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DIFFERENTIAL OPERATORS OF THE FIRST ORDER WITH DEGENERATE PRINCIPAL SYMBOLS

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Abstract. Let there be given a differential operator on $\mathbb{R}^n$ of the form $D = \sum_{i,j=1}^n a_{ij} \cdot x_j\partial/\partial x_i + \mu$, where $A = (a_{ij})$ is a real matrix and μ is a complex number. We study the following question: To what extent the mapping $D : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is surjective? We shall give some conditions on A and μ which assure the surjectivity of D.

1. Introduction. Let $\mathcal{T}$ be a space of functions or distributions on $\mathbb{R}^n$. A differential operator D on $\mathbb{R}^n$ is called *globally solvable* in $\mathcal{T}$ if the equation $Du = f$ has a solution $u \in \mathcal{T}$ for any $f \in \mathcal{T}$. By the classical theorems of Malgrange, Ehrenpreis and Hörmander a differential operator D with constant coefficients is globally solvable in the spaces $C^\infty(\mathbb{R}^n)$, $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. However, if D has non-constant coefficients, then in general D is not globally solvable in any of these spaces. In general, D is not even *locally* solvable. Here D is called *locally solvable* at a point $p \in \mathbb{R}^n$ if there exists a neighborhood $U(p)$ of p such that for any $f \in C_c^\infty(U)$ the equation $Du = f$ is solved on U by a distribution $u \in \mathcal{D}'(U)$. For example, Lewy's operator is not locally solvable at any point $p \in \mathbb{R}^n$.

In the present lecture, I shall study solvability questions for differential operators of the form

$$D = D_\mu^A := \sum_{i,j=1}^n a_{ij} x_j \frac{\partial}{\partial x_i} + \mu$$

with $A = (a_{ij}) \in M(n, \mathbb{R})$ and $\mu \in \mathbb{C}$. Clearly, the principal symbol of D is degenerate at the point $x = 0$. Therefore, all the difficulties which can arise manifest themselves at $x = 0$ (or at least at those points x where $\sum_{j=1}^n a_{ij} x_j = 0$ for all $i = 1, \ldots, n$). At other points, D can be locally transformed to an operator with constant coefficients by the Picard–Lindelöf theorem and all local solvability problems disappear.

Clearly, we cannot expect that the equation $Du = f$ always has a C^∞-solution u for any C^∞-function f; in fact, for $\mu = 0$ the left hand side is 0 at the origin whenever u is a C^∞-function. Therefore, in general we have to look for a distribution solution.

Our intention is to obtain results on global solvability in the space $\mathcal{S}'$ of tempered distributions. This problem can be compared with the division problem for distributions. Hörmander's famous division theorem ([4]) says that the multiplication operator $T \mapsto P \cdot T$, $T \in \mathcal{S}'$, where P is a polynomial, is surjective in $\mathcal{S}'$. As a consequence, by taking the Fourier transform, we get the surjectivity of any differential operator with constant coefficients in $\mathcal{S}'$. Now, the operator $D = D_\mu^A$ is an operator with "polynomial" coefficients and seems to be a kind of combination of a multiplication operator and a differential operator with constant coefficients. And now we ask if it can be "divided" by D, so to speak. Of course, in view of Lewy's example we cannot expect that *any* differential operator with polynomial coefficients is surjective in $\mathcal{S}'$.

There are good reasons to investigate our problem in the space $\mathcal{S}'$ rather than in the space of $\mathcal{D}'$ of all distributions. First, the space $\mathcal{S}'$ seems to have a better behavior towards our problem ([1], Ex. 2). Furthermore, the space $\mathcal{S}$ of Schwartz functions is a Fréchet space and therefore by functional analysis principles ([7], Ch. IV, §7) $D : \mathcal{S}' \to \mathcal{S}'$ is surjective if and only if the transpose operator $D^t : \mathcal{S} \to \mathcal{S}$ is injective and has closed range.

It is easily verified that D^t is of the form D_μ^A again; in fact, we have $(D_\mu^A)^t = D_{\mu-\mathrm{tr}(A)}^{-A}$ where $\mathrm{tr}(A)$ denotes the trace of A. Therefore we have to study, for $D = D_\mu^A$, under which conditions on A and μ

(a) $D : \mathcal{S} \to \mathcal{S}$ is injective, and
(b) $D\mathcal{S} \subset \mathcal{S}$ is closed.

We shall get the following

Theorem. *$D = D_\mu^A$ is globally solvable in $\mathcal{S}'(\mathbb{R}^n)$ whenever one of the following conditions holds*:

(i) *A has an eigenvalue λ with* $\operatorname{Re}\lambda \neq 0$;
(ii) $\operatorname{Re}\mu \neq 0$;
(iii) *A is nilpotent with $A \neq 0$.*

In a recent paper ([6]), D. Müller and F. Ricci have studied solvability questions for homogeneous left-invariant differential operators of second order on the Heisenberg group H_n. In particular, they have given necessary and sufficient conditions for local solvability of operators of the form

$$\sum_{i,j=1}^{n} a_{ij} Y_j X_i + \mu Z\,,$$

where $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, Z is the standard basis of the Lie algebra of H_n.

But these operators are transformed to the operators iD_μ^A by the Schrödinger representation. Now the conditions (i), (ii) and (iii) of the Theorem are contained in the sufficient conditions of Müller and Ricci. Thus it can be conjectured that it is possible to find a close relation between the Theorem and the Müller–Ricci result, using the Schrödinger representation.

