PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 27
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1992

ON A NON-STATIONARY FREE BOUNDARY TRANSMISSION PROBLEM WITH CONTINUOUS EXTRACTION AND CONVECTION, ARISING IN INDUSTRIAL PROCESSES

BUI AN TON

Department of Mathematics, University of British Columbia
Vancouver, B.C., Canada V6T 1Y4

GRZEGORZ ŁUKASZEWICZ

Institute of Applied Mathematics and Mechanics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland

Abstract. The existence of a weak solution of a non-stationary free boundary transmission problem arising in the production of industrial materials is established. The process is governed by a coupled system involving the Navier–Stokes equations and a non-linear heat equation. The stationary case was studied in [7].

Introduction. In this paper we establish the existence of a weak solution of a free boundary transmission problem with convection and continuous extraction, arising in the production of different industrial materials. The Bridgman crystal growth system of the semi-conductor industry and the casting of metal ingots are some of the examples of the type of problems considered.

In [7], Rodrigues has studied the stationary case, extending an earlier work of Cannon, Di Benedetto and Knightly [5], where there is no extraction. The existence of a weak solution for a non-stationary two-dimensional Stefan problem without extraction, and where the liquid phase is governed by the Stokes equations, has been established by Cannon, Di Benedetto and Knightly [4].

We shall consider the non-stationary free boundary transmission problem in $\Omega \times (0,T)$, $\Omega \subset \mathbb{R}^2$, with a continuous extraction, and with a liquid phase governed by the Navier–Stokes equations. The presence of continuous extraction generates some additional non-linearities in the heat equation and in the jump condition.

The mathematical formulation of the problem, as well as the main result of the paper, are given in Section 1. A linearized Navier–Stokes equation with a temperature-dependent penalty function is considered in Section 2. A non-linear heat equation is studied in Section 3. The method of retarded mollifiers is used in Section 4 to establish the existence of a weak solution for a coupled problem, involving the non-linear heat equation and the penalized Navier–Stokes equations. The proof of the equicontinuity of the solution of the penalized heat equation is given in Section 5. The main result of the paper is proved in Section 6.

1. Formulation of the problem and the main result. We shall formulate the solid-liquid free boundary problem with a natural convection in the fluid part and a given extraction velocity.

Let Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz boundary and let Γ be a regular curve dividing Ω into two simply connected sets $\Omega^\pm$ with $\partial\Omega^\pm\cap\partial\Omega$ non-empty. Let Γ_t, Ω_t^+ and Ω_t^- be the interface and the domains occupied by the liquid and the solid, respectively, at time t. Denote by $\partial\Omega_{\mathrm{F}}^\pm$ some fixed parts of the boundary $\partial\Omega^\pm\cap\partial\Omega$. Throughout the paper we shall assume that

$$\partial\Omega_{\mathrm{F}}^\pm\times(0,T)\subset\bigcup_{0<t<T}(\partial\Omega_t^\pm\setminus\Gamma_t)\,.$$

In the solid region the temperature $\theta^-(x,t)$ is governed by the initial boundary value problem

$$\text{(1.1)}\quad\begin{cases}\dfrac{\partial\theta^-}{\partial t}-k^-\Delta\theta^-+be\cdot\nabla\theta^-=0 \text{ in } \Omega_t^-\,, \quad \theta^-(x,t)<0 \text{ in } \Omega_t^-\,,\\ \dfrac{\partial\theta^-}{\partial n}(x,t)=0 \text{ on } \partial\Omega_t^-\setminus\Gamma_t\,, \quad \theta^-(x,0)=\theta_0^-(x) \text{ in } \Omega^-\,, \quad 0<t<T\,.\end{cases}$$

Here n denotes the unit exterior normal to $\partial\Omega$. (The thermal conductivity k and the heat capacity c ($k^-=k/c$) are assumed to be positive constants.) The scalar b represents the rate of extraction and is a positive constant. The vector function e satisfies the following assumption.

Assumption I. 1) *e is a $C^2(\overline{Q}_T)$ vector function*, $Q_T=\Omega\times(0,T)$, $\operatorname{div}(e)=0$ *and* $|e|\le 1$.

2) $\quad -1\le e\cdot n\le 0 \quad$ *on* $\partial\Omega_{\mathrm{F}}^+\times(0,T)\,,$

$\qquad 0\le e\cdot n\le 1 \quad$ *on* $\partial\Omega_{\mathrm{F}}^-\times(0,T)\,.$

3) $e\cdot n=0$ *on* $\partial Q_T\setminus\{\partial\Omega_{\mathrm{F}}^+\times(0,T)\}\cup\{\partial\Omega_{\mathrm{F}}^-\times(0,T)\}$ *where* $\partial Q_T=\partial\Omega\times(0,T)$.

In the liquid region the temperature $\theta^+(x,t)$ is determined by the problem

$$\text{(1.2)}\quad\begin{cases}\dfrac{\partial\theta^+}{\partial t}-k^+\Delta\theta^++\widetilde{u}\cdot\nabla\theta^+=0\,, \quad \theta^+(x,t)>0 \text{ in } \Omega_t^+\,, \quad 0<t<T\,,\\ \dfrac{\partial\theta^+}{\partial n}=0 \text{ on } \partial\Omega_t^+\setminus\Gamma_t\,, \quad \theta^+(x,0)=\theta_0^+(x) \text{ in } \Omega^+\,.\end{cases}$$

Again the thermal conductivity and the heat capacity are assumed to be positive constants. The velocity of the fluid is denoted by $\widetilde{u}$.

The motion of the fluid is governed by the Navier–Stokes equations [8]

$$\begin{cases} \dfrac{\partial\widetilde{u}}{\partial t} - \mu\Delta\widetilde{u} + (\widetilde{u}\cdot\nabla)\widetilde{u} + \nabla p = f(\theta^+) \quad \text{in } \Omega_t^+ , \\ \nabla\cdot\widetilde{u} = 0 \text{ in } \Omega_t^+ , \quad \widetilde{u} = be \text{ on } \partial\Omega_t^+ , \quad 0 < t < T , \\ \widetilde{u}(x,0) = u_0(x) \quad \text{in } \Omega^+ , \end{cases} \tag{1.3}$$

where $f(\theta^+)$ is the buoyancy force and μ is the viscosity. We assume that the viscosity is a positive constant. The general case, when μ depends on θ^+, may be treated in the same way and does not present any difficulty. It suffices to replace $-\mu\Delta\widetilde{u}$ by $-\nabla\{\mu(\theta^+)[\nabla\widetilde{u} + (\nabla\widetilde{u})^T]\}$.

At the interface we have the usual transmission boundary conditions

$$\begin{cases} \theta^+ = \theta^- = 0 , \\ k^- \dfrac{\partial\theta^-}{\partial\nu} - k^+ \dfrac{\partial\theta^+}{\partial\nu} = -\lambda\widetilde{u}\cdot\nu = -\lambda b(e\cdot\nu) \quad \text{on } \Gamma_t , \quad 0 < t < T . \end{cases} \tag{1.4}$$

Here $\lambda > 0$ is the latent heat, and ν is the unit normal to Γ_t oriented towards the liquid phase.

ASSUMPTION II. 1) *$\theta_0^\pm$ is in $L^\infty(\Omega)$; $\theta_0^+(x) > 0$ in Ω^+, $\theta_0^-(x) < 0$ in Ω^- and $\theta_0^+ = \theta_0^- = 0$ on Γ.*

2) *u_0 is in $L^2(\Omega)$, $\nabla\cdot u_0 = 0$ in Ω, $u_0 = be$ in $\Omega^- \cup \partial\Omega^+$.*

3) *f is a uniformly Lipschitz continuous function from $\mathbb{R}$ to $\mathbb{R}$ with $f(0) = 0$.*

In order to formulate the notion of weak solutions of the free boundary problem (1.1)–(1.4), we shall define the various function spaces used in the paper.

Let k be a non-negative integer and $1 < p < \infty$. We denote by $W^{k,p}(\Omega)$ the Sobolev space [6]

$$W^{k,p}(\Omega) = \{u : u \text{ and } D^\alpha u \text{ in } L^p(\Omega) ,\ |\alpha| \le k\} .$$

It is a Banach space with the norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big\{ \sum_{|\alpha|\le k} \|D^\alpha u\|^p_{L^p(\Omega)} \Big\}^{1/p} .$$

We shall write $H^k(\Omega)$ for $W^{k,2}(\Omega)$, and denote by $\|\cdot\|$ the $L^2(\Omega)$-norm.

The completions of the set of all $C_0^\infty(\Omega)$ vector functions φ such that $\operatorname{div}(\varphi) = 0$ in the $H^1(\Omega)$-norm and in the $L^2(\Omega)$-norm are denoted by $V^1(\Omega)$ and $V(\Omega)$, respectively.

$L^2(0,T;H^k(\Omega))$ is the set of equivalence classes of functions $u(\cdot,t)$ from $(0,T)$ to $H^k(\Omega)$ which are L^2-integrable. It is a Hilbert space with the norm

$$\|u\|_{L^2(0,T;H^k(\Omega))} = \Big\{ \int_0^T \|u(\cdot,t)\|^2_{H^k(\Omega)} dt \Big\}^{1/2}$$

and the obvious inner product. $L^\infty(0,T;H^k(\Omega))$ is defined with the usual modification.

Our aim now is to define the notion of a weak solution of (1.1)–(1.4). Set

$$\theta = \theta^\pm \quad \text{in } \Omega_t^\pm\,, \qquad g(\theta) = k^\pm\theta^\pm \quad \text{in } \Omega_t^\pm\,, \qquad u = \begin{cases} \widetilde{u} & \text{in } \Omega_t^+\,, \\ be & \text{in } \Omega_t^-\,, \end{cases}$$

and $\theta_0 = \theta_0^\pm$ in $\Omega^\pm$.

