,. \end{aligned}$$

Assume additionally that $\varphi_\varepsilon \to \varphi$ uniformly in $[0,\alpha]$. From (11) it follows that

$$\frac{1}{2}\int\limits_0^{\|\nabla w_\varepsilon(\varphi_\varepsilon)\|^2} \varphi_\varepsilon(s)\,ds - (f, w_\varepsilon(\varphi_\varepsilon)) \le \Pi_\varepsilon^{\varphi_\varepsilon}(\overline{w})\,.$$

Letting $\varepsilon \to 0$ with fixed $\overline{w} \in K_2$ we have

$$\frac{1}{2}\int\limits_0^{\|\nabla w\|^2} \varphi(s)\,ds - (f, w) \le \Pi_0^{\varphi}(\overline{w})\,. \tag{12}$$

By Lemma 2, we conclude that (12) is satisfied for every $\overline{w} \in K_1$. Therefore,

$$\varphi(\|\nabla w\|^2)(\nabla w, \nabla\overline{w} - \nabla w) \ge (f, \overline{w} - w) \quad \forall \overline{w} \in K_1\,.$$

This means that $w = w(\varphi)$ and, consequently,

$$\liminf E_\varepsilon(\varphi_\varepsilon) \ge E_0(\varphi)\,. \tag{13}$$

On the other hand, for any fixed $\varphi \in \Phi$, and possibly for a subsequence, $E_\varepsilon(\varphi) \to E_0(\varphi)$. Indeed, from the variational inequality

$$w_\varepsilon(\varphi) \in K_2, \quad \partial\Pi_\varepsilon^{\varphi}(w_\varepsilon(\varphi))(\overline{w} - w_\varepsilon(\varphi)) \ge 0 \quad \forall \overline{w} \in K_2 \tag{14}$$

we get

$$\varepsilon\|\Delta w_\varepsilon(\varphi))\|^2 + \|\nabla w_\varepsilon(\varphi))\| \le c$$

uniformly in ε. Taking a subsequence if necessary, we may assume

$$\begin{aligned} w_\varepsilon(\varphi) &\to \widetilde{w} \quad \text{weakly in } H_0^1(\Omega),\ \text{strongly in } L^2(\Omega)\,, \\ \varepsilon w_\varepsilon(\varphi) &\to 0 \quad \text{weakly in } H^{2,0}(\Omega)\,. \end{aligned}$$

Let $\varepsilon \to 0$ in (14), as in (10), to obtain

$$\widetilde{w} \in K_1, \quad \varphi(\|\nabla\widetilde{w}\|^2)(\nabla\widetilde{w}, \nabla\overline{w} - \nabla\widetilde{w}) \ge (f, \overline{w} - \widetilde{w}) \quad \forall \overline{w} \in K_1\,.$$

It follows that $\widetilde{w} = w(\varphi)$, so that

$$E_\varepsilon(\varphi) \equiv \|w_\varepsilon(\varphi) - w_0\| + \|\varphi\|_1 \to E_0(\varphi)\,.$$

If now $\widetilde{\varphi}$ is a solution of the optimal control problem (6), we have

$$E_\varepsilon(\varphi_\varepsilon) \le E_\varepsilon(\widetilde{\varphi})\,.$$

Therefore

$$\limsup E_\varepsilon(\varphi_\varepsilon) \le E_0(\widetilde{\varphi})\,.$$

Together with (13), this concludes the proof.

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