2. Injectivity of D. First, let us give a different description of $D = D_\mu^A$. It is immediately verified that

$$D\varphi(x) = \frac{d}{dt} e^{\mu t} \varphi(e^{tA}x)\Big|_{t=0}, \qquad \varphi \in C^1(\mathbb{R}^n)\,. \tag{2.1}$$

Thus, apart from the factor $e^{\mu t}$, our operator D is the infinitesimal generator of the flow

$$(t, x) \mapsto e^{tA}x\,. \tag{2.2}$$

Therefore, a C^1-function φ is annihilated by D if and only if φ is relatively invariant under this flow, i.e.

$$e^{\mu t} \varphi(e^{tA}x) = \varphi(x) \tag{2.3}$$

for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. From this observation, we get the following

PROPOSITION 2.1. *Let C_∞^k be the space of all C^k-functions vanishing at ∞. The mapping $D : C_\infty^1 \to C^0$ is injective whenever one of the following conditions holds*:

(i) $\operatorname{Re}\mu \neq 0$;
(ii) *A is not similar to a skew-symmetric matrix.*

Proof. Let $\varphi \in C_\infty^1$ be given with $D\varphi = 0$. If $\operatorname{Re}\mu \neq 0$, the equation (2.3) gives $\varphi(x) = 0$.

Now let $\operatorname{Re}\mu = 0$. By assumption, A is not similar to a skew-symmetric matrix. Therefore $\{e^{tA}x \mid t \in \mathbb{R}\}$ is unbounded for all x in a dense subset M of $\mathbb{R}^n$. Since φ vanishes at ∞ we conclude from (2.3) that $\varphi(x) = 0$ for all $x \in M$. Thus $\varphi = 0$.

Remark 2.2. We cannot expect that D is injective for any A and μ. Namely, if $n = 2$,

$$A = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \tag{2.4}$$

and $\mu/\beta \in i\mathbb{Z}$, then (2.3) can be satisfied by some test function $\varphi \neq 0$, for example by

$$\varphi(z) := (z/|z|)^k \varepsilon(|z|)\,, \qquad z \in \mathbb{C} \cong \mathbb{R}^2\,, \tag{2.5}$$

where $k = i\mu/\beta$ and ε is a non-vanishing test function on $\mathbb{R}$ whose support does not contain the origin.

3. Closedness of $D\mathcal{S}$. The aim of this chapter is to give conditions on A and μ under which $D\mathcal{S}$ is closed in $\mathcal{S}$. First we look for an inversion formula for the equation $D\varphi = f$.

LEMMA 3.1 *Let* $\varphi, f \in \mathcal{S}$ *satisfy*

$$D\varphi = f\,. \tag{3.1}$$

For a given point $x \in \mathbb{R}^n$, *assume that one of the following conditions holds*:

(i) $\operatorname{Re}\mu > 0$;
(ii) $\operatorname{Re}\mu = 0$ *and the set* $\{e^{sA}x \mid s \le 0\}$ *is unbounded*;
(iii) $\operatorname{Re}\mu < 0$ *and there are* $c, \gamma > 0$ *such that* $|e^{sA}x| \ge ce^{-\gamma s}$ *for all* $s \le 0$.

Then

$$\varphi(x) = \int\limits_{-\infty}^{0} e^{\mu s} f(e^{sA}x)\, ds \tag{3.2}$$

where the integral converges absolutely.

Proof. By (2.1) we derive from (3.1) the equation

$$e^{\mu s} f(e^{sA}x) = \frac{d}{dt} e^{\mu(s+t)} \varphi(e^{(s+t)A}x) = \frac{d}{ds} e^{\mu s} \varphi(e^{sA}x)\,. \tag{3.3}$$

Using the Jordan canonical form of A we observe that by the conditions (i), (ii) or (iii) the integral in (3.2) converges absolutely and that

$$\lim_{s\to-\infty} e^{\mu s} \varphi(e^{sA}x) = 0\,. \tag{3.4}$$

Therefore we get

$$\int\limits_{-\infty}^{0} e^{\mu s} f(e^{sA}x)\, ds = e^{\mu s}\varphi(e^{sA}x)\big|_{s=-\infty}^{s=0} = \varphi(x)\,, \tag{3.5}$$

as was to be shown.

Now we observe, again by using the Jordan canonical form of A, that each of the conditions (i)–(iii) of Lemma 3.1 holds for all x in a Zariski open set if it holds for one x. In this case we get an almost everywhere defined function by setting

$$S_\mu^A f(x) = \int\limits_{-\infty}^{0} e^{\mu s} f(e^{sA}x)\, ds \tag{3.6}$$

for $f \in \mathcal{S}$.

In some situations it is useful to have an alternate formula defining $S_\mu^A f(x)$. We regard the transpose operator D^t as an operator in $\mathcal{S}'$. Then we observe that the closure $\overline{D\mathcal{S}}$ of $D\mathcal{S}$ in $\mathcal{S}$ is just the orthogonal complement of the kernel of D^t.

LEMMA 3.2. *For a given point* $x \in \mathbb{R}^n$, *assume that one of the following conditions holds*:

(i) $\operatorname{Re}\mu > 0$ *and there are* $c, \gamma > 0$ *such that* $|e^{sA}x| \geq ce^{\gamma s}$ *for all* $s \geq 0$;
(ii) $\operatorname{Re}\mu = 0$ *and the sets* $\{e^{sA}x \mid s \leq 0\}$ *and* $\{e^{sA}x \mid s \geq 0\}$ *are unbounded*;
(iii) $\operatorname{Re}\mu < 0$ *and there are* $c, \gamma > 0$ *such that* $|e^{sA}x| \geq ce^{-\gamma s}$ *for all* $s \leq 0$.