Let χ be a $C^\infty(\overline{Q}_T)$ scalar function. Formally, from (1.1) and (1.4) we obtain, by an integration by parts,

$$\begin{gathered} \int_0^T \int_{\Omega_t^-} \left\{ \frac{\partial\theta^-}{\partial t}\chi + k^-\nabla\theta^-\nabla\chi + b(e\cdot\nabla\theta^-)\chi \right\} dx\,dt \\ = \int_0^T \int_{\Gamma_t} k^- \frac{\partial\theta^-}{\partial\nu}\chi\,dS\,dt\,, \\ \theta^-(x,0) = \theta_0^- \quad \text{in } \Omega^-\,. \end{gathered} \tag{1.5}$$

Similarly from (1.2)–(1.4) we have

$$\begin{gathered} \int_0^T \int_{\Omega_t^+} \left\{ \frac{\partial\theta^+}{\partial t}\chi + k^+\nabla\theta^+\nabla\chi + (\widetilde{u}\cdot\nabla\theta^+)\chi \right\} dx\,dt \\ = -\int_0^T \int_{\Gamma_t} k^+ \frac{\partial\theta^+}{\partial\nu}\chi\,dS\,dt\,, \\ \theta^+(x,0) = \theta_0^+ \quad \text{in } \Omega^+\,. \end{gathered} \tag{1.6}$$

Thus, (1.5)–(1.6) gives

$$\begin{gathered} \int_0^T \int_{\Omega} \left\{ \frac{\partial\theta}{\partial t}\chi + \nabla g(\theta)\nabla\chi + (u\cdot\nabla\theta)\chi \right\} dx\,dt \\ = -\lambda b \int_0^T \int_{\Gamma_t} (e\cdot\nu)\chi\,dS\,dt\,, \\ \theta(x,0) = \theta_0(x) \quad \text{in } \Omega\,. \end{gathered} \tag{1.7}$$

Let K_θ^+ be the characteristic function corresponding to the fluid phase. Then, since $\operatorname{div}(e) = 0$, we have

$$\int_0^T \int_{\Omega} \lambda b K_\theta^+(e\cdot\nabla\chi)\,dx\,dt = -\int_0^T \int_{\Gamma_t} \lambda b(e\cdot\nu)\chi\,dS\,dt + \int_0^T \int_{\partial\Omega_{\mathrm{F}}^+} \lambda b(e\cdot n)\chi\,dS\,dt\,.$$

We may rewrite (1.7) as

$$\int_0^T\int_\Omega \left\{\frac{\partial\theta}{\partial t}\chi+\nabla g(\theta)\nabla\chi+(u\cdot\nabla\theta)\chi-\lambda bK_\theta^+(e\cdot\nabla\chi)\right\}dx\,dt$$

$$=-\int_0^T\int_{\partial\Omega_{\mathrm F}^+}\lambda b(e\cdot n)\chi\,dS\,dt\,, \tag{1.8}$$

$$\theta(x,0)=\theta_0(x)\quad \text{in } \Omega\,.$$

To introduce a weak form of the equation (1.3) it is convenient to make a change of variables. Set $v=\widetilde{u}-be$, $v_0=u_0-be(x,0)$ and let φ be a $C^\infty(Q_T)$ vector function with $\operatorname{div}(\varphi)=0$, $\varphi(x,T)=0$ and $\varphi=0$ near $\bigcup_{0<t<T}\partial\Omega_t^+$. Then, from (1.3) we have, after some formal calculations,

$$\int_0^T\int_{\Omega_t^+}\left\{-v\cdot\frac{\partial\varphi}{\partial t}+\mu\nabla v\cdot\nabla\varphi-(v+be)\cdot[(v+be)\cdot\nabla]\varphi\right\}dx\,dt \tag{1.9}$$

$$=\int_0^T\int_{\Omega_t^+}\left\{f(\theta^+)\cdot\varphi+b\left(e\cdot\frac{\partial\varphi}{\partial t}-\mu\nabla e\cdot\nabla\varphi\right)\right\}dx\,dt-\int_\Omega u_0\cdot\varphi(x,0)\,dx\,.$$

Let

$$Q^+=\{(x,t)\in Q_T:\theta(x,t)>0\}\,.$$

The equation (1.9) may be rewritten as

$$\int_{Q^+}\left\{-v\cdot\frac{\partial\varphi}{\partial t}+\mu\nabla v\cdot\nabla\varphi-(v+be)\cdot[(v+be)\cdot\nabla]\varphi-f(\theta)\cdot\varphi\right\}dx\,dt \tag{1.10}$$

$$=-\int_{\Omega^+}u_0\varphi(x,0)\,dx+\int_{Q^+}b\left\{e\cdot\frac{\partial\varphi}{\partial t}-\mu\nabla e\cdot\nabla\varphi\right\}dx\,dt\,.$$

DEFINITION 1.1. Suppose that Assumptions I, II are satisfied. Then $\{\theta,v,K\}$ is said to be a *weak solution* of (1.1)–(1.4) if:

(i) $\theta\in L^2(0,T;H^1(\Omega))\cap C(Q_T)$ and $\partial\theta/\partial t\in L^2(0,T;(H^1(\Omega))^*)$,

(ii) $v\in L^\infty(0,T;V(\Omega))\cap L^2(0,T;V^1(\Omega))$ and $v=0$ a.e. in

$$Q^-=\{(x,t)\in Q_T:\theta(x,t)<0\}\,,$$

(iii) $K\in L^\infty(Q_T)$ and $K=0$ on the set $\{(x,t)\in Q_T:\theta(x,t)=0\}$. Moreover, $0\le K_\theta^+\le K\le 1-K_\theta^-\le 1$ a.e. in Q_T where $K_\theta^\pm$ is the characteristic function of the set $Q^\pm$.

(iv) $\{\theta,v,K\}$ satisfies (1.10) for all φ in $L^2(0,T;V^1(\Omega))$ with $\partial\varphi/\partial t$ in $L^2(0,T;L^2(\Omega))$ such that $\varphi(\cdot,T)=0$ and $\operatorname{supp}\varphi(\cdot,t)\subset\{x\in\Omega:\theta(x,t)>0\}$, as well as (cf. (1.8))

$$\int_0^T \left\langle \frac{\partial\theta}{\partial t}, \psi \right\rangle dt + \int_0^T \int_\Omega \{\nabla g(\theta)\cdot\nabla\psi + ((v+be)\cdot\nabla\theta)\psi - \lambda bK(e\cdot\nabla\psi)\}\,dx\,dt = -\int_0^T \int_{\partial\Omega_{\mathrm{F}}^+} \lambda b(e\cdot n)\psi\,dS\,dt$$

for all ψ in $L^2(0,T;H^1(\Omega))$, with $\theta(x,0)=\theta_0(x)$ in Ω. Here $\langle\cdot,\cdot\rangle$ is the pairing between $(H^1(\Omega))^*$ and $H^1(\Omega)$.

The main result of the paper is the following theorem.

THEOREM 1.1. *Suppose that Assumptions* I, II *are satisfied. Then there exists a weak solution* $\{\theta, v, K\}$ *of the free boundary problem* (1.1)–(1.4) *in the sense of Definition* 1.1.

Remark. 1) The case $b=0$, i.e. when there is no extraction, can easily be obtained from Theorem 1.1 by letting $b\to 0^+$.

2) The problem of the unicity of the solution is open.

2. A linearized penalized Navier–Stokes equation. In this section we consider an initial boundary value problem for a linearized Navier–Stokes equations with a temperature-dependent penalty function.

Let β_ε be the function from $\mathbb{R}$ to $\mathbb{R}^+\cup\{0\}$ defined by

$$\beta_\varepsilon(\theta) = \begin{cases} 1, & -\infty<\theta\le -2\varepsilon,\\ -1-\varepsilon^{-1}\theta, & -2\varepsilon\le\theta<-\varepsilon,\\ 0, & -\varepsilon\le\theta<\infty. \end{cases} \tag{2.1}$$

Let w be in $C^\infty(\overline{Q}_T)$ with $\operatorname{div}(w)=0$ and consider the initial boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t} - \mu\Delta v + \{(w+be)\cdot\nabla\}v + \nabla p + \varepsilon^{-1}\beta_\varepsilon(\sigma)v \\ \qquad = f(\sigma) - b\dfrac{\partial e}{\partial t} + \mu b\Delta e - b\{(w+be)\cdot\nabla\}e \quad \text{in } Q_T,\\ v=0 \text{ on } \partial Q_T, \quad \nabla\cdot v = 0 \text{ in } Q_T, \quad v(x,0)=v_0(x) \text{ in } \Omega. \end{cases} \tag{2.2}$$

THEOREM 2.1. *Suppose that Assumptions* I, II *are satisfied. Let* $\{f,w,\sigma\}$ *be in* $L^2(Q_T)\times C^\infty(\overline{Q}_T)\times L^\infty(Q_T)$ *with* $\operatorname{div}(w)=0$. *Then there exists a unique* v_ε *in* $C(0,T;V(\Omega))\cap L^2(0,T;V^1(\Omega))$ *which is a weak solution of* (2.2). *Furthermore,*

$$\|v_\varepsilon\|^2_{C(0,T;V(\Omega))} + \|v_\varepsilon\|^2_{L^2(0,T;V^1(\Omega))} + \varepsilon^{-1}\int_{Q_T} \beta_\varepsilon(\sigma)|v_\varepsilon(x,t)|^2\,dx\,dt \le M\{\|v_0\|^2 + \|f\|^2_{L^2(Q_T)} + (1+b^2)\|e\|^2_{C^2(Q_T)} + \|w\|^2_{L^2(Q_T)}\}$$

where M is independent of ε, w *and* σ.

