1. Let \( G \) be an open set in \( \mathbb{R}^n \) and let \( F \) be its boundary. Let \( \Gamma \) be some part of \( F \) which is a smooth \((n - 1)\)-dimensional submanifold. Let \( A \) be a closed subset of \( \Gamma \).

Let \( u \) be a function harmonic in \( G \) satisfying the boundary condition \( D_v u = 0 \) on \( \Gamma \setminus A \), where \( v \) is the outer normal to \( \Gamma \). When can we say that \( D_v u = 0 \) on \( \Gamma \), i.e. when the singularity of \( u \) on \( A \) is removable? It is evident that the answer depends on the structure of \( A \) and on the behaviour of \( u \) in a neighbourhood of \( A \). For instance, if \( A \) is a single point, then the singularity is removable if \( |u(x)| = o(r^{2-n}) \) as \( r \to 0 \), where \( r \) is the distance from \( A \), and can be non-removable if \( n > 2 \) and \( |u(x)| = O(r^{2-n}) \).

Indeed, let \( f \in C_0^\infty(\Gamma) \). We show that if \( |u(x)| = o(r^{2-n}) \), then

\[
\int_{\Gamma} f(x) D_v u(x) \, dS = 0.
\]

Let \( A \) be the origin. Let \( h \in C_0^\infty(\Gamma) \), \( h(A) = 1 \). Then

\[
\int_{\Gamma} f(x) D_v u(x) \, dS = \int_{\Gamma} f(x) D_v u(x) h(x/\varepsilon) \, dS
\]

for any \( \varepsilon > 0 \). We can extend \( f \) and \( h \) in such a way that they vanish outside some neighbourhood of \( A \) and \( D_v f = D_v h = 0 \) on \( \Gamma \). By the Green formula we have

\[
\int_{\Gamma} f(x) D_v u(x) h(x/\varepsilon) \, dS = \int_G (f(x) h(x/\varepsilon) \Delta u(x) - u(x) \Delta (f(x) h(x/\varepsilon))) \, dx
\]

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and therefore,
\[ \int f(x) D_v u(x) \, dS = - \lim_{\varepsilon \to 0} \int \frac{u(x) \Delta (f(x) h(x/\varepsilon))}{G} \, dx. \]

It is clear that \( |\Delta (f(x) h(x/\varepsilon))| \leq C \varepsilon^{-2} \). Therefore from the condition \( |u(x)| = o(r^{2-n}) \) it follows that \( \int f(x) D_v u(x) \, dS = 0 \). The same is true if \( u \in L^p_{\text{loc}}(G) \), where \( p = n/(n-2) \), \( n > 2 \). This can be seen immediately if we apply Hölder’s inequality.

On the other hand, if \( \Gamma \) coincides locally with the plane \( x_n = 0 \), then for the fundamental solution \( E(x) \) of the Laplace operator we have \( D_n E(x) = \delta(x') \) when \( x_n = 0 \), where \( x' = (x_1, \ldots, x_{n-1}) \), and we can see that the singularity of the solution is non-removable if \( n > 2 \) and \( |u(x)| = O(r^{2-n}) \).

2. Now let \( P(x, D) \) be a linear differential operator of order \( m \) with coefficients smooth in \( G \), and suppose that another differential operator \( B(x, D) \), which also has smooth coefficients, is defined on \( \Gamma \). We do not make any assumptions about the type of the operator \( P \).

Consider the following problem: when from the conditions: \( Pu = 0 \) in \( G \), \( Bu = 0 \) on \( \Gamma \setminus A \) does it follow that \( Bu = 0 \) on \( \Gamma ? \) We state a number of sufficient conditions. All these conditions are sharp, which can be shown by suitable examples.

Our results can be easily transferred to boundary-value problems for linear systems of differential equations. The conditions on the smoothness of the coefficients of the operators \( P \) and \( B \), and on the smoothness of the manifold \( \Gamma \) can, of course, be made essentially weaker.

3. Our main assumption is the validity of the Green formula:
\[ \int_G (P u \cdot v - u \cdot P' v) \, dx = \int \sum_{j=1}^N B_j(x, D) u \cdot S_j(x, D) v \, dS \]
for smooth functions \( u \) and \( v \), if \( v = 0 \) in a neighbourhood of \( F \setminus \Gamma \). Here \( P' \) is the operator transposed to \( P \), \( B_j \) and \( S_j \) are differential operators with smooth coefficients, and one of the \( B_j \), say \( B_1 \), coincides with the original operator \( B \).

Assume also that
\[ S_1(x, D) = Q(x, D) D^k_v + S'_1(x, D), \]
where \( D_v \) is differentiation in the normal direction, \( Q \) acts in the directions tangent to \( \Gamma \), and \( k \) is some number, \( 0 \leq k \leq m - 1 \).

Suppose that the operators \( S'_1, S_2, \ldots, S_N \) do not involve the derivative \( D^k_v \) (but they can involve \( D^i_v \) for \( i < k \) and for \( i > k \)) and that the equation \( Qw = g \) has a solution \( w \in C^m(\Gamma) \) for a set \( M \) of functions \( g \), which is dense in \( C^\infty_0(\Gamma) \).
4. **Theorem 1.** Let \( A \) be a single point. If \( u(x) = o(r^{m-n-k}) \), where \( r \) is the distance of \( x \) from \( A \), then \( Bu = 0 \) on \( \Gamma' \).