Then

$$u \mapsto \int_{\mathbb{R}} e^{\mu s} u(e^{sA}x)\, ds\,, \qquad u \in \mathcal{S}\,, \tag{3.7}$$

is a tempered distribution which belongs to $\ker D^t$, *and for* $f \in \overline{D\mathcal{S}}$ *we have*

$$S_\mu^A f(x) = -\int_0^\infty e^{\mu s} f(e^{sA}x)\, ds\,. \tag{3.8}$$

Proof. By Lemma 3.1 it is clear that (3.7) defines a tempered distribution. To see that it belongs to $\ker D^t$ we have to show that

$$\int e^{\mu s} D\varphi(e^{sA}x)\, ds = 0 \tag{3.9}$$

for all $\varphi \in \mathcal{S}$. This can be derived from (3.3). Now (3.8) follows from $\overline{D\mathcal{S}} = (\ker D^t)^\perp$, and our lemma is proved.

Our method for proving closedness of $D\mathcal{S}$ is to prove

$$S_\mu^A f \in \mathcal{S} \tag{3.10}$$

for $f \in \overline{D\mathcal{S}}$. For this we use the Sobolev embedding theorem ([3]). Given $m \in \mathbb{N}$ we consider the Sobolev space H^m of all functions u on $\mathbb{R}^n$ such that for all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfying $|\alpha| \leq m$ the derivatives $\partial^\alpha u$ (in the sense of distributions) belong to $L^2(\mathbb{R}^n)$. The norm in H^m is given by

$$\|u\|_{H^m} = \Big(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^2}^2\Big)^{1/2}. \tag{3.11}$$

Clearly, we have $\mathcal{S} \subsetneq \bigcap_{m\in\mathbb{N}} H^m$. Therefore we want to modify H^m. For $b, m \in \mathbb{N}$ we consider the space H_b^m of all functions u such that $x^\beta \partial^\alpha u(x)$ belongs to L^2 for all multi-indices α, β satisfying $|\alpha| \leq m$, $|\beta| \leq b$. The norm in H_b^m is defined by

$$\|u\|_{m,b} := \Big(\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq b}} \|x^\beta \partial^\alpha u\|_{L^2}^2\Big)^{1/2}. \tag{3.12}$$

Now Sobolev's embedding theorem gives

$$\bigcap_{m,b\in\mathbb{N}} H_b^m = \mathcal{S}\,. \tag{3.13}$$

Given $f \in \mathcal{S}$ and $s \in \mathbb{R}$, we define a distribution T_f^s on $\mathbb{R}^n$ by

$$\langle T_f^s, u\rangle := \int f(e^{sA}x) u(x)\, dx\,. \tag{3.14}$$

Denoting by $a_{ij}(s)$ the matrix coefficients of e^{sA} we have

$$\frac{\partial}{\partial x_j} T_f^s = \sum_{i=1}^{n} a_{ij}(s) T^s_{\partial f/\partial x_i}\,. \tag{3.15}$$

Denoting by $\widetilde{a}_{ij}(s)$ the matrix coefficients of e^{-sA} we have

$$x_j T_f^s = \sum_{i=1}^{n} \widetilde{a}_{ji}(s) T^s_{x_i f}\,. \tag{3.16}$$

Generalizing (3.6), for a measurable weight function $w : \,]-\infty, 0] \to \mathbb{C}$ we define the distribution

$$S_w^A f := \int\limits_{-\infty}^{0} w(s) T_f^s \, ds \tag{3.17}$$

provided that this integral converges in the space of tempered distributions. In this case $S_w^A f$ is called *well-defined*.

The following lemma is obvious:

LEMMA 3.3. *Assume that there are $c, \gamma > 0$ such that*

$$|w(s)| \le c e^{\gamma s}\,. \tag{3.18}$$

If γ is sufficiently large, then for any $f \in \mathcal{S}$ the distributions $S_w^A f$, $S^A_{wa_{ij}}(\partial f/\partial x_i)$ and $S^A_{w\tilde{a}_{ji}}(x_i f)$ are well-defined and

$$\frac{\partial}{\partial x_j}(S_w^A f) = \sum_{i=1}^{n} S^A_{wa_{ij}}\left(\frac{\partial f}{\partial x_i}\right) \tag{3.19}$$

$$x_j(S_w^A f) = \sum_{i=1}^{n} S^A_{w\tilde{a}_{ji}}(x_i f)\,. \tag{3.20}$$

LEMMA 3.4. *For given $b, m \in \mathbb{N}$ there exists $\gamma > 0$ with the following property: If (3.18) holds for some $c > 0$, then for any $f \in \mathcal{S}$ the distribution $S_w^A f$ belongs to H_b^m and S_w^A defines a continuous operator from $\mathcal{S}$ to H_b^m.*

Proof. By iterating Lemma 3.3, we only need to show that for sufficiently large γ

$$\|S^A_{wa}(x^\beta \partial^\alpha f)\|^2_{L^2} < \varepsilon \tag{3.21}$$

for all f in a 0-neighborhood U in $\mathcal{S}$, where a is a product of functions a_{ij} and $\widetilde{a}_{ji}$. Writing for a small $\delta > 0$

$$w(s)a(s) = e^{\delta s/2}(w(s)e^{-\delta s/2}a(s)) \tag{3.22}$$

and using the Cauchy–Schwarz inequality, we can estimate the expression in (3.21) by

$$\int_{\mathbb{R}^n}\int_{-\infty}^{0} e^{\delta s}\,ds \int_{-\infty}^{0} |w(s)|^2 e^{-\delta s}|a(s)|^2 |e^{sA}x|^{2|\beta|}|\partial^\alpha f(e^{sA}x)|^2\,ds\,dx\,. \tag{3.23}$$

Using the change of variables formula, obviously it is enough to show that

$$\int_{-\infty}^{0} |w(s)|^2 e^{-\delta s}|a(s)|^2 e^{-s\,\mathrm{tr}(A)} \int_{\mathbb{R}^n} |x|^{2|\beta|}|\partial^\alpha f(x)|^2\,dx\,ds < \varepsilon \tag{3.24}$$

for all $f \in U$. Of course, this holds provided that γ is sufficiently large, and the proof is complete.

