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A PRIORI ESTIMATES IN GEOMETRY AND SOBOLEV SPACES ON OPEN MANIFOLDS

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Introduction. For bounded domains in $\mathbb{R}^n$ satisfying the cone condition there are many embedding and module structure theorem for Sobolev spaces which are of great importance in solving partial differential equations. Unfortunately, most of them are wrong on arbitrary unbounded domains or on open manifolds. On the other hand, just these theorems play a decisive role in foundations of nonlinear analysis on open manifolds and in solving partial differential equations. This was pointed out by the author in particular in [4]. But if the open Riemannian manifold (M^n, g) and the considered Riemannian vector bundle $(E, h) \to M$ have bounded geometry of sufficiently high order then most of the Sobolev theorems can be preserved. The key for this are a priori estimates for the connection coefficients and the exponential map coming from curvature bounds. By means of uniform charts and trivializations and a uniform decomposition of unity the local euclidean arguments remain applicable. Only the compactness of embeddings is no more valid. This is the content of our main section 4.

1. A priori estimates in geometry. Let (M^n, g) be open, complete. Consider the following two conditions $(\mathrm{B}_k) = (\mathrm{B}_k(M))$ and (I):

$$(\mathrm{B}_k) \qquad |\nabla^i R| \equiv |(\nabla^g)^i R^g| \le C_i\,,\ 0 \le i \le k\,,$$

$$(\mathrm{I}) \qquad r_{\mathrm{inj}}(M^n, g) = \inf_{x \in M} r_{\mathrm{inj}}(x) > 0\,,$$

where $R = R^g$ denotes the Riemannian curvature tensor, $|\ |$ the pointwise norm and $r_{\mathrm{inj}}(x)$ the injectivity radius at x, i.e. the distance between x and the cut locus. We say (M^n, g) *has bounded geometry up to order* k if it satisfies the conditions (B_k) and (I). Given any open manifold M^n and $k \in \mathbb{Z}$, $k \ge 0$, there exists a

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complete Riemannian metric g an M^n satisfying (I) and (B_k) i.e. these conditions do not restrict the topological type. This was proved be Green in [7]. Natural examples of manifolds of bounded geometry are Riemannian coverings of closed Riemannian manifolds or Riemannian homogeneous spaces.

Let $(E,h) \xrightarrow{\pi} M$ be a Riemannian vector bundle with metric connection $\nabla = \nabla^h$. It satisfies the condition $(\mathrm{B}_k(E,\nabla))$ if

$$(\mathrm{B}_k(E,\nabla)) \qquad |\nabla^i R| \equiv |(\nabla^h)^i R^h| \le C_i', \quad 0 \le i \le k,$$

where R^h denotes the curvature of (E,∇).

The meaning of a priori estimates consists in giving bounds for the metric g_{ij}, the Christoffel symbols Γ_{ij}^m and their derivatives in normal coordinates which depend on bounds for R and its covariant derivatives.

Proposition 1.1. *If (M^n,g) satisfies (B_k) and if $\mathfrak{U}$ is an atlas of normal coordinate charts of radius $\le r_0$, then there exist constants C_α, C'_β such that*

$$|D^\alpha g_{ij}| \le C_\alpha, \quad |\alpha| \le k, \tag{1.1}$$

$$|D^\beta \Gamma_{ij}^m| \le C'_\beta, \quad |\beta| \le k-1, \tag{1.2}$$

where C_α, C'_β are independent of the base points of the normal charts and depend only on r_0 and on curvature bounds including bounds for the derivatives.

For the rather long and technical proof which uses comparison theorems for iterated inhomogeneous equations we refer to [5]. ■

Corollary 1.2. $\exp_p$, $d\exp_p$, $\nabla d\exp_p$, …, $\nabla^k d\exp_p$ *are bounded by a constant independent of $p \in M$.* ■