If \( A \) is an infinite set, it is convenient to apply the Hausdorff measure for its description. The \( d \)-dimensional Hausdorff measure of \( A \), denoted by \( H_d(A) \), is defined as \( \lim_{\epsilon\to 0} \inf \sum r_j^d \), where the infimum is taken over all coverings of \( A \) by countable collections of balls with radii \( r_j \leq \epsilon \).

5. **Theorem 2.** Let \(-\infty < l < m, 1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1\). If \( Pu = 0 \) in \( G, Bu = 0 \) on \( \Gamma \setminus A \), \( u \in W^l_p(G) \) and \( H_{n-q(m-k-l)}(A) < \infty \), then \( Bu = 0 \) on \( \Gamma \).

If \( u \in W^l_p(G) \), then the same is true if \( H_{n-m+k+l}(A) = 0 \).

Here \( W^l_p(G) = L_p(G) \) and \( W^l_q(G) \) for \( l \) natural is the space of functions whose derivatives of orders \( \leq l \) are in \( L_p(G) \). For negative integers \( l \) this space consists of distributions of the form \( \sum D^j f_i \) for \( |i| \leq -l, f_i \in L_p(G) \).

6. **Theorem 3.** Let \( Pu = 0 \) in \( G, Bu = 0 \) on \( \Gamma \setminus A \) and \( u \in C^l(G \cup \Gamma) \). Assume that the order of the operator \( B \) is greater than \( l \). If \( H_{n-m+k+l}(A) = 0 \), then \( Bu = 0 \) on \( \Gamma \).

Here the space \( C^l(M) \) for \( l \) natural consists of functions whose derivatives of orders \( \leq l - 1 \) are continuous and satisfy the Lipschitz condition in \( M \). If \( l > 0 \) is not an integer, then this is a space of functions whose derivatives of orders \( \leq \lfloor l \rfloor \) satisfy the H"older condition with exponent \( l - \lfloor l \rfloor \) (here \( \lfloor l \rfloor \) is the integer part of \( l \)). Finally, if \( l < 0 \), then \( C^l(M) \) consists of distributions of the form \( \sum D^j f_i \), where \( |i| \leq -\lfloor l \rfloor, f_i \in C^{\lfloor l \rfloor - l}(M) \).

7. The results for the Neumann problem, stated in the first section, are not sharp in the case \( n = 2 \). It is well known that in this case the condition on \( u \) must have the form \( |u(x)| = o(\ln r) \). We state a similar sharp result for a general elliptic boundary-value problem.

Let \( A \) be a smooth submanifold in \( \Gamma \) of dimension \( d = n-m \). Suppose \( z_1, \ldots, z_d \) are local coordinates on \( A \), and \( y_1, \ldots, y_{n-d} \) are coordinates in the complementary space, so that the \( y_{n-d} = x_n \) axis is transversal to \( \Gamma \) and \( y_1, \ldots, y_{n-d-1} \) are the inner coordinates in \( \Gamma \).

Assume that \( m = 2k \) and the operators \( P, B_1, \ldots, B_k \) define a regular elliptic problem. Assume also that \( m_1 < m_2 < \ldots < m_k = m-1 \), where \( m_j \) is the order of \( B_j \). By the construction of the parametrix of this problem (see [1]),

\[
u(x) = Q Pu + \sum_{j=1}^k Q_j B_j [u \otimes \delta(x_n)] + Tu,\]

where \( Q, Q_j, T \) are pseudo-differential operators of orders \( -m, -m_j, -1 \), respectively. Let \( Q_0 \) be the operator with symbol \( 1/p_0(x, \xi) \) and \( g_j = B_j u - B_j Q_0 u \). Let \( x \in \Gamma \). Let \( r_1, \ldots, r_k \) be the roots of the equation \( p_0(x, \xi', r) = 0 \) with positive
imaginary parts. Let $R(x, \xi') = B(x, \xi')^{-1}$ where $B$ is the matrix with elements $b_{jl}(x, \xi') = b_j(x \cdot \xi', r_l(x, \xi'))$. Then

$$u(x) = Q_0 Pu + (2\pi)^{-n+1} k \sum_{j,l=1}^k r_{jl}(x, \xi') F[g_l](\xi') \exp(ir_j(x, \xi')x_n + ix'\xi') \, d\xi' + Tu$$

where $F[g]$ is the Fourier transform of $g$. Therefore the principal symbol of $Q_j$ is

$$\sum_l r_{lj}(x, \xi') \exp(ir_j(x, \xi')x_n)$$

and the order of homogeneity of $r_{lj}$ in $\xi'$ is $-m_j$. Let

$$r(y, z, \eta, \zeta) = r(x, \xi') = \sum r_{ji}(x, \xi').$$

The order of this function in $\xi'$ is $1 - m$.

**Theorem 4.** Let $Pu = 0$ in $G$ and $B_j u = 0$ on $\Gamma \setminus A$ for $j = 1, \ldots, k$, and $|u(x)| = o(\ln r)$, where $r$ is the distance of $x$ from $A$. Let

$$\int_{|\eta|=1} r(0, z, \eta, 0) \, dS_\eta \neq 0 \quad \text{for } z \in A.$$

Then $B_j u = 0$ on $\Gamma$ and $u \in C^\infty(G \cup \Gamma)$.

The proof is based on a construction from [3].

**References**