Remark 3.5. Let $\sigma(A)$ denote the spectrum of A. If $\sigma(A) \subseteq i\mathbb{R}$, then S_w^A defines a continuous operator from $\mathcal{S}$ to H_b^m whenever (3.18) is satisfied at least for one $\gamma > 0$. This follows immediately from the proof of Lemma 3.4 keeping in mind that $a_{ij}(s)$ and $\widetilde{a}_{ij}(s)$ are bounded by a polynomial in this case. By (3.13) we conclude that $\varphi := S_\mu^A f \in \mathcal{S}$ for any $f \in \mathcal{S}$ whenever $\operatorname{Re}\mu > 0$. The techniques of Lemma 3.1 give $D_\mu^A \varphi = f$. Therefore, replacing A and μ by $-A$ and $-\mu$ in case of need we get

PROPOSITION 3.6. *If* $\sigma(A) \subseteq i\mathbb{R}$ *and* $\operatorname{Re}\mu \neq 0$, *the mapping* $D_\mu^A : \mathcal{S} \to \mathcal{S}$ *is bijective.*

Let λ be an eigenvalue of A. Then, of course, λ is also an eigenvalue of the transpose matrix A^t. We can view A^t as an endomorphism of the dual space $(\mathbb{R}^n)'$ of $\mathbb{R}^n$. Take an eigenvector $l : \mathbb{R}^n \to \mathbb{C}$ of A^t for the eigenvalue λ. We may assume that l is real-valued if $\lambda \in \mathbb{R}$. Obviously, we have

$$l(Ax) = \lambda l(x)\,, \tag{3.25}$$

and therefore

$$l(e^{tA}x) = e^{\lambda t} l(x)\,, \tag{3.26}$$

$$D_\mu^A(l\varphi) = l D_{\mu+\lambda}^A \varphi\,. \tag{3.27}$$

By Hörmander's division theorem the mapping $T \mapsto lT$, $T \in \mathcal{S}'$, is surjective. Therefore the mapping $\varphi \mapsto l\varphi$, $\varphi \in \mathcal{S}$, is a topological homomorphism, and we can derive the equation

$$\overline{D_\mu^A(l\mathcal{S})} = l\overline{D_{\mu+\lambda}^A \mathcal{S}}\,. \tag{3.28}$$

Now, for any $r \in \mathbb{N}$ we define the function space

$$E_r := \overline{D_\mu^A(l^r\mathcal{S})} = l^r\overline{D_{\mu+r\lambda}^A \mathcal{S}}\,. \tag{3.29}$$

Let $z_\lambda = x_\lambda + iy_\lambda \in \mathbb{C}^n$ be an eigenvector for λ. If $\lambda \in \mathbb{R}$ we take $y_\lambda = 0$. Now we define a differential operator $\overline{d}$ by

$$\overline{d}\varphi(x) := \frac{1}{2}\cdot\frac{d}{dt}[\varphi(x + tx_\lambda) + i\varphi(x + ty_\lambda)]\Big|_{t=0}\,, \qquad \varphi \in C^\infty(\mathbb{R}^n)\,. \tag{3.30}$$

It is easily seen that

$$\overline{d}(\varphi \circ e^{tA}) = e^{\lambda t}(\overline{d}\varphi) \circ e^{tA}\,. \tag{3.31}$$

By differentiation with respect to t at $t = 0$ we get

$$\overline{d} D_\mu^A \varphi = D_{\mu+\lambda}^A \overline{d}\varphi\,. \tag{3.32}$$

Consequently, we have

$$\overline{d}\,\overline{D_\mu^A \mathcal{S}} = \overline{D_{\mu+\lambda}^A \overline{d}\mathcal{S}}\,. \tag{3.33}$$

Now we select $x_l \in \mathbb{R}^n$ such that $l(x_l) = 1$. If $\lambda \notin \mathbb{R}$ we select $y_l \in \mathbb{R}^n$ such that $l(y_l) = i$. If $\lambda \in \mathbb{R}$ we put $y_l := 0$. Let $z_l := x_l + iy_l$ and $\widetilde{z} := Az_l - \overline{\lambda} z_l =: \widetilde{x} + i\widetilde{y}$ with $\widetilde{x}, \widetilde{y} \in \mathbb{R}^n$. It is easily seen that $\widetilde{x}, \widetilde{y} \in V$. Putting

$$\overline{\partial}\varphi(x) := \frac{1}{2} \cdot \frac{d}{dt}[\varphi(x + tx_l) + i\varphi(x + ty_l)]\bigg|_{t=0}\,, \tag{3.34}$$

$$\widetilde{\partial}\varphi(x) := \frac{1}{2} \cdot \frac{d}{dt}[\varphi(x + t\widetilde{x}) + i\varphi(x + t\widetilde{y})]\bigg|_{t=0} \tag{3.35}$$

for $\varphi \in C^\infty(\mathbb{R}^n)$ we calculate

$$\overline{\partial} D_0^A = D_0^A \overline{\partial} + \overline{\lambda}\,\overline{\partial} + \widetilde{\partial}\,. \tag{3.36}$$