(1.2) carries over to the case of vector bundles of bounded geometry. Let $p \in M$, $(x^1,\ldots,x^n) \overset{\Phi}{\leftrightarrow} x^1X_1 + \ldots + x^nX_n = \exp_p^{-1} : U_r(p) \to B_r(0) \subset T_pM$ be a system of normal coordinates, and $e_1,\ldots,e_n \in \pi^{-1}(p) = E_p \subset E$ an orthonormal frame in E_p which defines by parallel transport along radial geodesics a field of orthonormal frames in $E|_U$. This defines locally a flat connection ∇^0 on $E|_{U'}$, by requiring $e_1,\ldots,e_N$ to be parallel sections, $\nabla^0(f\cdot e) = df \otimes e$, $f \in C^\infty(U)$, $e \in C^\infty(E|_U)$. $\Gamma = \nabla - \nabla^0$ is a 1-form with values in $\mathfrak{g}_E$ (= bundle of skew symmetric endomorphisms of E) and can be described by

$$dx^i \otimes \Gamma_{\alpha i}^\beta \otimes e^\alpha \otimes e_\beta = \theta_\alpha^\beta e^\alpha \otimes e_\beta,$$

where $\nabla_{X_i} e_\alpha = \Gamma_{\alpha i}^\beta e_\beta$ and e^α is dual to e_α with respect to the metric in E.

Proposition 1.3. *Assume* $(\mathrm{B}_k(M)), (\mathrm{B}_k(E,\nabla))$, $k \ge 1$, *and* $\Gamma_{\alpha i}^\beta$ *are as above. Then*

$$|D^\gamma \Gamma_{\alpha i}^\beta| \le C_\gamma, \quad |\gamma| \le k-1, \ \alpha,\beta = 1,\ldots,N, \ i = 1,\ldots,n, \tag{1.3}$$

where C_γ is a constant depending on curvature bounds and r and is independent of p.

We refer to [5] for the proof. ■

(1.3) is a typical a priori estimate in geometry. In physical language, bounds for the field strength and its derivatives imply bounds for the gauge potential of one order less. Moreover, (1.3) holds in analogous manner for the connection form ω in a principal fibre bundle setting since the basic equations $R = d\omega + [\omega, \omega]$ in a principal fibre bundle and $R = d\theta + [\theta, \theta]$ in a vector bundle are equivalent.

2. Sobolev spaces. Sobolev spaces, embedding theorems, norm inequalities and module structures play a key role in global analysis on compact manifolds. Almost nothing of them remains valid after going over to noncompact manifolds. Nearly everything—except compactness properties—remains valid if we are working on manifolds and in bounded geometry. This is one of the striking reasons—among many others—for our restriction to bounded geometry. Since we need them later on, we will give now precise definitions.

Assume (M^n, g) to be open and complete, and let $(E, h) \to M$ be a Riemannian vector bundle with metric connection $\nabla^E = \nabla^h$. The Levi-Cività connections ∇^g and ∇^E define connections in all tensor boundles $T^q_r \otimes E$, in particular in $\Lambda^q T^*M \otimes E$, where $\Lambda^q T^*M \subset T^q_0$. We denote by $\Omega^q(E)$ and $\Omega(T^q_r \otimes E) \equiv \Omega^0(T^q_r \otimes E)$ the spaces of smooth q-forms and tensor fields with values in E, respectively. For the sake of brevity, we consider here as main object forms with values in E. The other case is quite parallel. $\Omega^q_0(E) \subset \Omega^q(E)$ is the subspace of forms with compact support. Then for $p \in \mathbb{R}$, $1 \le p < \infty$, and r a nonnegative integer we define

$$\Omega^{q,p}_r(E) = \Big\{ \varphi \in \Omega^q(E) \,\Big|\, |\varphi|_{p,r} := \Big(\sum_{i=0}^{r} \int |\nabla^i \varphi|^p \, d\,\mathrm{vol} \Big)^{1/p} < \infty \Big\},$$

$$\overline{\Omega}^{q,p,r}(E) = \text{completion of } \Omega^{q,p}_r(E) \text{ with respect to } |\ |_{p,r}\,,$$

$$\mathring{\Omega}^{q,p,r}(E) = \text{completion of } \Omega^q_0(E) \text{ with respect to } |\ |_{p,r}\,,$$

$$\Omega^{q,p,r}(E) = \{\varphi \mid \varphi \text{ a measurable regular distributional } q\text{-form with } |\varphi|_{p,r} < \infty\}.$$

