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THE SPACE OF EXPONENTIALLY DECREASING ENTIRE FUNCTIONS AND ITS APPLICATION TO SOLVABILITY

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Introduction. It is well known that the Mizohata equation

$$\frac{\partial u}{\partial t} + it\frac{\partial u}{\partial x} = f, \qquad f \in C_0^\infty(\mathbb{R}^2)\,,$$

has no solution in the space $\mathcal{D}'(\mathbb{R}^2)$ of distributions, or in the space $B(\mathbb{R}^2)$ of hyperfunctions (see [T] for the historical backgrounds). On the other hand, N. Aronszajn introduced an abstract Fréchet space, the Aronszajn space of the traces of the analytic solutions of the heat equations in $\mathbb{C}^n \times \{t \in \mathbb{C}^1,\ \operatorname{Re} t > 0\}$, and M. S. Baouendi [B] simplified the complicated arguments and showed that this equation has a solution in the Aronszajn space.

In this talk, we introduce a much simpler space $\mathcal{X}(\mathbb{R}^n)$ of real analytic and exponentially decreasing functions and show that the Fourier transformation is an isomorphism on $\mathcal{X}(\mathbb{R}^n)$, and also on its strong dual $\mathcal{X}'(\mathbb{R}^n)$. As an application, applying the Fourier transformation only we show that the Mizohata operator is solvable in $C^\infty(\mathbb{R}_t; \mathcal{X}'(\mathbb{R}))$.

The complete proofs will be published elsewhere.

1. Definitions and basic properties. We introduce a space of exponentially decreasing functions and its strong dual.

Definition 1.1. We denote by $\mathcal{X}$ or $\mathcal{X}(\mathbb{R}^n)$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such

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that for any $k,\, h > 0$

$$|\phi|_{k,h} = \sup_{\substack{x\in\mathbb{R}^n\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty \tag{1.1}$$

where $\mathbb{N}_0$ is the set of all nonnegative integers. The topology in $\mathcal{X}$ defined by the semi-norms in (1.1) makes $\mathcal{X}$ a Fréchet space. In fact, it is the projective limit topology over all $h > 0$ and $k > 0$.

Furthermore, the space $\mathcal{X}$ is a Fréchet nuclear space and therefore it is reflexive. Also, it is easy to show that the space $\mathcal{X}(\mathbb{R}^n)$ is dense in $\mathcal{S}$.

We show the Fourier transformation is an isomorphism of $\mathcal{X}(\mathbb{R}^n)$.

THEOREM 1.2. *The Fourier transformation $\mathcal{F} : \phi \to \widehat{\phi}$ is a topological isomorphism of $\mathcal{X}(\mathbb{R}^n)$ with inverse given by the Fourier inversion formula.*

DEFINITION 1.3. We denote by $\mathcal{X}'$ the strong dual of $\mathcal{X}$. In other words, $u \in \mathcal{X}'$ if and only if there exist $k, h > 0$ and $C = C(k,h) > 0$ such that

$$|u(\phi)| \le C|\phi|_{k,h}, \qquad \varphi \in \mathcal{X}. \tag{1.2}$$

It is clear that the space $\mathcal{S}'$ of tempered distributions is a subclass of $\mathcal{X}'$ by Theorem 1.2. Finally, we have the following theorem.

THEOREM 1.4. *The Fourier transformation is an isomorphism of $\mathcal{X}'(\mathbb{R}^n)$.*

2. Applications. We are in a position to state the main result of our talk.

THEOREM 2.1. *The Mizohata equation*

$$\frac{\partial u}{\partial t} + it^k \frac{\partial u}{\partial x} = f\,, \qquad f \in C_0^\infty(\mathbb{R}^2)\,, \tag{2.1}$$

has a solution in the space $C^\infty(\mathbb{R}_t; \mathcal{X}'(\mathbb{R}))$.

Finally, we show that the space $\mathcal{X}(\mathbb{R}^n)$ is stable under local operators.

THEOREM 2.2. *Let $P(x,D) = \sum_{|\alpha|=0}^\infty a_\alpha(x) D^\alpha$ be a local operator, i.e., the differential operator of infinite order with the property that for any $M > 0$ there exist $L > 0$ and $B > 0$ such that*

$$\sup_{x\in\mathbb{R}^n} |D^\beta a_\alpha(x)| \le BM^{|\beta|}\beta! L^{|\alpha|}/\alpha!$$

for all α and β. Then the operator $P(x,D) : \mathcal{X} \to \mathcal{X}$ is continuous.

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