We conclude

$$\overline{\partial} D_\mu^A = D_{\mu+\bar{\lambda}}^A \overline{\partial} + \widetilde{\partial} \tag{3.37}$$

and thus

$$\overline{\partial}^k D_\mu^A = D_{\mu+k\bar{\lambda}}^A \overline{\partial}^k + k\overline{\partial}^{k-1}\widetilde{\partial} \tag{3.38}$$

for $k = 0, 1, 2, \ldots$

Because of (3.25) the kernel V of l is A-invariant, and thus we can define a differential operator D_μ^0 on V by

$$D_\mu^0 \varphi^0(x^0) = \frac{d}{dt} e^{\mu t} \varphi^0(e^{tA} x^0)\bigg|_{t=0}\,, \qquad \varphi^0 \in C^\infty(V)\,. \tag{3.39}$$

LEMMA 3.7. *Assume that $(D^0_{\mu+k\lambda+k'\bar{\lambda}})^t$ is globally solvable in $\mathcal{S}'(V)$ for $k, k' = 0, 1, 2, \ldots$ Let $r \in \mathbb{N}$ be given. Then for each $f \in \overline{D_\mu^A \mathcal{S}(\mathbb{R}^n)}$ there exists $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$f - D_\mu^A \psi \in E_r\,. \tag{3.40}$$

Proof. Having proved the assertion for $r = 1$, we can get it for all r by iteration.

Thus, let $f \in \overline{D_\mu^A \mathcal{S}}$ be given. Let $\varphi \mapsto \varphi^0$ be the restriction map from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(V)$. Then f^0 belongs to $\overline{D_\mu^0 \mathcal{S}(V)}$. Since $\mathcal{S}(V)$ is a Fréchet space, by assumption D_μ^0 is a topological isomorphism from $\mathcal{S}(V)$ onto its (closed) range in

$\mathcal{S}(V)$, and therefore

$$f^0 = D^0_\mu \psi^0 \tag{3.41}$$

for some $\psi^0 \in \mathcal{S}(V)$. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ is an extension of ψ^0, the function $f - D^A_\mu \psi$ vanishes on V. Now, in case of real λ the subspace V has codimension one and thus

$$f - D^A_\mu \psi = l f_1 \tag{3.42}$$

for some $f_1 \in \mathcal{S}(\mathbb{R}^n)$ (see [8], Chap. V, §5).

In case of non-real λ it is harder to get equation (3.42). Let $f = \lim_{\nu\to\infty} D^A_\mu \varphi_\nu$ with $\varphi_\nu \in \mathcal{S}(\mathbb{R}^n)$, $\nu \in \mathbb{N}$. First we shall prove by induction on k that the sequence $(\overline{\partial}^k \varphi_\nu)^0$ converges in $\mathcal{S}(V)$ for each k. For $k = 0$, the assertion follows from the equation

$$f^0 = \lim_{\nu\to\infty} D^0_\mu \varphi^0_\nu \tag{3.43}$$

keeping in mind that D^0_μ is a topological isomorphism from $\mathcal{S}(V)$ onto its range. Now assume by induction hypothesis that

$$\lim_{\nu\to\infty} (\overline{\partial}^{k-1}\varphi_\nu)^0 = \psi^0_{k-1} \tag{3.44}$$

and let $\psi_{k-1} \in \mathcal{S}(\mathbb{R}^n)$ be an extension of ψ^0_{k-1}. By (3.38) we have

$$\begin{aligned}(\overline{\partial}^k f)^0 &= \lim_{\nu\to\infty} (D^A_{\mu+k\bar\lambda} \overline{\partial}^k \varphi_\nu + k \overline{\partial}^{k-1} \widetilde{\partial} \varphi_\nu)^0 \\ &= \lim_{\nu\to\infty} D^0_{\mu+k\bar\lambda} (\overline{\partial}^k \varphi_\nu)^0 + k(\widetilde{\partial}\psi_{k-1})^0 \,.\end{aligned} \tag{3.45}$$

Since $D^0_{\mu+k\bar\lambda}$ is a topological isomorphism, it follows that

$$\lim_{\nu\to\infty} (\overline{\partial}^k \varphi_\nu)^0 =: \psi^0_k \,. \tag{3.46}$$

Now a suitable version of Borel's theorem ([5]) gives $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$(\overline{\partial}^k \psi)^0 = \psi^0_k \,. \tag{3.47}$$

Using (3.38) again we conclude that for each k

$$\begin{aligned}(\overline{\partial}^k (f - D^A_\mu \psi))^0 &= \lim_{\nu\to\infty} (D^A_{\mu+k\bar\lambda} \overline{\partial}^k (\varphi_\nu - \psi) + k\overline{\partial}^{k-1} \widetilde{\partial}(\varphi_\nu - \psi))^0 \\ &= \lim_{\nu\to\infty} (D^0_{\mu+k\bar\lambda}((\overline{\partial}^k \varphi_\nu)^0 - \psi^0_k) + k\widetilde{\partial}((\overline{\partial}^{k-1}\varphi_\nu)^0 - \psi^0_{k-1})) = 0\end{aligned} \tag{3.48}$$

From [2], Lemma 2.7, we get $f - D^A_\mu \psi = l f_1$ for some $f_1 \in \mathcal{S}(\mathbb{R}^n)$, and (3.42) is proved for non-real λ, too.