Furthermore, we define

$$^{\mathrm{b},m}\Omega^q(E) = \Big\{ \varphi \,\Big|\, \varphi \text{ a } C^m\text{-form and } {}^{\mathrm{b},m}|\varphi| := \sum_{i=0}^{m} \sup_{x \in M} |\nabla^i \varphi|_x < \infty \Big\},$$

$$^{\mathrm{b},m}\mathring{\Omega}^q(E) = \text{completion of } \Omega^q_0(E) \text{ with respect to } {}^{\mathrm{b},m}|\ |.$$

$^{\mathrm{b},m}\Omega(E)$ equals the completion of

$$^{\mathrm{b}}_{m}\Omega^q(E) = \{\varphi \in \Omega^q(E) \mid {}^{\mathrm{b},m}|\varphi| < \infty\}$$

with respect to $^{\mathrm{b},m}|\ |$. All the defined spaces $\mathring{\Omega}^{q,p,r}(E)$, $\overline{\Omega}^{q,p,r}(E)$, $\Omega^{q,p,r}(E)$, $^{\mathrm{b},m}\mathring{\Omega}^q(E)$, $^{\mathrm{b},m}\Omega^q(E)$ are Banach spaces and there are inclusions

$$\mathring{\Omega}^{q,p,r}(E) \subseteq \overline{\Omega}^{q,p,r}(E) \subseteq \Omega^{q,p,r}(E)\,, \tag{2.1}$$

and

$$ {}^{\mathrm{b},m}\mathring{\Omega}^q(E) \subsetneq {}^{\mathrm{b},m}\Omega^q(E)\,. $$

If $p=2$ then $\mathring{\Omega}^{q,p,r}(E)$, $\overline{\Omega}^{q,p,r}(E)$, $\Omega^{q,p,r}$ are Hilbert spaces.

On an arbitrary open Riemannian manifold all three spaces in (2.1) are different and it is not clear which one we should use.

PROPOSITION 2.1. *If* (M^n,g) *satisfies* (I) *and* (B_k) *then*

$$ \mathring{\Omega}^{q,p,r}(E) = \overline{\Omega}^{q,p,r}(E) = \Omega^{q,p,r}(E), \qquad 0 \le r \le k+2\,. \tag{2.2} $$

We refer to [6] for the proof. ∎

3. Classical Sobolev theorems. In this section we list up a series of important Sobolev theorems which shall be generalized in the next section to bundles and manifolds of bounded geometry. Denote by $B=B^n\subset\mathbb{R}^n$ an open euclidean ball, consider the product bundle $B\times\mathbb{C}^N$ with standard flat connection and write $\Omega^{0,p,r}(B\times\mathbb{C})=\Omega^{p,r}(B)$, $\Omega^{0,p,r}(B\times\mathbb{C}^N)=\Omega^{p,r}(\theta^N)$.

PROPOSITION 3.1. (a) *If* $p'<\infty$ *and* $p>1$, $r-n/p\ge r'-n/p'$, $r>r'$, *then there exists a continuous embedding*

$$ \mathring{\Omega}^{p,r}(B) \hookrightarrow \mathring{\Omega}^{p',r'}(B)\,. \tag{3.1} $$

(b) *If* $p'=\infty$ *or* $p=1$, *then we have to assume* $r-n/p>r'-n/p'$, *and* $r>r'$ *to obtain the same assertion. In particular*, *for* $p'=\infty$,

$$ \mathring{\Omega}^{p,r}(B) \hookrightarrow {}^{\mathrm{b},r'}\mathring{\Omega}(B) \subset {}^{\mathrm{b},r'}\Omega(B)\,. \tag{3.2} $$

A special case of 3.1 with $r'=0$ is

PROPOSITION 3.2. *If* $n/p-n/q<r$ *then there exists a continuous embedding*

$$ \mathring{\Omega}^{p,r}(B) \hookrightarrow \Omega^{q,0}(B)\,. \tag{3.3} $$

PROPOSITION 3.3. *If* $r_i>n/p_i$, $r_1,r_2\ge r$ *and* $r_1-n/p_1+r_2-n/p_2\ge r-n/p$, *then there exists a continuous embedding*

$$ \mathring{\Omega}^{p_1,r_1(B)} \otimes \mathring{\Omega}^{p_2,r_2}(B) \hookrightarrow \mathring{\Omega}^{p,r}(B) \tag{3.4} $$

given by $f_1\otimes f_2\to f_1\cdot f_2$.