P r o o f. It is clear that we only have to establish the estimate. We have

$$\frac{1}{2}\|v_\varepsilon(\cdot,t)\|^2+\mu\|\nabla v_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))}+\varepsilon^{-1}\int\limits_0^t\int\limits_\Omega \beta_\varepsilon(\sigma)|v_\varepsilon(x,s)|^2\,dx\,ds$$
$$\le \frac{1}{2}\|v_0\|^2+\int\limits_0^t\int\limits_\Omega\Big\{fv_\varepsilon-\frac{\partial e}{\partial t}\cdot v_\varepsilon+b\Delta e\cdot v_\varepsilon-[(w+be)\cdot\nabla]e\cdot v_\varepsilon\Big\}\,dx\,ds$$
$$\le \frac{1}{2}\|v_0\|^2+\frac{1}{2}\int\limits_0^t\|v_\varepsilon(\cdot,s)\|^2\,ds$$
$$+M\{\|f\|^2_{L^2(Q_T)}+\|w\|^2_{L^2(Q_T)}+(1+b^2)\|e\|^2_{C^2(Q_T)}\}\,.$$

Since β_ε is non-negative, the Gronwall lemma gives the estimate of the theorem.

3. A non-linear heat equation. Let β_ε be given by (2.1) and let $\{\omega,\sigma\}$ be in $L^2(0,T;V^1(\Omega))\times L^\infty(Q_T)$. In this section we shall show the existence of a unique θ_ε such that

$$\begin{aligned}&\int\limits_0^T\Big\langle\frac{\partial\theta_\varepsilon}{\partial t},\psi\Big\rangle\,dt\\&\quad+\int\limits_{Q_T}\{g'(\sigma)\nabla\theta_\varepsilon\cdot\nabla\psi+((\omega+be)\cdot\nabla\theta_\varepsilon)\psi-\lambda b\beta_\varepsilon(-\theta_\varepsilon)\nabla\psi\cdot e\}\,dx\,dt\\&\qquad=-\int\limits_0^T\int\limits_{\partial\Omega_{\mathrm F}^+}\lambda b(e\cdot n)\psi\,dS\,dt\,,\\&\theta_\varepsilon(x,0)=\theta_0\quad\text{in }\Omega\,,\end{aligned}\tag{3.1}$$

for all ψ in $L^2(0,T;H^1(\Omega))$. Here $\langle\cdot,\cdot\rangle$ denotes the pairing between $H^1(\Omega)$ and its dual $(H^1(\Omega))^*$, and

$$g'(\sigma)=\begin{cases}k^+&\text{for }\sigma>0,\\k^-&\text{for }\sigma<0,\end{cases}\qquad g'(0)=k_0=\min\{k^+,k^-\}\,.$$

THEOREM 3.1. *Suppose that Assumptions* I, II *are satisfied and let* $\{\omega,\sigma\}$ *be in* $L^2(0,T;V^1(\Omega))\times L^\infty(Q_T)$. *There exists a unique* θ_ε *in* $L^\infty(Q_T)\cap L^2(0,T;H^1(\Omega))$ *with* $\partial\theta_\varepsilon/\partial t$ *in* $L^2(0,T;(H^1(\Omega))^*)$ *which is a solution of* (3.1). *Moreover*,

$$\|\theta_\varepsilon\|^2_{L^\infty(Q_T)}+k_0\|\theta_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))}\le M\{1+\|\theta_0\|^2_{L^\infty(\Omega)}\}$$

where M *is independent of* ε, σ *and* ω.

The key assertion of the theorem is the $L^\infty(Q_T)$-uniform boundedness of θ_ε.

Consider the auxiliary problem

$$\Big\langle\frac{\partial\theta}{\partial t}(\cdot,t),\chi\Big\rangle+\eta(|\nabla\theta|^2\nabla\theta,\nabla\chi)+(g'(\sigma)\nabla\theta,\nabla\chi)$$

$$
\begin{aligned}
&+((\omega+be)\cdot\nabla\theta,\chi)-\lambda b(e\beta_\varepsilon(-\theta),\nabla\chi)=-\int_{\partial\Omega_{\mathrm F}^+}\lambda b(e\cdot n)\chi\,dS\,,\\
&\theta(x,0)=\theta_0(x)\quad\text{in }\Omega\,,\qquad 0<\varepsilon,\eta<1\,,
\end{aligned}
\tag{3.2}
$$

for almost all t in $(0,T)$ and for all χ in $W^{1,4}(\Omega)$. Here $\langle\cdot,\cdot\rangle$ is the pairing between $W^{1,4}(\Omega)$ and its dual.

We seek $\theta_{\varepsilon\eta}=\theta$ in $L^4(0,T;W^{1,4}(\Omega))$ with $\partial\theta/\partial t$ in $\{L^4(0,T;W^{1,4}(\Omega))\}^*$ which is a solution of (3.2).

LEMMA 3.1. *Under the hypotheses of Theorem* 3.1, *for each* $\eta>0$, *there exists* $\theta_{\varepsilon\eta}=\theta$ *in* $L^4(0,T;W^{1,4}(\Omega))$ *with* $\partial\theta/\partial t$ *in* $\{L^4(0,T;W^{1,4}(\Omega))\}^*$ *which is a solution of* (3.2). *Furthermore*,

$$
\|\theta\|^2_{L^\infty(Q_T)}+\eta\|\theta\|^4_{L^4(0,T;W^{1,4}(\Omega))}+k_0\|\theta\|^2_{L^2(0,T;H^1(\Omega))}\le M\{1+\|\theta_0\|^2_{L^\infty(\Omega)}\}
$$

where M *is independent of* ε, η, σ *and* ω.

Proof. 1) Since $\beta_\varepsilon(s)$ is continuous in s and $0\le\beta_\varepsilon(s)\le 1$, the existence of a weak solution θ of (3.2) in $L^4(0,T;W^{1,4}(\Omega))$ with $\partial\theta/\partial t$ in $\{L^4(0,T;W^{1,4}(\Omega))\}^*$ follows from the standard theory of pseudo-monotone operators.

The Sobolev theorem [6] gives $W^{1,4}(\Omega)\subset L^\infty(\Omega)$, and $W^{1,4}(\Omega)$ is an algebra. Taking $\chi=\theta^{2s-1}$ in (3.2), with $2\le s<\infty$, we obtain

$$
\begin{aligned}
(2s)^{-1}\frac{d}{dt}\|\theta(\cdot,t)\|^{2s}_{L^{2s}(\Omega)}&+\eta(2s-1)\int_\Omega\theta^{2s-2}|\nabla\theta|^4\,dx\\
&+(2s-1)k_0\int_\Omega\theta^{2s-2}|\nabla\theta|^2\,dx\\
&+(2s)^{-1}\int_\Omega(\omega+be)\cdot\nabla(\theta^{2s})\,dx\\
&\le\lambda b\int_\Omega e\cdot\beta_\varepsilon(-\theta)\nabla(\theta^{2s-1})\,dx-\lambda b\int_{\partial\Omega_{\mathrm F}^+}(e\cdot n)\theta^{2s-1}\,dS\,.
\end{aligned}
\tag{3.3}
$$

2) Since $\omega=0$ on $\partial\Omega$ and $\operatorname{div}(\omega)=0$, we get

$$
(2s)^{-1}\Big|\int_\Omega(\omega+be)\cdot\nabla(\theta^{2s})\,dx\Big|\le\frac b2\Big\{\int_\Omega\theta^{2s-2}|\nabla\theta|^2\,dx+\int_\Omega\theta^{2s}dx\Big\}\,.
\tag{3.4}
$$

On the other hand,

$$
\begin{aligned}
\int_\Omega e\cdot\beta_\varepsilon(-\theta)\nabla(\theta^{2s-1})\,dx&=\int_{\partial\Omega}(e\cdot n)\beta_\varepsilon(-\theta)\theta^{2s-1}\,dS-\int_\Omega\theta^{2s-1}\nabla\cdot\{e\beta_\varepsilon(-\theta)\}\,dx\\
&=\int_{\partial\Omega_{\mathrm F}^+}(e\cdot n)\beta_\varepsilon(-\theta)\theta^{2s-1}\,dS+\int_\Omega\theta^{2s-1}(e\cdot\nabla\theta)\beta'_\varepsilon(-\theta)\,dx
\end{aligned}
$$

and

$$\lambda b \int\limits_{\partial\Omega_{\mathrm{F}}^{+}} (e\cdot n)(\beta_\varepsilon(-\theta)-1)\theta^{2s-1}dS \le \lambda b \int\limits_{\partial\Omega_{\mathrm{F}}^{+}} (2\varepsilon)^{2s-1}dS \le M(2\varepsilon)^{2s-1}.$$

Since $|\beta_\varepsilon'(-\theta)| = \varepsilon^{-1}$ for $\varepsilon<\theta<2\varepsilon$ and $\beta_\varepsilon'(-\theta)=0$ outside $[\varepsilon,2\varepsilon]$, $|\theta\beta_\varepsilon'(-\theta)|\le 2$ for $\theta\notin\{\varepsilon,2\varepsilon\}$. Observe that

$$\int\limits_{\Omega}\theta^{2s-2}dx \le \frac{2s-2}{2s}\int\limits_{\Omega}\theta^{2s}dx + s^{-1}\operatorname{mes}\Omega.$$

Hence

$$\Big|\int\limits_{\Omega}\theta^{2s-1}(e\cdot\nabla\theta)\beta_\varepsilon'(-\theta)\,dx\Big| \le \int\limits_{\Omega}\theta^{2s-2}|\nabla\theta|^2\,dx + \int\limits_{\Omega}\theta^{2s}\,dx + s^{-1}\operatorname{mes}\Omega. \tag{3.5}$$

From (3.3)–(3.5) we get, for large s,

$$\frac{d}{dt}\|\theta(\cdot,t)\|_{L^{2s}(\Omega)}^{2s} \le sM\|\theta\|_{L^{2s}(\Omega)}^{2s} + sM(2\varepsilon)^{2s-1}+M.$$

Therefore, by Gronwall's lemma,

$$\|\theta(\cdot,t)\|_{L^{2s}(\Omega)}^{2s} \le \exp(sMT)\{\|\theta_0\|_{L^{2s}(\Omega)}^{2s} + sM(2\varepsilon)^{2s-1}+M\},$$

hence

$$\|\theta(\cdot,t)\|_{L^{2s}(\Omega)} \le \exp(MT)\{\|\theta_0\|_{L^{2s}(\Omega)} + (2sM)^{1/(2s)}(2\varepsilon)^{(2s-1)/(2s)} + (2M)^{1/(2s)}\}.$$

Letting $s\to\infty$ we have

$$\|\theta(\cdot,t)\|_{L^\infty(\Omega)} \le M_1(\|\theta_0\|_{L^\infty(\Omega)}+1).$$

All the other assertions of the lemma are easy to establish.