The proof is complete when we show that

$$f_1 \in \overline{D^A_{\mu+\lambda}\mathcal{S}} \,. \tag{3.49}$$

Because $\overline{D^A_\mu \mathcal{S}}$ is the orthogonal complement of $\ker(D^A_\mu)^t$ and $l f_1 \in \overline{D^A_\mu \mathcal{S}}$, it is enough to verify that

$$\ker(D^A_{\mu+\lambda})^t \subseteq l \ker(D^A_\mu)^t \,. \tag{3.50}$$

So let $S \in \ker(D^A_{\mu+\lambda})^t$ be given. By division of distributions, take $T \in \mathcal{S}'$ such that

$$lT = S\,. \tag{3.51}$$

By (3.27) we get

$$l(D^A_\mu)^t T = (D^A_{\mu+\lambda})^t S = 0\,. \tag{3.52}$$

If λ is real, then $(D^A_\mu)^t T$ is the trivial extension R^0 of a distribution $R \in \mathcal{S}'(V)$. (The mapping $R \mapsto R^0$ from $\mathcal{S}'(V)$ to $\mathcal{S}'(\mathbb{R}^n)$ is just the transpose of the restriction map $\varphi \mapsto \varphi^0$.) By assumption there is a distribution $W \in \mathcal{S}'(V)$ such that

$$R = (D^0_\mu)^t W\,. \tag{3.53}$$

We conclude

$$(D^A_\mu)^t W^0 = R^0 = (D^A_\mu)^t T \tag{3.54}$$

and thus

$$T - W^0 \in \ker(D^A_\mu)^t\,. \tag{3.55}$$

Clearly, we have

$$l(T - W^0) = lT = S\,, \tag{3.56}$$

and (3.50) is proved in case of real λ.

If λ is non-real, then by [2], Bemerkung 2.6, we have

$$(D^A_\mu)^t T = \sum_{k=0}^m \overline{\partial}^k R^0_k \tag{3.57}$$

for some distributions $R_k \in \mathcal{S}'(V)$. Using the assumption and formula (3.37), it is easily seen by induction on k that for any $R \in \mathcal{S}'(V)$ the distribution $\overline{\partial}^k\ R^0$ can be written in the form

$$\overline{\partial}^k R^0 = (D^A_{\mu+\nu\bar\lambda})^t \Big(\sum_{j=0}^k \overline{\partial}^j \widetilde{W}^0_j \Big) \tag{3.58}$$

for any $\nu = 0, 1, 2, \dots$, where $\widetilde{W}_j \in \mathcal{S}'(V)$. Therefore we can derive from (3.57) that there are distributions $W_j \in \mathcal{S}'(V)$ such that

$$(D^A_\mu)^t T = (D^A_\mu)^t \Big(\sum_{j=0}^m \overline{\partial}^j W^0_j \Big)\,. \tag{3.59}$$

Thus

$$T - \sum_{j=0}^m \overline{\partial}^j W^0_j \in \ker(D^A_\mu)^t\,. \tag{3.60}$$

Furthermore, we have

$$l\Big(T-\sum_{j=0}^{m}\overline{\partial}^j W_j^0\Big)=lT=S \tag{3.61}$$

since

$$l(\overline{\partial}^j W_j^0)=\overline{\partial}^j(lW_j^0)=0 \tag{3.62}$$

for each j. In (3.62) the relation $\overline{\partial}l=0$ has been used. Now (3.50) is proved also for non-real λ, and the proof of our lemma is complete.

PROPOSITION 3.8. *If A has an eigenvalue λ with* $\operatorname{Re}\lambda\neq 0$, *then $D_\mu^A\mathcal{S}\subseteq\mathcal{S}$ is closed for all $\mu\in\mathbb{C}$.*

Proof. Replacing A and μ by $-A$ and $-\mu$ in case of need, we may assume that $\operatorname{Re}\lambda>0$.

We proceed by induction on the dimension n. Take $l:\mathbb{R}^n\to\mathbb{C}$ according to (3.25) and put $V:=\ker l$. First assume that the restriction A^0 of A to V has an eigenvalue with non-vanishing real part or that $\operatorname{Re}(\mu+k\lambda)\neq 0$ for all $k=0,1,2,\ldots$ In this case, the assumption of Lemma 3.7 is satisfied by induction hypothesis and Prop. 3.6, respectively, as well as by Prop. 2.1.

Now let $f\in\overline{D_\mu^A\mathcal{S}}$ be given. We have to prove that there exists $\varphi\in\mathcal{S}$ such that $f=D_\mu^A\varphi$. By Lemma 3.7, there exist $\psi_r\in\mathcal{S}$ and $f_r\in\overline{D_{\mu+r\lambda}^A\mathcal{S}}$ such that

$$f=D_\mu^A\psi_r+l^r f_r\,, \tag{3.63}$$

$r=0,1,2,\ldots$ If we can show for some r that $f_r=D_{\mu+r\lambda}^A\varphi_r$ for some $\varphi_r\in\mathcal{S}$, then by (3.27) we have

$$f=D_\mu^A(\psi_r+l^r\varphi_r) \tag{3.64}$$

and the proof is done. Thus, for a given $M>0$, we may assume that $\operatorname{Re}\mu>M$. By Lemma 3.4 we conclude that $\varphi:=S_\mu^A f$ belongs to C_∞^1 and then, by the techniques of Lemma 3.1, that $D_\mu^A\varphi=f$. Furthermore, by Prop. 2.1 we get

$$\varphi=\psi_r+l^r\varphi_r \tag{3.65}$$

for all $r=0,1,2,\ldots$, where $\varphi_r:=S_{\mu+r\lambda}^A f_r$. Since φ_r is arbitrarily smooth for sufficiently large r by Lemma 3.4, we conclude that $\varphi\in C^\infty$.