The proofs of 3.1–3.2 are contained in any good presentation of Sobolev spaces (cf. [1]). For 3.3 we refer to [3], pp. 42–44.

3.1–3.3 immediately imply

PROPOSITION 3.4. *All assertions remain valid if we replace* $\mathring{\Omega}^{p,r}(B)$ *by* $\mathring{\Omega}^{p,r}(\theta^N)$ *or even* $\mathring{\Omega}^{q,p,r}(\theta^N)$. *In* 3.3 *we have to read*

$$ \mathring{\Omega}^{p_1,r_1}(\theta^N) \otimes \Omega^{p_2,r_2}(\theta^N) \hookrightarrow \Omega^{p,r}(\theta^N\otimes\theta^N)\,. $$

4. Extension to manifolds and bundles of bounded geometry. Now we come to the main results of this paper.

THEOREM 4.1. *Let (M^n, g) be open, complete, of bounded geometry up to order k, and let $(E, \nabla^E) \to M$ be a Riemannian vector bundle satisfying* $(\mathrm{B}_k(E, \nabla^E))$. *Then every Sobolev embedding theorem and theorem concerning the module structure of Sobolev spaces of order $r \le k$, which is valid for an open euclidean n-ball B, is also valid for the corresponding Sobolev spaces on (M^n, g).*

Proof. According to an unpublished but very often used result of Calabi, for manifolds satisfying (I) and (B_0) there exists a uniformly locally finite cover of M by normal charts of radius $0 < \delta_M < r_{\mathrm{inj}}(M)$. Let $B = B_{\delta_M}$ and let $e_1, \ldots, e_N$ be a synchronous frame over a normal chart (U, Φ) as in Section 1. According to Corollary 1.2 and (1.3),

$$\mathring{\Omega}^{q,p,r}(E|_U) \xrightarrow{\cong} \mathring{\Omega}^{q,p,r}(B \times \mathbb{C}^N)\,, \tag{4.1}$$

$$\varphi = \varphi^\alpha e_\alpha \to (\varphi^1 \circ \Phi^{-1}, \ldots, \varphi^N \circ \Phi^{-1})\,, \tag{4.2}$$

is an equivalence of Sobolev spaces where the constants in the equivalence (4.1) depend on ${}^{\mathrm{b},k}|R^M|$, ${}^{\mathrm{b},k}|R^E|$, δ_M and are independent of $p \in M$. Now let $\mathfrak{U} = \{(U_i, \Phi_i)\}_i$ be a uniformly locally finite cover of M by normal charts. There exists an associated partition of unity $\{\eta_i\}_i$ such that $d\eta_i$, $\nabla d\eta_i, \ldots, \nabla^{k+1} d\eta_i$ are uniformly bounded (cf. [2]). Define

$$ {}^{\mathfrak{U}}\mathring{\Omega}^{q,p,r}(E) := \sum_i \mathring{\Omega}^{q,p,r}(E|_{U_i})$$

to be a sum of Banach spaces (i.e. direct sum and completion). Then according to (4.1) and the independence of the constants of p_i, Φ_i,

$$ {}^{\mathfrak{U}}\mathring{\Omega}^{q,p,r}(E) \cong \sum_i \mathring{\Omega}^{q,p,r}(B \times \mathbb{C}^N)$$

as equivalent Banach spaces. Let $\varphi \in \Omega^{q,p,r}(E)$. Then $\varphi = \sum_i \eta_i \varphi$ and $\varphi \to \{\eta_i \varphi\}_i$ is a bounded map

$$\Omega^{q,p,r}(E) \to {}^{\mathfrak{U}}\mathring{\Omega}^{q,p,r}(E) = \sum_i \mathring{\Omega}^{q,p,r}(B \times \mathbb{C}^N)$$

since $d\eta_i, \nabla d\eta_i, \ldots, \nabla^{r-1} d\eta_i$ are uniformly bounded. We infer that every continuous embedding

$$\mathring{\Omega}^{q,p,r}(B \times \mathbb{C}^N) \hookrightarrow \mathring{\Omega}^{q,p',r'}(B \times \mathbb{C}^N)$$