Proof of Theorem 3.1. Let $\theta_{\varepsilon\eta}=\theta_\eta$ be as in Lemma 3.1. From the estimates of the lemma we obtain, possibly for subsequences, $\theta_\eta\to\theta$ in the weak* topology of $L^\infty(Q_T)$, $\{\theta_\eta,\eta^{1/4}\theta_\eta,\partial\theta_\eta/\partial t\}\to\{\theta,0,\partial\theta/\partial t\}$ weakly in $L^2(0,T;H^1(\Omega))\times L^4(0,T;W^{1,4}(\Omega))\times\{L^4(0,T;W^{1,4}(\Omega))\}^*$ as $\eta\to 0$. It follows from Aubin's theorem [1] that (for a subsequence and some θ) $\theta_\eta\to\theta$ in $L^2(0,T;L^2(\Omega))$. Moreover, $\theta_\eta\to\theta$ in $C(0,T;(W^{1,4}(\Omega))^*)$.

The estimates for θ stated in the theorem are an immediate consequence of those of Lemma 3.1. It is now easy to check that θ is a solution of (3.1). Moreover, $\partial\theta/\partial t$ is in $L^2(0,T;(H^1(\Omega))^*)$ and thus θ belongs to $C(0,T;L^2(\Omega))$.

The solution obtained is unique. Indeed, suppose that φ is another solution of (3.1) with all the stated properties. Then

$$\frac12\frac{d}{dt}\|\theta-\varphi\|^2 + k_0\|\nabla(\theta-\varphi)\|^2 + \int\limits_{\Omega}(\omega+be)\cdot(\theta-\varphi)\nabla(\theta-\varphi)\,dx$$
$$= \lambda b\int\limits_{\Omega}(\beta_\varepsilon(-\theta)-\beta_\varepsilon(-\varphi))e\cdot\nabla(\theta-\varphi)\,dx.$$

Since $\|\beta_\varepsilon(-\theta)-\beta_\varepsilon(-\varphi)\| \le C_\varepsilon\|\theta-\varphi\|$ we get

$$\frac{1}{2}\frac{d}{dt}\|\theta-\varphi\|^2 + k_0\|\nabla(\theta-\varphi)\|^2 \le \frac{k_0}{2}\|\nabla(\theta-\varphi)\|^2 + M\|\theta-\varphi\|^2 .$$

Hence, by the Gronwall lemma, $\theta=\varphi$. This completes the proof of Theorem 3.1.

4. An auxiliary coupled parabolic system. In this section we shall use the method of retarded mollifiers to establish the existence of $\{\theta, v\}$ in

$$\{L^\infty(Q_T)\cap L^2(0,T;H^1(\Omega))\}\times\{L^\infty(0,T;V(\Omega))\cap L^2(0,T;V^1(\Omega))\}$$

with $\partial\theta/\partial t$ in $L^2(0,T;(H^1(\Omega))^*)$ which is a solution of the following problem:

$$\int_0^T \left\langle \frac{\partial\theta}{\partial t},\psi\right\rangle dt + \int_{Q_T}\{g'(\theta)\nabla\theta\cdot\nabla\psi + (v+be)\cdot\nabla\theta\psi - \lambda b\beta_\varepsilon(-\theta)e\cdot\nabla\psi\}\,dx\,dt$$

$$= -\lambda b\int_0^T\int_{\partial\Omega_{\mathrm{F}}^+}(e\cdot n)\psi\,dS\,dt\,, \tag{4.1}$$

$$\theta(x,0)=\theta_0(x) \quad \text{in } \Omega\,,$$

for all ψ in $L^2(0,T;H^1(\Omega))$, with

$$\int_{Q_T}\left\{-v\cdot\frac{\partial\varphi}{\partial t} + \mu\nabla v\cdot\nabla\varphi - (v+be)\cdot[(v+be)\cdot\nabla]\varphi + \varepsilon^{-1}\beta_\varepsilon(\theta)\varphi\right\}dx\,dt \tag{4.2}$$

$$= \int_{Q_T}\left\{f(\theta)\varphi + be\cdot\frac{\partial\varphi}{\partial t} - \mu\nabla e\cdot\nabla\varphi\right\}dx\,dt + \int_\Omega v_0\varphi(x,0)\,dx$$

for all φ in $L^2(0,T;V^1(\Omega))$, $\partial\varphi/\partial t$ in $L^2(0,T;V(\Omega))$ and $\varphi(x,T)=0$. Here β_ε is given by (2.1).

Let $J(x,t)$ be a non-negative smooth function in $\mathbb{R}^3$ with support in $\{(x,t): |x|<t^{1/2},\ 1<t<2\}$ and such that

$$\int_{\mathbb{R}^3} J(x,t)\,dx\,dt = 1\,.$$

Let u be in $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and set [3]

$$J_\delta u(x,t) = \delta^{-3}\int_{\mathbb{R}^3} J(y\delta^{-1}, s\delta^{-1})u(x-y,t-s)\,dy\,ds$$

for $\delta>0$. Then $J_\delta u$ is called the *retarded mollifier* of u.

Let u be in $L^q(0,T;L^p(\Omega))$, $1\le p,q<\infty$, and let $u=0$ outside Q_T. It is easy to prove that:

1) $J_\delta u$ is in $C^\infty(\overline{Q}_T)$, $J_\delta u\to u$ in $L^q(0,T;L^p(\Omega))$,
2) $J_\delta u(x,t)$ depends only on $u(\cdot,s)$ for s in $(t-2\delta, t-\delta)$,
3) if u is in $L^1(0,T;V^1(\Omega))$, then $\mathrm{div}(J_\delta u)=0$.

Let $\delta = T/N$, $N = 1, 2, \ldots$, and set $\widetilde{u}_N(x,t) = J_\delta u$. Consider the following problems:

$$(4.3)\qquad \begin{cases} \left\langle \dfrac{\partial \theta_N}{\partial t}(\cdot,t),\chi \right\rangle + (g'(\widetilde{\theta}_N)\nabla\theta_N,\nabla\chi) + ((v_N+be)\cdot\nabla\theta_N,\chi) \\ \qquad\qquad - \lambda b(e\beta_\varepsilon(-\theta_N),\nabla\chi) = -\lambda b \displaystyle\int_{\partial\Omega_{\mathrm{F}}^+} (e\cdot n)\chi\,dS\,, \\ \theta_N(x,0) = \theta_0(x) \quad \text{in } \Omega\,, \end{cases}$$

for almost all t in $(0,T)$ and for all χ in $L^2(0,T;H^1(\Omega))$ with

$$(4.4)\qquad \begin{cases} \dfrac{\partial v_N}{\partial t} - \mu\Delta v_N + \{(\widetilde{v}_N + be)\cdot\nabla\}v_N + \nabla p + \varepsilon^{-1}\beta_\varepsilon(\widetilde{\theta}_N) = f(\widetilde{\theta}_N) \quad \text{in } Q_T, \\ \nabla\cdot v_N = 0, \quad v_N = 0 \text{ on } \partial Q_T, \quad v_N(x,0) = v_0(x) \text{ in } \Omega\,. \end{cases}$$

Lemma 4.1. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied and let β_ε be given by* (2.1). *Then for each N, there exists a unique $\{\theta_N, v_N\} = \{\theta_{N,\varepsilon}, v_{N,\varepsilon}\}$ which is a weak solution of* (4.3)–(4.4)). *Moreover*, *for $0 < \varepsilon \le 1$*,

1) $\|\theta_N\|_{L^\infty(Q_T)} + \|\theta_N\|_{L^2(0,T;H^1(\Omega))} + \|\partial\theta_N/\partial t\|_{L^2(0,T;(H^1(\Omega))^*)} \le M$,
2) $\|v_N\|_{L^\infty(0,T;V(\Omega))} + \|v_N\|_{L^2(0,T;V^1(\Omega))} + \varepsilon^{-1}\int_{Q_T} \beta_\varepsilon(\widetilde{\theta}_N)v_N^2\,dx\,dt \le M$,
3) $\|\partial v_N/\partial t\|_{L^2(0,T;(V^2(\Omega))^*)} \le C(\varepsilon)$

where M is independent of ε and N, and $C(\varepsilon)$ is independent of N.

Proof. Let $I_j = (jT/N, (j+1)T/N)$ with $0 \le j \le N-1$. From the properties of the retarded mollifier, the values of $\widetilde{v}_N$, $\widetilde{\theta}_N$ on $\Omega\times I_j$ depend only on the values of v_N, θ_N on $\Omega\times I_{j-1}$.

First we apply Theorem 2.1 to solve (4.4) on $I_0 = (0, T/N)$ and then use Theorem 3.1 to solve (4.3). The values $v_N(\cdot, T/N)$ and $\theta_N(\cdot, T/N)$ are uniquely determined.