However, we have to prove that $\varphi\in\mathcal{S}$. For this it is enough to show that $\varphi\in H_b^m$ for each $b,m\in\mathbb{N}$, i.e.

$$\|x^\beta\partial^\alpha\varphi\|_{L^2}^2<\infty \tag{3.66}$$

for all multi-indices α, β satisfying $|\alpha|\le m$, $|\beta|\le b$. First, we take r such that $\varphi_r=S_{\mu+r\lambda}^A f_r\in H_b^m$. Then we take a constant $c(p)>0$ such that

$$|x|^{|\beta|}|\partial^\alpha f_r(x)|\le\frac{c(p)}{(1+|x|^2)^{n/2}(1+|l(x)|^2)^{p/2}} \tag{3.67}$$

for all $|\alpha| \leq m$, $|\beta| \leq b$. In the course of the proof we shall determine $p \geq r$ sufficiently large for our need.

We have to show that

$$\int_{\{l(x)\leq 1\}} |x^\beta \partial^\alpha (l^r S^A_{\mu+r\lambda} f_r)|^2 < \infty \,, \tag{3.68}$$

$$\int_{\{l(x)>1\}} |x^\beta \partial^\alpha (l^r S^A_{\mu+r\lambda} f_r)|^2 < \infty \,. \tag{3.69}$$

Clearly, (3.68) holds because $S^A_{\mu+r\lambda} f_r \in H^m_b$. For (3.69) we use (3.8). Estimating in a similar way to the proof of Lemma 3.4 we have to show that

$$\int_0^\infty e^{2s\,\mathrm{Re}(\mu+r\lambda)} |a(s)|^2 \int_{\{l(x)>1\}} |l(x)|^{2q} |e^{sA}x|^{2|\beta|} |(\partial^\alpha f_r)(e^{sA}x)|^2 \, dx\, ds < \infty \tag{3.70}$$

for $|\alpha| \leq m$, $|\beta| \leq b$, $q \leq r$. Using (3.67), we can estimate this expression by

$$\begin{aligned} &\int_0^\infty e^{2s\,\mathrm{Re}(\mu+r\lambda)} |a(s)|^2 \\ &\qquad \times \int_{\{l(x)>1\}} \frac{|l(x)|^{2q}}{(1+|e^{2\lambda s}||l(x)|^2)^p} \cdot \frac{c(p)^2}{(1+|e^{sA}x|^2)^n} \, dx\, ds \\ &\quad \leq \int_0^\infty e^{2s\,\mathrm{Re}(\mu+r\lambda-p\lambda)} |a(s)|^2 e^{-s\,\mathrm{tr}\,A} \int_{\mathbb{R}^n} \frac{c(p)^2}{(1+|x|^2)^n} \, dx\, ds \,. \end{aligned} \tag{3.71}$$

Obviously, the last expression is finite provided that p is taken sufficiently large. This proves $f \in D^A_\mu \mathcal{S}$.

It remains to prove our proposition for $\sigma(A^0) \subseteq i\mathbb{R}$ and $\mathrm{Re}(\mu + k\lambda) = 0$ for some k. Given $f \in \overline{D^A_\mu \mathcal{S}}$, by (3.33) we have

$$\overline{d}^{k+1} f \in \overline{D^A_{\mu+(k+1)\lambda} \overline{d}^{k+1} \mathcal{S}} \,. \tag{3.72}$$

By the proof just given, the space $D^A_{\mu+(k+1)\lambda}\mathcal{S}$ is closed, and therefore in view of Prop. 2.1

$$D^A_{\mu+(k+1)\lambda} : \mathcal{S} \to D^A_{\mu+(k+1)\lambda}\mathcal{S} \tag{3.73}$$

is a topological isomorphism. Since the set $\overline{d}^{k+1}\mathcal{S} \subseteq \mathcal{S}$ is closed, so is $D^A_{\mu+(k+1)\lambda}\overline{d}^{k+1}\mathcal{S}$. Therefore we have

$$\overline{d}^{k+1} f = D^A_{\mu+(k+1)\lambda} \overline{d}^{k+1} \varphi = \overline{d}^{k+1} D^A_\mu \varphi \tag{3.74}$$

for some $\varphi \in \mathcal{S}$. It follows that $f = D^A_\mu \varphi$, and the proof is complete.

Now we shall discuss the case $\sigma(A) \subseteq i\mathbb{R}$ and $\mathrm{Re}\,\mu = 0$. We assume that A is not similar to a skew-symmetric matrix. Then, by linear algebra, there is at least one eigenvalue λ such that

$$A^t k - \lambda k = l \tag{3.75}$$

for some linear mapping $k : \mathbb{R}^n \to \mathbb{C}$. We conclude

$$k(e^{sA}x) = e^{\lambda s}(k(x) + sl(x)), \quad x \in \mathbb{R}^n. \tag{3.76}$$

Whenever s and $\operatorname{Re} k(x)\overline{l(x)}$ have the same sign, it follows that

$$|k(e^{sA}x)|^2 \geq |k(x)|^2 + s^2|l(x)|^2. \tag{3.77}$$

LEMMA 3.9. *For all $b, m \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that $S_\mu^A f$ belongs to H_b^m for each $f \in E_r$. Moreover, S_μ^A defines a continuous operator from E_r to H_b^m.*

Proof. We have to show that there is $r \in \mathbb{N}$, a 0-neighborhood U in $\overline{D_{\mu+r\lambda}^A \mathcal{S}}$ and a constant c such that for any $f_r \in U$ and for any multiindices α, β satisfying $|\alpha| \leq m$, $|\beta| \leq b$

$$\int_{\mathbb{R}^n} |x^\beta \partial^\alpha S_\mu^A(l^r f_r)(x)|^2 \, dx < c. \tag{3.78}$$

