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REMOVABLE SINGULARITIES IN THE BOUNDARY CONDITIONS

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1. Let G be an open set in $\mathbb{R}^n$ and let F be its boundary. Let Γ be some part of F which is a smooth $(n-1)$-dimensional submanifold. Let A be a closed subset of Γ.

Let u be a function harmonic in G satisfying the boundary condition $D_v u = 0$ on $\Gamma \setminus A$, where v is the outer normal to Γ. When can we say that $D_v u = 0$ on Γ, i.e. when the singularity of u on A is removable? It is evident that the answer depends on the structure of A and on the behaviour of u in a neighbourhood of A. For instance, if A is a single point, then the singularity is removable if $|u(x)| = o(r^{2-n})$ as $r \to 0$, where r is the distance from A, and can be non-removable if $n > 2$ and $|u(x)| = O(r^{2-n})$.

Indeed, let $f \in C_0^\infty(\Gamma)$. We show that if $|u(x)| = o(r^{2-n})$, then

$$\int_\Gamma f(x) D_v u(x)\, dS = 0\,.$$

Let A be the origin. Let $h \in C_0^\infty(\Gamma)$, $h(A) = 1$. Then

$$\int_\Gamma f(x) D_v u(x)\, dS = \int_\Gamma f(x) D_v u(x) h(x/\varepsilon)\, dS$$

for any $\varepsilon > 0$. We can extend f and h in such a way that they vanish outside some neighbourhood of A and $D_v f = D_v h = 0$ on Γ. By the Green formula we have

$$\int_\Gamma f(x) D_v u(x) h(x/\varepsilon)\, dS = \int_G (f(x)h(x/\varepsilon)\Delta u(x) - u(x)\Delta(f(x)h(x/\varepsilon)))\, dx$$

and therefore,

$$\int_\Gamma f(x)D_v u(x)\,dS = -\lim_{\varepsilon\to 0}\int_G u(x)\Delta(f(x)h(x/\varepsilon))\,dx\,.$$

It is clear that $|\Delta(f(x)h(x/\varepsilon))| \le C\varepsilon^{-2}$. Therefore from the condition $|u(x)| = o(r^{2-n})$ it follows that $\int f(x)D_v u(x)\,dS = 0$. The same is true if $u \in L_{p,\mathrm{loc}}(G)$, where $p = n/(n-2)$, $n > 2$. This can be seen immediately if we apply Hölder's inequality.

On the other hand, if Γ coincides locally with the plane $x_n = 0$, then for the fundamental solution $E(x)$ of the Laplace operator we have $D_n E(x) = \delta(x')$ when $x_n = 0$, where $x' = (x_1, \dots, x_{n-1})$, and we can see that the singularity of the solution is non-removable if $n > 2$ and $|u(x)| = O(r^{2-n})$.

2. Now let $P(x, D)$ be a linear differential operator of order m with coefficients smooth in $\overline{G}$, and suppose that another differential operator $B(x, D)$, which also has smooth coefficients, is defined on Γ. We do not make any assumptions about the type of the operator P.

Consider the following problem: when from the conditions: $Pu = 0$ in G, $Bu = 0$ on $\Gamma \setminus A$ does it follow that $Bu = 0$ on Γ? We state a number of sufficient conditions. All these conditions are sharp, which can be shown by suitable examples.

Our results can be easily transferred to boundary-value problems for linear systems of differential equations. The conditions on the smoothness of the coefficients of the operators P and B, and on the smoothness of the manifold Γ can, of course, be made essentially weaker.

3. Our main assumption is the validity of the Green formula:

$$\int_G (Pu \cdot v - u \cdot P'v)\,dx = \int_\Gamma \sum_{j=1}^{N} B_j(x, D)u \cdot S_j(x, D)v\,dS$$

for smooth functions u and v, if $v = 0$ in a neighbourhood of $F \setminus \Gamma$. Here P' is the operator transposed to P, B_j and S_j are differential operators with smooth coefficients, and one of the B_j, say B_1, coincides with the original operator B.

Assume also that

$$S_1(x, D) = Q(x, D)D_v^k + S_1'(x, D)\,,$$

where D_v is differentiation in the normal direction, Q acts in the directions tangent to Γ, and k is some number, $0 \le k \le m - 1$.

Suppose that the operators $S_1', S_2, \dots, S_N$ do not involve the derivative D_v^k (but they can involve D_v^i for $i < k$ and for $i > k$) and that the equation $Qw = g$ has a solution $w \in C^m(\Gamma)$ for a set M of functions g, which is dense in $C_0^\infty(\Gamma)$.