Now we re-apply Theorem 2.1 on $I_1 = (T/N, 2T/N)$ to solve (4.4) and Theorem 3.1 to solve (4.3) with initial data $v_N(\cdot, T/N)$ and $\theta_N(\cdot, T/N)$. By induction we get $\{\theta_N, v_N\}$ on $(0,T)$. The estimates of the lemma follow from those of Theorems 2.1 and 3.1.

Theorem 4.1. *Suppose all the hypotheses of Theorem* 1.1 *are satisfied and let β_ε be given by* (2.1). *Then there exists $\{\theta_\varepsilon, v_\varepsilon\}$ which is a solution of* (4.1)–(4.2). *Moreover*, *for $0 < \varepsilon \le 1$*,

1) $\|\theta_\varepsilon\|_{L^\infty(Q_T)} + \|\theta_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\partial\theta_\varepsilon/\partial t\|_{L^2(0,T;(H^1(\Omega))^*)} \le M$,
2) $\|v_\varepsilon\|_{L^\infty(0,T;V(\Omega))} + \|v_\varepsilon\|_{L^2(0,T;V^1(\Omega))} + \varepsilon^{-1}\int_{Q_T} \beta_\varepsilon(\theta_\varepsilon)|v_\varepsilon|^2\,dx\,dt \le M$,

where M is independent of ε.

P r o o f. Let $\{\theta_{N,\varepsilon}, v_{N,\varepsilon}\}$ be as in Lemma 4.1. From the estimates of the lemma, we deduce, possibly for subsequences, that $\theta_{N,\varepsilon} \to \theta_\varepsilon$ weakly in $L^2(0,T;H^1(\Omega))$ and in the weak* topology of $L^\infty(Q_T)$, and $\partial\theta_{N,\varepsilon}/\partial t \to \partial\theta_\varepsilon/\partial t$ weakly in $L^2(0,T;(H^1(\Omega))^*)$ as $N \to \infty$. It follows from Aubin's theorem that $\theta_{N,\varepsilon} \to \theta_\varepsilon$ in $L^2(0,T;L^2(\Omega))$. We have

$$\begin{aligned}
&\|J_\delta(\theta_{N,\varepsilon}) - \theta_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad \le \|J_\delta(\theta_{N,\varepsilon} - \theta_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} + \|J_\delta\theta_\varepsilon - \theta_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad \le C\|\theta_{N,\varepsilon} - \theta_\varepsilon\|_{L^2(0,T;L^2(\Omega))} + \|J_\delta\theta_\varepsilon - \theta_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\,.
\end{aligned}$$

Thus $\widetilde{\theta}_{N,\varepsilon} = J_\delta(\theta_{N,\varepsilon}) \to \theta_\varepsilon$ in $L^2(0,T;L^2(\Omega))$ as $N \to \infty$.

Similarly, possibly for subsequences, $v_{N,\varepsilon} \to v_\varepsilon$ weakly in $L^2(0,T;V^1(\Omega))$ and in the weak* topology of $L^\infty(0,T;V(\Omega))$, and $\partial v_{N,\varepsilon}/\partial t \to \partial v_\varepsilon/\partial t$ weakly in $L^2(0,T;(V^2(\Omega))^*)$.

By Aubin's theorem, $v_{N,\varepsilon} \to v_\varepsilon$ in $L^2(0,T;V(\Omega))$ and a.e. in $\Omega \times (0,T)$.

As above, $\widetilde{v}_{N,\varepsilon} \to v_\varepsilon$ in $L^2(0,T;V(\Omega))$ and a.e. in $\Omega \times (0,T)$.

Since $\beta_\varepsilon(s)$ is continuous in s,

$$\beta_\varepsilon(\widetilde{\theta}_{N,\varepsilon}) \to \beta_\varepsilon(\theta_\varepsilon)\,, \qquad \beta_\varepsilon(-\theta_{N,\varepsilon}) \to \beta_\varepsilon(-\theta_\varepsilon)$$

a.e. in $\Omega \times (0,T)$ as $N \to \infty$. Hence

$$\varepsilon^{-1} \int\limits_{Q_T} \beta_\varepsilon(\theta_\varepsilon) v_\varepsilon^2\, dx\, dt \le C$$

where C is independent of ε.

All the other estimates of the theorem are trivial consequences of those of Lemma 4.1.

In view of the above convergence of $\{\theta_{N,\varepsilon}, v_{N,\varepsilon}\}$ to $\{\theta_\varepsilon, v_\varepsilon\}$, it is not difficult to check that $\{\theta_\varepsilon, v_\varepsilon\}$ is indeed a solution of (4.1)–(4.2).

5. The equicontinuity of θ_ε. In this section we prove the equicontinuity in Q_T of the family (θ_ε), $\varepsilon > 0$. This will allow us, in the next section, to choose a subsequence $(\theta_{\varepsilon'})$, $\varepsilon' \to 0$, converging uniformly to a continuous function θ.

We follow the method presented in [3] (see also [4]). It is based on two basic estimates (Lemmas 5.1 and 5.2 below); the main result of the this section is Theorem 5.1.

Let $\{\theta_\varepsilon, v_\varepsilon\}$ be as in Theorem 4.1. For simplicity we shall write $\{\theta, v\}$ for $\{\theta_\varepsilon, v_\varepsilon\}$ throughout this section.

First we shall introduce some notations. Set, for $h > 0$,

$$\theta_h(x,t) = h^{-1} \int\limits_t^{t+h} \theta(x,s)\, ds \qquad \text{(Steklov average of } \theta)\,.$$

Let (x_0, t_0) be an arbitrary point of Q_T and set

$$B(x_0, R) = \{x \in \Omega : |x - x_0| < R\}, \qquad A_k(t) = \{x \in \Omega : \theta(x,t) > k\},$$
$$Q_R(s) = B(x_0, R) \times [t_0 - sR^2, t_0], \qquad A_{k,R}(t) = B(x_0, R) \cap A_k(t).$$

We define

$$\theta_+^{(k)}(x,t) = \max\{\theta(x,t) - k, 0\}, \quad \theta_-^{(k)}(x,t) = \max\{-(\theta(x,t) - k), 0\},$$
$$M(k,R) = \operatorname*{ess\,sup}_{Q_R(s)} \theta_\pm^{(k)}.$$

LEMMA 5.1. *Let $\theta = \theta_\varepsilon$ be the solution of* (4.1) *given by Theorem* 4.1, $Q_R(s) \subset Q_T$ *and k be an arbitrary real number. Then*

$$\max_{[t_0 - s(1-\sigma_2)R^2, t_0]} \|\theta_\pm^{(k)}(\cdot,t)\|^2_{L^2(B(R-\sigma_1 R))} + \int\limits_{t_0 - s(1-\sigma_2)R^2}^{t_0} \int\limits_{B(R-\sigma_1 R)} |\nabla \theta_\pm^{(k)}|^2 \, dx\, dt$$
$$\le \gamma\{(\sigma_1 R)^{-2} + (\sigma_2 s R^2)^{-1}\} \int\limits_{t_0 - sR^2}^{t_0} \int\limits_{B(R)} |\theta_\pm^{(k)}|^2 dx\, dt + \frac{c}{\sigma_1}\{M(k,R)\}^2 R^2 + \frac{c}{\sigma_1} R^3$$

where γ and c depend only on the data, and σ_1, σ_2 are such that $0 < \sigma_1, \sigma_2 < 1$.

Proof. 1) Let $\zeta(x,t)$ be a cut-off function in $Q_R(s) \subset Q_T$ such that:

(i) $\zeta(\cdot,t) \in C_0(B(x_0,R))$, $0 \le \zeta \le 1$, $|\nabla \zeta| \le (\sigma_1 R)^{-1}$,
(ii) $\zeta(x, t_0 - sR^2) = 0$, $x \in B(x_0, R)$,
(iii) $0 \le \partial\zeta/\partial t \le (s\sigma_2 R^2)^{-1}$,
(iv) $\zeta(x,t) = 1$ for (x,t) in $B(x_0, R - \sigma_1 R) \times [t_0 - s(1-\sigma_2)R^2, t_0]$.

Let $\psi(x,t)$ be in $L^2(t_0 - sR^2, t_0; H_0^1(B(R)))$, where we write $B(R)$ for $B(x_0,R)$. From (4.1) it follows (cf. [6]) that for all $t \in [t_0 - sR^2, t_0]$ we have (with $\beta = \beta_\varepsilon$)

$$\int\limits_{t_0 - sR^2}^{t} \left\langle \frac{\partial\theta}{\partial t}, \psi \right\rangle dt + \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} \{g'(\theta)\nabla\theta \cdot \nabla\psi + ((v + be)\cdot\nabla\theta)\psi - \lambda b \beta(-\theta) e \cdot \nabla\psi\}\, dx\, dt = 0. \tag{5.1}$$

Take $\psi = \theta_\pm^{(k)}\zeta^2$. We shall carry out the calculations for $\theta_+^{(k)}\zeta^2$. Those for $\theta_-^{(k)}\zeta^2$ are similar.