implies a continuous embedding

$$\begin{aligned}\Omega^{q,p,r}(E) \to {}^{\mathfrak{U}}\mathring{\Omega}^{q,p,r}(E) &= \sum_i \mathring{\Omega}^{q,p,r}(B \times \mathbb{C}^N)\\ &\hookrightarrow \sum_i \mathring{\Omega}^{q,p',r'}(B \times \mathbb{C}^N) \to {}^{\mathfrak{U}}\mathring{\Omega}^{q,p',r'}(E) \to \Omega^{q,p',r'}(E)\end{aligned}$$

given by

$$\varphi \to \{\eta_i\varphi\}_i \to \{(\eta_i\varphi)^1 \circ \Phi_i^{-1}, \ldots, (\eta_i\varphi)^N \circ \Phi_i^{-1}\}_i \in \sum_i \mathring{\Omega}^{q,p,r}(B \times \mathbb{C}^N)$$
$$\to \{(\eta_i\varphi)^1 \circ \Phi_i^{-1}, \ldots, (\eta_i\varphi)^N \circ \Phi_i^{-1}\}_i \in \sum_i \mathring{\Omega}^{q,p',r'}(B \times \mathbb{C}^N)$$
$$\to \{\eta_i\varphi\}_i \in {}^{\mathfrak{U}}\mathring{\Omega}^{q,p',r'}(E) \to \sum_i \eta_i\varphi \in \Omega^{q,p',r'}(E)\,.$$

Module structure theorems can be considered as special cases of embedding theorems and are proved by the same sequence of constructions and conclusions (assuming their validity for the euclidean ball). ∎

COROLLARY 4.2. *Assume* (M^n, g), (E, ∇^E) *are as above*, $p' < \infty$, $p > 1$, $r - n/p \geq r' - n/p'$, $r > r'$. *Then there exists a continuous embedding* $\Omega^{q,p,r,}(E) \hookrightarrow \Omega^{q,p',r'}(E))$. ∎

COROLLARY 4.3. *Assume* (M^n, g), (E, ∇^E) *are as above*, $r > n/p + r'$. *Then there exists a continuous embedding* $\Omega^{q,p,r,}(E) \hookrightarrow {}^{\mathrm{b},r'}\Omega(E)$. ∎

COROLLARY 4.4. *Assume* (M^n, g), (E, ∇^E) *are as above and* $n/p - n/p' < r$. *Then there exists a continuous embedding* $\Omega^{q,p,r,}(E) \hookrightarrow \Omega^{q,p',0}(E)$. ∎

COROLLARY 4.5. *Assume* (M^n, g), (E_1, ∇^{E_1}), (E_2, ∇^{E_2}) *are as above and* $r_1 - n/p_1 + r_2 - n/p_2 \geq r - n/p$, $r_1, r_2 \geq r$, $r_i > n/p_i$, $i = 1, 2$. *Then there exists a continuous embedding*

$$\Omega^{p_1,r_1}(E, \nabla^{E_1}) \otimes \Omega^{p_2,r_2}(E_2, \nabla^{E_2}) \hookrightarrow \Omega^{p,r}(E_1 \otimes E_2, \nabla^{E_1} \otimes \nabla^{E_2})\,. \blacksquare$$

Remark. As mentioned in 4.1, we assume in all corollaries $r, r_1, r_2 \leq k$.

References

[1] T. Aubin, *Nonlinear Analysis on Manifolds. Monge–Ampère Equations*, Springer, New York 1982.
[2] J. Dodziuk, *Sobolev spaces of differential forms and de Rham–Hodge isomorphism*, J. Differential Geom. 16 (1981), 63–73.
[3] D. Ebin, *Espace des métriques Riemanniennes et mouvement des fluides via les variétés d'applications*, Lecture notes, Paris 1972.
[4] J. Eichhorn, *Gauge theory on open manifolds of bounded geometry* , Internat. J. Modern Physics, to appear.
[5] —, *The boundedness of connection coefficients and their derivatives*, Math. Nachr. 152 (1991), 145–158.
[6] —, *Elliptic differential operators on noncompact manifolds*, in: Teubner-Texte zur Math. 106, Teubner, 1988, 4–169.
[7] R. Greene, *Complete metrics of bounded curvature on noncompact manifolds*, Arch. Math. (Basel) 31 (1978), 89–95.