We shall only integrate over $\{\operatorname{Re}(k(x)\overline{l(x)}) < 0\}$ in the proof of (3.78), because the estimation for $\{\operatorname{Re}(k(x)\overline{l(x)}) > 0\}$ works in the same way by using (3.8). Then we can use (3.77). Taking notice of (3.19) and (3.20) we have to prove that for $f_r \in U$ and $r - m \leq q \leq r$

$$\int_{\{\operatorname{Re}(k(x)\overline{l(x)})<0\}} |S_w^A(x^\beta l^q \partial^\alpha f_r)(x)|^2 \, dx < c' \tag{3.79}$$

where $w(s)$ is a polynomially bounded weight function depending on m and b. We take U in such a manner that each $f_r \in U$ satisfies

$$|x^\beta \partial^\alpha f_r(x)| \leq (1 + |x|^2)^{-n/2}(1 + |k(x)|^2)^{-p/2}(1 + |l(x)|^2)^{-r/2} \tag{3.80}$$

where p is to be determined in the course of the proof. After putting $\widetilde{w}(s) := (1 + s^2)w(s)$ and using the Cauchy–Schwarz inequality as well as (3.77) we have to estimate the expression

$$\int \int_{-\infty}^{0} |\widetilde{w}(s)|^2 \frac{|l(x)|^{2q}}{(1 + |e^{sA}x|^2)^n (1 + s^2|l(x)|^2)^p (1 + |l(x)|^2)^r} \, ds \, dx. \tag{3.81}$$

Since $\sigma(A) \subset i\mathbb{R}$ there is a polynomial $P(s) \geq 1$ such that

$$|e^{-sA}x| \leq P(s)|x| \tag{3.82}$$

and therefore

$$|e^{sA}x| \geq |x|/P(s). \tag{3.83}$$

Then we have

$$(1 + |e^{sA}x|^2)^{-n} \leq P(s)^{2n}(1 + |x|^2)^{-n}. \tag{3.84}$$

Now, for $|l(x)| \le 1$, the integrand in (3.81) can be estimated by

$$\frac{|l(x)|^{2(q-p)}}{(1+|x|^2)^n} \cdot \frac{|\widetilde{w}(s)|^2 P(s)^{2n}}{(1+s^2)^p} \tag{3.85}$$

and, for $|l(x)| > 1$, by

$$\frac{|\widetilde{w}(s)|^2 P(s)^{2n}}{(1+|x|^2)^n (1+s^2)^p} \cdot \frac{|l(x)|^{2q}}{(1+|l(x)|^2)^r}\,. \tag{3.86}$$

To get the conclusion of our lemma we only have to take p sufficiently large in dependence on the growth of $|\widetilde{w}(s)|^2 P(s)^{2n}$ and to take $r \ge m + p$.

PROPOSITION 3.10. *Let $\sigma(A) \subset i\mathbb{R}$, let A be not similar to a skew-symmetric matrix and let $\operatorname{Re}\mu = 0$. Assume that $(D^0_{\mu+k\lambda})^t$ is globally solvable in $\mathcal{S}'(V)$ for each $k \in \mathbb{Z}$. Then $D^A_\mu \mathcal{S} \subset \mathcal{S}$ is closed.*

Proof. Let $f \in \overline{D^A_\mu \mathcal{S}}$ be given. We only have to show that $\varphi := S^A_\mu f$ belongs to H^m_b for all $b, m \in \mathbb{N}$.

For $b, m \in \mathbb{N}$ we take $r \in \mathbb{N}$ according to Lemma 3.9. By Lemma 3.7 there exist $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $f_r \in E_r$ such that

$$f = D^A_\mu \psi + f_r\,. \tag{3.87}$$

By Lemma 3.9 we have $S^A_\mu f_r \in H^m_b$ and thus $S^A_\mu f = \psi + S^A_\mu f_r \in H^m_b$. This proves our proposition.

COROLLARY 3.11. *Let A be nilpotent, $A \ne 0$. Then D^A_μ is globally solvable in $\mathcal{S}'$ for every $\mu \in \mathbb{C}$.*

Proof. If $\operatorname{Re}\mu \ne 0$, the assertion is given by Prop. 3.6. If $\operatorname{Re}\mu = 0$, we proceed by induction on n using Prop. 3.10: If $\mu \ne 0$, the assertion can be proved for any nilpotent A, also for $A = 0$; in fact, it is true for $n = 1$ and the conclusion follows by induction. If $\mu = 0$, we only have to prove closedness of $D^A_\mu \mathcal{S}$ for nilpotent matrices A of rank 1.

Let $f \in \overline{D^A_\mu \mathcal{S}}$ be given. Clearly, there is a vector $v \in \ker A$ and a linear form l on $\mathbb{R}^n$ such that

$$Ax = l(x)v\,, \qquad x \in \mathbb{R}^n\,. \tag{3.88}$$

If $x \in \ker l = \ker A$, the Dirac measure δ_x belongs to $\ker(D^A_\mu)^t$ and thus $f(x) = \langle \delta_x, f\rangle = 0$. Therefore, f can be written as

$$f = l f_1 \tag{3.89}$$

with some $f_1 \in \mathcal{S}$. If $x \notin \ker A$, by Lemma 3.2 we have

$$\int_{\mathbb{R}} f_1(x + s l(x) v)\, ds = 0\,. \tag{3.90}$$

We conclude that there is $\varphi \in \mathcal{S}$ such that

$$f_1(x) = \frac{d}{dt}\varphi(x+tv)\,|_{t=0}\ . \tag{3.91}$$

(See [8], Chap. II, §5.) Then $D^A_\mu \varphi = l f_1 = f$, and the proof is complete.

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