Since $\{\theta, \partial\theta/\partial t\}$ is in $L^2(0,T;H^1(\Omega)) \times L^2(0,T;(H^1(\Omega))^*)$ we have $\{\zeta\theta, \partial(\zeta\theta)/\partial t\}$ in $L^2(0,T;H^1(\Omega)) \times L^2(0,T;(H^1(\Omega))^*)$. Hence

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\zeta\theta_+^{(k)}\|^2_{L^2(B(R))} &= \left\langle \frac{\partial}{\partial t}(\zeta\theta_+^{(k)}), \zeta\theta_+^{(k)} \right\rangle \\
&= \left\langle \zeta\frac{\partial}{\partial t}(\theta_+^{(k)}), \zeta\theta_+^{(k)} \right\rangle + \left\langle \frac{\partial\zeta}{\partial t}\theta_+^{(k)}, \zeta\theta_+^{(k)} \right\rangle \\
&= \left\| \left(\zeta\frac{\partial\zeta}{\partial t}\right)^{1/2} \theta_+^{(k)} \right\|^2_{L^2(B(R))} + \left\langle \frac{\partial\theta_+^{(k)}}{\partial t}, \zeta^2\theta_+^{(k)} \right\rangle
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the pairing between $H^1(\Omega)$ and its dual $(H^1(\Omega))^*$. Thus,

$$\int\limits_{t_0-sR^2}^{t}\left\langle\frac{\partial\theta}{\partial t},\zeta^2\theta_+^{(k)}\right\rangle dt=\int\limits_{t_0-sR^2}^{t}\left\langle\frac{\partial}{\partial t}\theta_+^{(k)},\zeta^2\theta_+^{(k)}\right\rangle dt \tag{5.2}$$

$$=\frac{1}{2}\|\zeta(\cdot,t)\theta_+^{(k)}(\cdot,t)\|^2_{L^2(B(R))}-\int\limits_{t_0-sR^2}^{t}\left\|\left(\zeta\frac{\partial\zeta}{\partial t}\right)^{1/2}\theta_+^{(k)}\right\|^2_{L^2(B(R))}dt\,.$$

2) Consider now the term involving $g(\theta)$. We have

$$\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)} g'(\theta)\nabla\theta\cdot\nabla(\zeta^2\theta_+^{(k)})\,dx\,dt$$

$$=\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)} g'(\theta)\{\zeta^2\nabla\theta_+^{(k)}+2\zeta\nabla\zeta\theta_+^{(k)}\}\cdot\nabla\zeta\theta\,dx\,dt$$

$$=\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)} g'(\theta)\nabla\theta_+^{(k)}\cdot\{\zeta^2\nabla\theta_+^{(k)}+2\zeta\nabla\zeta\theta_+^{(k)}\}\,dx\,dt\,.$$

Since $g'(\theta)\geq k_0$ and $0\leq\zeta\leq 1$, we obtain

$$\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)} g'(\theta)\nabla\theta\cdot\nabla(\zeta^2\theta_+^{(k)})\,dx\,dt\geq\frac{k_0}{2}\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)}\zeta^2|\nabla\theta_+^{(k)}|^2\,dx\,dt \tag{5.3}$$

$$-C(k_0)\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)}|\nabla\zeta|^2(\theta_+^{(k)})^2dx\,dt\,.$$

3) Since $\operatorname{div}(v+be)=0$, integration by parts yields

$$\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)}((v+be)\cdot\nabla\theta)\theta_+^{(k)}\zeta^2\,dx\,dt \tag{5.4}$$

$$=\frac{1}{2}\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)}(v+be)\zeta^2\nabla(\theta_+^{(k)})^2\,dx\,dt$$

$$=-\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)}(\theta_+^{(k)})^2(v+be)\zeta\cdot\nabla\zeta\,dx\,dt\equiv I\,.$$

4) We have

$$\int\limits_{t_0-sR^2}^{t}\int\limits_{B(R)}\lambda be\beta(-\theta)\cdot\nabla(\zeta^2\theta_+^{(k)})\,dx\,dt \tag{5.5}$$

$$= \int_{t_0-sR^2}^{t} \int_{B(R)} \lambda be\beta(-\theta)\zeta^2 \cdot \nabla\theta_+^{(k)}\, dx\, dt$$
$$+ 2 \int_{t_0-sR^2}^{t} \int_{B(R)} \lambda be\beta(-\theta)\theta_+^{(k)}\zeta \cdot \nabla\zeta\, dx\, dt \equiv K + J\,.$$

It now follows from (5.1)–(5.5) that

$$\tag{5.6} \frac{1}{2}\|\zeta(\cdot,t)\theta_+^{(k)}(\cdot,t)\|^2_{L^2(B(R))} + \frac{k_0}{2} \int_{t_0-sR^2}^{t} \int_{B(R)} \zeta^2|\nabla\theta_+^{(k)}|^2 dx\, dt$$
$$\le C(k_0) \int_{t_0-sR^2}^{t} \int_{B(R)} (\theta_+^{(k)})^2|\nabla\zeta|^2\, dx\, dt$$
$$+ \int_{t_0-sR^2}^{t} \int_{B(R)} (\theta_+^{(k)})^2\zeta\frac{\partial\zeta}{\partial t}\, dx\, dt + I + J + K\,.$$

5) From the properties of the cut-off function ζ, we obtain for each $t \in [t_0 - s(1-\sigma_2)R^2, t_0]$

$$\tag{5.7} \|\theta_+^{(k)}(\cdot,t)\|^2_{L^2(B(R-\sigma_1 R))} + k_0 \int_{t_0-sR^2}^{t} \int_{B(R)} \zeta^2|\nabla\theta_+^{(k)}|^2\, dx\, dt$$
$$\le 2(I+J+K) + \gamma\{(\sigma_1 R)^{-2} + (\sigma_2 sR^2)^{-1}\} \int_{t_0-sR^2}^{t_0} \int_{B(R)} |\theta_+^{(k)}|^2\, dx\, dt\,.$$

We shall now estimate I. From (5.4), we have

$$\tag{5.8} |I| \le c(\sigma_1 R)^{-1} \int_{t_0-sR^2}^{t_0} \int_{B} |\theta_+^{(k)}|^2\, |v+be|\, dx\, dt$$
$$\le c(\sigma_1 R)^{-1}\{\operatorname*{ess\,sup}_{Q_R(s)} \theta_+^{(k)}\}^2 \|v+be\|_{L^4(Q_T)} \Big(\int_{t_0-sR^2}^{t_0} \operatorname{mes} A_{k,R}(t)\, dt\Big)^{3/4}.$$

Since $v+be$ is in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$, it is also in $L^4(0,T;L^4(\Omega))$. Thus,

$$\tag{5.9} |I| \le c(\sigma_1 R)^{-1}\{M(k,R)\}^2\{1 + \|v\|_{L^2(0,T;H^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))}\}R^3$$
$$\le c_1(\sigma_1 R)^{-1}\{M(k,R)\}^2 R^3$$

by taking into account the estimates of Theorem 4.1; c_1 depends only on the data.

Let K be given by (5.5). Then it is easy to see that

$$|K| \le \eta \int_{t_0 - sR^2}^{t} \int_{B(R)} \zeta^2 |\nabla \theta_+^{(k)}|^2 \, dx \, dt + C(\eta) R^3 , \qquad \eta > 0 . \tag{5.10}$$

Finally, for J as in (5.5), we have

$$\begin{aligned} |J| &\le \int_{t_0 - sR^2}^{t_0} \int_{B(R)} \zeta^2 |\nabla \zeta| \, dx \, dt + \int_{t_0 - sR^2}^{t_0} \int_{B(R)} (\theta_+^{(k)})^2 |\nabla \zeta| \, dx \, dt \\ &\le c(\sigma_1 R)^{-1} R^4 + c(\sigma_1 R)^{-1} \{M(k,R)\}^2 \Big(\int_{t_0 - sR^2}^{t_0} \operatorname{mes} A_{k,R}(t) \, dt \Big)^{3/4} \\ &\le \frac{c_2}{\sigma_1} R^3 + \frac{c_2}{\sigma_1} \{M(k,R)\}^2 R^2 . \end{aligned} \tag{5.11}$$

Thus, from (5.7)–(5.11) we get

$$\max_{[t_0 - s(1-\sigma_2)R^2, t_0]} \|\theta_+^{(k)}(\cdot,t)\|^2_{L^2(B(R-\sigma_1 R))} + \int_{t_0 - s(1-\sigma_2)R^2}^{t_0} \int_{B(R-\sigma_1 R)} |\nabla \theta_+^{(k)}|^2 \, dx \, dt$$

$$\le \gamma \{ (\sigma_1 R)^{-2} + (\sigma_2 s R^2)^{-1} \} \int_{t_0 - sR^2}^{t_0} \int_{B(R)} |\theta_+^{(k)}|^2 \, dx \, dt + \frac{c}{\sigma_1} \{M(k,R)\}^2 R^2 + \frac{c}{\sigma_1} R^3 .$$

The lemma is proved.

We shall now establish a logarithmic estimate. The restriction to the planar domain plays a crucial role in the proof.

Lemma 5.2. *Let k be in $\mathbb{R}^+$, $\mu \ge \operatorname{ess\,sup}_{Q_R(s)} \theta_+^{(k)}$ and $0 < \eta < \mu$. Set*

$$\varphi(x,t) = \max \Big\{ \log \frac{\mu}{\mu - \theta_+^{(k)}(x,t) + \eta}, 0 \Big\} .$$

Then there exists $C = C(s)$ such that

$$\begin{aligned} &\int_{B(R-\sigma_1 R)} \varphi^2(x,t) \, dx \\ &\quad \le \int_{B(R)} \varphi^2(x, t_0 - sR^2) \, dx + C(1 + R\eta^{-1}) \sigma_1^{-2} \log(\mu \eta^{-1}) \cdot \operatorname{mes} B(R) \\ &\qquad + C\{1 + \log(\mu\eta^{-1})\} R^2 \eta^{-2} \operatorname{mes} B(R) \end{aligned}$$

for all t in $[t_0 - sR^2, t_0]$.

Proof. Let $\zeta(x)$ be a cut-off function in $Q_R(s)$ such that $\zeta(x) = 1$ on $B(R - \sigma_1 R)$, $|\nabla \zeta| \le (\sigma_1 R)^{-1}$, $0 \le \zeta \le 1$. Set $\widetilde{\varphi}(\theta_+^{(k)}) = \varphi(\theta_+^{(k)}(x,t))$. In the equation (5.1) we shall use the test function

$$\psi(x,t) = \zeta^2(x)\frac{\partial}{\partial\theta}(\widetilde{\varphi}(\theta))^2$$

where for simplicity of notations we write θ for $\theta_+^{(k)}$ when there is no confusion possible. It is clear that $(\widetilde{\varphi}^2)'' = 2(1+\widetilde{\varphi})(\widetilde{\varphi}')^2$ with $\widetilde{\varphi}' = \partial\widetilde{\varphi}(\theta)/\partial\theta$.