4. THEOREM 1. *Let A be a single point. If $u(x) = o(r^{m-n-k})$, where r is the distance of x from A, then $Bu = 0$ on Γ.*

If A is an infinite set, it is convenient to apply the Hausdorff measure for its description. The d-dimensional Hausdorff measure of A, denoted by $H_d(A)$, is defined as $\lim_{\varepsilon\to 0}\inf\sum r_j^d$, where the infimum is taken over all coverings of A by countable collections of balls with radii $r_j \le \varepsilon$.

5. THEOREM 2. *Let $-\infty < l < m$, $1 < p < \infty$, $1/p + 1/q = 1$. If $Pu = 0$ in G, $Bu = 0$ on $\Gamma \setminus A$, $u \in W_p^l(G)$ and $H_{n-q(m-k-l)}(A) < \infty$, then $Bu = 0$ on Γ. If $u \in W_\infty^l(G)$, then the same is true if $H_{n-m+k+l}(A) = 0$.*

Here $W_p^0(G) = L_p(G)$ and $W_p^l(G)$ for l natural is the space of functions whose derivatives of orders $\le l$ are in $L_p(G)$. For negative integers l this space consists of distributions of the form $\sum D^i f_i$ for $|i| \le -l$, $f_i \in L_p(G)$.

6. THEOREM 3. *Let $Pu = 0$ in G, $Bu = 0$ on $\Gamma \setminus A$ and $u \in C^l(G \cup \Gamma)$. Assume that the order of the operator B is greater than l. If $H_{n-m+k+l}(A) = 0$, then $Bu = 0$ on Γ.*

Here the space $C^l(M)$ for l natural consists of functions whose derivatives of orders $\le l-1$ are continuous and satisfy the Lipschitz condition in M. If $l > 0$ is not an integer, then this is a space of functions whose derivatives of orders $\le [l]$ satisfy the Hölder condition with exponent $l-[l]$ (here $[l]$ is the integer part of l). Finally, if $l \le 0$, then $C^l(M)$ consists of distributions of the form $\sum D^i f_i$, where $|i| \le -[l]$, $f_i \in C^{[l]-l}(M)$.

7. The results for the Neumann problem, stated in the first section, are not sharp in the case $n = 2$. It is well known that in this case the condition on u must have the form $|u(x)| = o(\ln r)$. We state a similar sharp result for a general elliptic boundary-value problem.

Let A be a smooth submanifold in Γ of dimension $d = n - m$. Suppose $z_1, \dots, z_d$ are local coordinates on A, and $y_1, \dots, y_{n-d}$ are coordinates in the complementary space, so that the $y_{n-d} = x_n$ axis is transversal to Γ and $y_1, \dots, y_{n-d-1}$ are the inner coordinates in Γ.

Assume that $m = 2k$ and the operators $P, B_1, \dots, B_k$ define a regular elliptic problem. Assume also that $m_1 < m_2 < \dots < m_k = m-1$, where m_j is the order of B_j. By the construction of the parametrix of this problem (see [1]),

$$u(x) = QPu + \sum_{j=1}^{k} Q_j B_j[u \otimes \delta(x_n)] + Tu\,,$$

where Q, Q_j, T are pseudo-differential operators of orders $-m$, $-m_j$, -1, respectively. Let Q_0 be the operator with symbol $1/p_0(x,\xi)$ and $g_j = B_j u - B_j Q_0 u$. Let $x \in \Gamma$. Let $r_1, \dots, r_k$ be the roots of the equation $p_0(x,\xi',r) = 0$ with positive imaginary parts. Let $R(x,\xi') = B(x,\xi')^{-1}$ where B is the matrix with elements $b_{jl}(x,\xi') = b_j(x \cdot \xi', r_l(x,\xi'))$. Then

$$u(x) = Q_0 Pu + (2\pi)^{-n+1} \int \sum_{j,l=1}^{k} r_{jl}(x,\xi') F[g_l](\xi') \exp(ir_j(x,\xi')x_n + ix'\xi')\, d\xi' + Tu$$

where $F[g]$ is the Fourier transform of g. Therefore the principal symbol of Q_j is

$$\sum_l r_{lj}(x,\xi') \exp(ir_j(x,\xi')x_n)$$

and the order of homogeneity of r_{lj} in ξ' is $-m_j$. Let

$$r(y,z,\eta,\zeta) = r(x,\xi') = \sum r_{jl}(x,\xi')\,.$$

The order of this function in ξ' is $1-m$.

THEOREM 4. *Let $Pu = 0$ in G and $B_j u = 0$ on $\Gamma \setminus A$ for $j = 1,\ldots,k$, and $|u(x)| = o(\ln r)$, where r is the distance of x from A. Let*

$$\int_{|\eta|=1} r(0,z,\eta,0)\, dS_\eta \neq 0 \qquad \textit{for } z \in A\,.$$

Then $B_j u = 0$ on Γ and $u \in C^\infty(G \cup \Gamma)$.

The proof is based on a construction from [3].

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