Since θ is in $L^2(0,T;H^1(\Omega))$, ψ belongs to $L^2(t_0 - sR^2, t_0; H^1(B(R)))$ and $\psi = 0$ on $\partial B(R) \times [t_0 - sR^2, t_0]$.

1) We now show that

$$\int\limits_{t_0-sR^2}^{t} \left\langle \frac{\partial\theta}{\partial t}, \frac{\partial}{\partial\theta}(\zeta^2\widetilde{\varphi}^2) \right\rangle dt = \int\limits_{B(R)} \zeta^2\widetilde{\varphi}^2(\theta)\,dx\Big|_{t_0-sR^2}^{t}. \tag{5.12}$$

With $\{\theta, \partial\theta/\partial t\}$ in $L^2(0,T;H^1(\Omega)) \times L^2(0,T;(H^1(\Omega))^*)$ we know that $\theta_h \to \theta$ in $L^2(0,T_1;H^1(\Omega))$ and $\partial\theta_h/\partial t \to \partial\theta/\partial t$ in $L^2(0,T_1;(H^1(\Omega))^*)$ as $h \to 0^+$ for any $T_1 < T$ (θ_h is the Steklov average of θ). Moreover, since θ belongs to $C(0,T;L^2(\Omega))$, $\theta_h \to \theta$ in $C(0,T_1;L^2(\Omega))$. We have

$$\begin{aligned}\int\limits_{t_0-sR^2}^{t} \left\langle \frac{\partial}{\partial t}\theta_h, \frac{\partial}{\partial\theta_h}(\zeta^2\widetilde{\varphi}^2(\theta_h)) \right\rangle dt &= \int\limits_{t_0-sR^2}^{t} \int\limits_{B(R)} \frac{\partial}{\partial t}(\zeta^2\widetilde{\varphi}^2(\theta_h))\,dx\,dt \\ &= \int\limits_{B(R)} \zeta^2\widetilde{\varphi}^2(\theta_h)\,dx\Big|_{t_0-sR^2}^{t}.\end{aligned}$$

A simple calculation gives

$$\begin{aligned}\left|\frac{\partial}{\partial\theta_h}(\widetilde{\varphi}^2(\theta_h))\right| &\le 2\eta^{-1}\log(\mu\eta^{-1}), \\ \left|\frac{\partial}{\partial x}\frac{\partial}{\partial\theta_h}(\widetilde{\varphi}^2(\theta_h))\right| &\le 2\eta^{-1}(\log(\mu\eta^{-1}) + \eta^{-1})\left|\frac{\partial\theta_h}{\partial x}\right|.\end{aligned} \tag{5.13}$$

Thus, possibly for subsequences we get

$$\frac{\partial}{\partial\theta_h}(\widetilde{\varphi}^2(\theta_h)) \to \frac{\partial}{\partial\theta}(\widetilde{\varphi}^2(\theta)) \quad \text{weakly in } L^2(0,T_1;H^1(\Omega)).$$

Hence

$$\int\limits_{t_0-sR^2}^{t} \left\langle \frac{\partial}{\partial t}\theta_h, \frac{\partial}{\partial\theta_h}(\zeta^2\widetilde{\varphi}^2(\theta_h)) \right\rangle dt \to \int\limits_{t_0-sR^2}^{t} \left\langle \frac{\partial}{\partial t}\theta, \frac{\partial}{\partial\theta}(\zeta^2\widetilde{\varphi}^2(\theta)) \right\rangle dt.$$

On the other hand, since $\partial\widetilde{\varphi}^2(\theta_h)/\partial\theta_h$ is a continuous function of θ_h and $\theta_h \to \theta$ in $C(0,T_1;L^2(\Omega))$, we obtain from (5.13)

$$\int\limits_{B(R)} \zeta^2\widetilde{\varphi}^2(\theta_h)\,dx\Big|_{t_0-sR^2}^{t} \to \int\limits_{B(R)} \zeta^2\widetilde{\varphi}^2(\theta)\,dx\Big|_{t_0-sR^2}^{t}.$$

Thus (5.12) is proved.

2) We now consider the remaining terms. We have, for all $\nu > 0$,

$$\int_{t_0-sR^2}^{t}\int_{B(R)} g'(\theta)\nabla\theta\cdot\nabla\psi\,dx\,dt \tag{5.14}$$
$$= \int_{t_0-sR^2}^{t}\int_{B(R)} g'(\theta)\nabla\theta\cdot\{2(1+\widetilde{\varphi})(\widetilde{\varphi}')^2\zeta^2\nabla\theta+(\widetilde{\varphi}^2)'\nabla\zeta^2\}\,dx\,dt$$
$$\geq 2k_0\int_{t_0-sR^2}^{t}\int_{B(R)}(1+\widetilde{\varphi})(\widetilde{\varphi}')^2\zeta^2|\nabla\theta|^2\,dx\,dt$$
$$-\nu\int_{t_0-sR^2}^{t}\int_{B(R)}(1+\widetilde{\varphi})(\widetilde{\varphi}')^2\zeta^2|\nabla\theta|^2\,dx\,dt$$
$$-C(\nu)\int_{t_0-sR^2}^{t}\int_{B(R)}\widetilde{\varphi}|\nabla\zeta|^2\,dx\,dt\,.$$

3) We now estimate the expressions involving the velocity vector. It is easy to see that

$$\int_{t_0-sR^2}^{t}\int_{B(R)}(v\cdot\nabla\theta)\psi\,dx\,dt \leq \nu\int_{t_0-sR^2}^{t}\int_{B(R)}(1+\widetilde{\varphi})(\widetilde{\varphi}')^2\zeta^2|\nabla\theta|^2\,dx\,dt \tag{5.15}$$
$$+C(\nu)\int_{t_0-sR^2}^{t}\int_{B(R)}\varphi|v|^2\zeta^2\,dx\,dt\,.$$

We have

$$\int_{t_0-sR^2}^{t}\int_{B(R)}(be\cdot\nabla\theta)\psi\,dx\,dt = 2b\int_{t_0-sR^2}^{t}\int_{B(R)}e\widetilde{\varphi}\zeta^2\cdot\nabla\widetilde{\varphi}\,dx\,dt \tag{5.16}$$
$$\leq \nu\int_{t_0-sR^2}^{t}\int_{B(R)}(1+\widetilde{\varphi})(\widetilde{\varphi}')^2|\nabla\theta|^2\zeta^2\,dx\,dt + C(\nu)\int_{t_0-sR^2}^{t}\int_{B(R)}|e|^2\widetilde{\varphi}\zeta^2\,dx\,dt\,.$$

In a similar fashion,

$$-\int_{t_0-sR^2}^{t}\int_{B(R)}\lambda b\beta(-\theta)e\cdot\nabla\psi\,dx\,dt \tag{5.17}$$
$$= -\lambda b\int_{t_0-sR^2}^{t}\int_{B(R)}(\beta(-\theta)e\cdot\nabla\theta)(1+\widetilde{\varphi})(\widetilde{\varphi}')^2\zeta^2dx\,dt$$

$$- \lambda b \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} \beta(-\theta) e (\widetilde{\varphi}^2)' \cdot \nabla \zeta^2 dx\, dt$$

$$\le \nu \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} (1 + \widetilde{\varphi})(\widetilde{\varphi}')^2 |\nabla \theta|^2 \zeta^2 dx\, dt$$

$$+ C(\nu) \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} (1 + \widetilde{\varphi})(\widetilde{\varphi}')^2 \zeta^2 dx\, dt$$

$$+ C_1 \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} \widetilde{\varphi}\widetilde{\varphi}' \zeta |\nabla \zeta|\, dx\, dt\,.$$

4) It follows from (5.12)–(5.17) and from (5.1) with $\psi = \zeta^2 \partial(\widetilde{\varphi}(\theta))^2/\partial\theta$ that

$$\|\zeta\varphi(\cdot,t)\|^2_{L^2(B(R))} + k_0 \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} (1 + \widetilde{\varphi})(\widetilde{\varphi}')^2 \zeta^2 |\nabla\theta|^2 dx\, dt \tag{5.18}$$

$$\le \|\zeta\varphi(\cdot, t_0 - sR^2)\|^2_{L^2(B(R))} + C \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} \widetilde{\varphi} |\nabla\zeta|^2 dx\, dt$$

$$+ C \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} \widetilde{\varphi}\zeta^2 \{|e|^2 + |v|^2\}\, dx\, dt$$

$$+ C \int\limits_{t_0 - sR^2}^{t} \int\limits_{B(R)} \{(1 + \widetilde{\varphi})(\widetilde{\varphi}')^2 \zeta^2 + \widetilde{\varphi}\widetilde{\varphi}'\zeta|\nabla\zeta|\}\, dx\, dt\,.$$

Since $0 \le \widetilde{\varphi} \le \log(\mu\eta^{-1})$, $0 \le \widetilde{\varphi}' \le \eta^{-1}$ and

$$L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \subset L^4(0,T; L^4(\Omega))\,,$$

Ω being a subset of the plane, we obtain from (5.18), by taking into account Theorem 4.1,

$$\int\limits_{B(R - \sigma_1 R)} |\varphi(x,t)|^2\, dx \tag{5.19}$$

$$\le \int\limits_{B(R)} \{\varphi(x, t_0 - sR^2)\}^2 dx + C\sigma_1^{-2} \log(\mu\eta^{-1}) \operatorname{mes} B(R)$$

$$+ C(1 + \log(\mu\eta^{-1})) R^2 \eta^{-2} \operatorname{mes} B(R)$$

$$+ C\sigma_1^{-2} \log(\mu\eta^{-1}) R\eta^{-1} \operatorname{mes} B(R)\,.$$

The lemma is proved.

The main result of the section is the following theorem.

THEOREM 5.1. *Let $\{\theta_\varepsilon, v_\varepsilon\}$ be as in Theorem* 4.1. *Then $\{\theta_\varepsilon\}$ is equicontinuous in Q_T, i.e. for every compact subset G of Q_T there exists a non-decreasing continuous function ω_G, $\omega_G(0)=0$, depending on* $\operatorname{dist}(G,\partial Q_T)$ *but independent of ε, such that*

$$|\theta_\varepsilon(x_1,t_1)-\theta_\varepsilon(x_2,t_2)|\le\omega_G(|x_1-x_2|+|t_1-t_2|^{1/2})$$

for any two points (x_1,t_1), (x_2,t_2) in G.

Proof. In view of Lemmas 5.1, 5.2 the theorem is an immediate consequence of a result of Di Benedetto [2] (see also [4]) where the inequality (2.8) of [2] with $\varkappa=1/2$ and $N=2$ is replaced by (5.19).

6. Proof of Theorem 1.1

1) Let $\{\theta_\varepsilon, v_\varepsilon\}$ be as in Theorem 4.1. From Theorem 5.1 and from the estimates of Theorem 4.1, we deduce that, possibly for subsequences, $\theta_\varepsilon\to\theta$ weakly in $L^2(0,T;H^1(\Omega))$ and in the weak* topology of $L^\infty(Q_T)$, and $\partial\theta_\varepsilon/\partial t\to\partial\theta/\partial t$ weakly in $L^2(0,T;(H^1(\Omega))^*)$. Moreover, $\theta_\varepsilon\to\theta$ uniformly on compact subsets of Q_T and thus in $L^2(Q_T)$ and a.e. in Q_T. Then $\beta_\varepsilon(-\theta_\varepsilon)$ converges to some K in the weak* topology of $L^\infty(Q_T)$. Similarly, $v_\varepsilon\to v$ weakly in $L^2(0,T;V^1(\Omega))$ and in the weak* topology of $L^\infty(0,T;V(\Omega))$.

2) Since θ is continuous in Q_T, the set $Q^+=\{(x,t)\in Q_T:\theta(x,t)>0\}$ is open in the relative topology of Q_T, and similarly for Q^-.

Recall that $\beta_\varepsilon(s)=1$ for $-\infty<s\le-2\varepsilon$, $\beta_\varepsilon(s)=-1-\varepsilon^{-1}s$ for $-2\varepsilon\le s\le-\varepsilon$ and $\beta_\varepsilon(s)=0$ otherwise. Let S^- be an arbitrary compact subset of Q^-. Then $\beta_\varepsilon(-\theta_\varepsilon)=0$ on S^- for $0<\varepsilon\le\varepsilon_0(S^-)$. Hence $K=0$ a.e. in Q^-, $K\le 1-K_\theta^-$, where K_θ^- is the characteristic function of the set Q^-.

Let S^+ be an arbitrary compact subset of Q^+. Then $\beta_\varepsilon(-\theta_\varepsilon)=1$ on S^+ for all $0<\varepsilon\le\varepsilon_0(S^+)$. Thus, $K=1$ a.e. on Q^+ and

$$K_\theta^+\le K\le 1-K_\theta^- .$$

3) We have

$$\varepsilon^{-1}\int_{S^-}v_\varepsilon^2(x,t)\,dx\,dt\le\varepsilon^{-1}\int_{Q_T}\beta_\varepsilon(\theta_\varepsilon)v_\varepsilon^2(x,t)\,dx\,dt\le C .$$

Therefore $v=0$ a.e. in S^-. Since S^- is an arbitrary compact subset of Q^-, we get $v=0$ a.e. in Q^-.

We have

$$\int_0^T\left\langle\frac{\partial}{\partial t}\theta_\varepsilon,\psi\right\rangle dt+\int_{Q_T}\{g'(\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\psi-(v_\varepsilon\cdot\nabla\psi)\theta_\varepsilon+b\psi(e\cdot\nabla\theta_\varepsilon)$$
$$-\lambda b\beta_\varepsilon(-\theta_\varepsilon)e\cdot\nabla\psi\}\,dx\,dt=-\lambda b\int_0^T\int_{\partial\Omega_{\mathrm F}^+}(e\cdot n)\psi\,dS\,dt ,$$

$$\theta_\varepsilon(x,0) = \theta_0(x) \quad \text{in } \Omega\,,$$

for all ψ in $L^2(0,T;H^1(\Omega))$.

With $g(\theta_\varepsilon) = k^+\theta_\varepsilon$ for $\theta_\varepsilon \ge 0$ and $g(\theta_\varepsilon) = k^-\theta_\varepsilon$ for $\theta_\varepsilon \le 0$ we deduce from the uniform boundedness of θ_ε in $L^2(0,T;H^1(\Omega))$ and the uniform convergence of θ_ε to θ on compact subsets of Q_T that

$$g'(\theta_\varepsilon)\nabla\theta_\varepsilon = \nabla\{g(\theta_\varepsilon)\} \to g'(\theta)\nabla\theta$$

weakly in $L^2(0,T;L^2(\Omega))$.

From the first two parts, we obtain

$$\int_0^T \left\langle \frac{\partial\theta}{\partial t}, \psi \right\rangle dt + \int_{Q_T} \{g'(\theta)\nabla\theta\cdot\nabla\psi - (v\cdot\nabla\psi)\theta + b\psi(e\cdot\nabla\theta)$$
$$-\lambda b(e\cdot\nabla\psi)K_\theta^+\}\,dx\,dt = -\lambda b\int_0^T\int_{\partial\Omega_{\mathrm{F}}^+}(e\cdot n)\psi\,dS\,dt$$

for all ψ in $L^2(0,T;H^1(\Omega))$.

Since $\theta_\varepsilon \to \theta$ in $C(0,T;(H^1(\Omega))^*)$, $\theta(x,0) = \theta_0$.

4) It remains to show the convergence for the penalized Navier–Stokes equations. Since Q^+ is an open subset of $\mathbb{R}^3$, we have the Whitney decomposition:

$$Q^+ = \bigcup_{j=1}^{\infty}(\overline{\Omega}_j \times \bar{I}_j)$$

where Ω_j are squares whose sides are parallel to the axes and I_j are intervals with

$$\mathrm{Int}\{\overline{\Omega}_j \times \bar{I}_j\} \cap \mathrm{Int}\{\overline{\Omega}_k \times \bar{I}_k\} = \emptyset \quad \text{ if } j \ne k\,.$$

Let S be any compact subset of Q^+. Then

$$S \subset \bigcup_{j=1}^{N}(\overline{\Omega}_j \times \bar{I}_j) \subset Q^+\,.$$

For φ in $L^2(0,T;V^2(\Omega_j))$ with $\partial\varphi/\partial t$ in $L^2(0,T;L^2(\Omega_j))$ and $\varphi(\cdot,T) = 0$, we get $\beta_\varepsilon(\theta_\varepsilon)\varphi(x,t) = 0$ for $\varepsilon < \varepsilon_0(j)$.

It then follows from (4.2) and from the estimates of Theorem 4.1 that

$$\|\partial v_\varepsilon/\partial t\|_{L^2(I_j;[V^2(\Omega_j)]^*)} \le M$$

where M is independent of j and of ε. An application of Aubin's theorem gives $v_\varepsilon \to v$ in $L^2(I_j;L^2(\Omega_j))$ and a.e. in $\Omega_j \times I_j$. By the Cantor diagonalization process we get a subsequence, denoted again by v_ε, such that $v_\varepsilon \to v$ in $L^2(S)$ and a.e. in S.

Since S is an arbitrary compact subset of Q^+, from (4.2) and from the above arguments we obtain

$$\int_{Q^+} \left\{ -v\cdot\frac{\partial\varphi}{\partial t} + \mu\nabla v\cdot\nabla\varphi - [(v+be)\cdot\nabla]\varphi\cdot(v+be) - f(\theta)\varphi \right\} dx\,dt = \int_{Q^+} v_0\cdot\varphi(x,0)\,dx + \int_{Q^+} b\left\{ e\cdot\frac{\partial\varphi}{\partial t} - \mu\nabla e\cdot\nabla\varphi\right\} dx\,dt$$

for all φ in $C^\infty(Q^+)$, $\operatorname{div}(\varphi)=0$, $\varphi=0$ near ∂Q^+ and $\varphi(\cdot,T)=0$. The theorem is proved.

References

[1] J. P. Aubin, *Un théorème de compacité*, C. R. Acad. Sci. Paris 256 (1963), 5042–5044.
[2] E. Di Benedetto, *Continuity of weak solutions to certain singular parabolic equations*, Ann. Mat. Pura Appl. (4) 130 (1982), 131–176.
[3] A. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier–Stokes equations*, Comm. Pure Appl. Math. 35 (6) (1982), 771–831.
[4] J. R. Cannon, E. Di Benedetto and G. H. Knightly, *The bidimensional Stefan problem with convection: the time dependent case*, Comm. Partial Differential Equations 8 (14) (1983), 1549–1604.
[5] —, —, —, *The steady state Stefan problem with convection*, Arch. Rational Mech. Anal. 73 (1980), 79–97.
[6] A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'tseva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc. Transl. Math. Monographs 23, Providence, R.I., 1968.
[7] J.-F. Rodrigues, *A steady-state Boussinesq–Stefan problem with continuous extraction*, Ann. Mat. Pura Appl. 144 (1986), 203–218.
[8] R. Temam, *Navier–Stokes Equations*, North-Holland, Amsterdam 1977.