PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 27
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1992

SOME PROPERTIES OF EXPONENTIALLY HARMONIC MAPS

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1. The equation. Let (M,g) and (N,h) be two compact Riemannian manifolds, and $\phi : M \to N$ a smooth map. A classical definition asserts that ϕ is *harmonic* iff it is an extremal of the energy functional

$$E(\phi) = \int_M e(\phi) v_g \,,$$

where $e(\phi) = \frac{1}{2}|d\phi|^2$ is the energy density of ϕ and v_g the Riemannian volume element. The map ϕ is harmonic iff it satisfies the Euler–Lagrange system

$$\tau(\phi) = \operatorname{div}(d\phi) = 0\,.$$

In local coordinates

$$e(\phi) = \frac{1}{2} g^{ij} h_{\alpha\beta}(\phi) \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}\,,$$

and

$$\tau(\phi)^\alpha = g^{ij}\left(\frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij} \frac{\partial \phi^\alpha}{\partial x^k} + {}^N\Gamma^\alpha_{\beta\gamma}(\phi) \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} \right)$$

where the Γ's are the Christoffel symbols of the connections.

Research partially supported by EEC contract SC1-0105-C(AM).

The existence problem for harmonic maps is the following: given two Riemannian manifolds and a homotopy class $\mathcal{H}$ of maps from M to N, when is there a harmonic map in $\mathcal{H}$?

If M has a boundary, the same problem can be posed for homotopy classes relative to Dirichlet data.

This problem has been extensively studied, and the answer depends on the manifolds and the homotopy class (see e.g. [7]). In particular, the dimension of M plays a critical rôle.

To obtain existence of solutions in all dimensions, without conditions on the manifolds, we consider another problem of calculus of variations as follows.

Define the *exponential-energy* of ϕ as

$$\mathbb{E}(\phi) = \int_M \exp(\tfrac{1}{2}|d\phi|^2)v_g\,,$$

and say that a smooth extremal of $\mathbb{E}$ is an *exponentially harmonic map*.

Equivalently, we could write

$$\mathbb{E}(\phi) = \int_M \exp(\tfrac{1}{2}\operatorname{trace}(g^{-1}.\phi^* h)).v_g = \int_M (\det\exp(g^{-1}.\phi^* h))^{1/2} v_g\,,$$

where $\phi^* h$ is the pull-back of h by ϕ.

The Euler–Lagrange equation of this problem can be written

$$\not\tau(\phi) = \operatorname{div}(\exp e(\phi).d\phi) = \exp e(\phi).(\tau(\phi) + d\phi.\nabla e(\phi)) = 0\,.$$

It is an elliptic—but not uniformly elliptic—system of partial differential equations. Even in the case of maps between Euclidean spaces, it is non-linear and the second order terms are coupled. Indeed, in that case, it reads

$$\sum_{i=1}^{m} \frac{\partial^2\phi^\alpha}{\partial x^i \partial x^i} + \sum_{i,j=1}^{m}\sum_{\beta=1}^{n} \frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\frac{\partial^2\phi^\beta}{\partial x^i\partial x^j} = 0\,,$$

for $\alpha = 1, \ldots, n$.

In the case of maps from $\mathbb{R}^2$ to $\mathbb{R}$, this system reduces to the equation

$$\left(1 + \left(\frac{\partial\phi}{\partial x^1}\right)^2\right)\frac{\partial^2\phi}{\partial x^1\partial x^1} + 2\frac{\partial\phi}{\partial x^1}\frac{\partial\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x^1\partial x^2} + \left(1 + \left(\frac{\partial\phi}{\partial x^2}\right)^2\right)\frac{\partial^2\phi}{\partial x^2\partial x^2} = 0\,,$$

which is cited in [11, p. 431] as an example of a non-uniformly elliptic equation which is regularly elliptic.

We note for further use that if $e(\phi)$ is constant, then $\not\tau(\phi) = 0$ iff $\tau(\phi) = 0$, so that ϕ is exponentially harmonic iff it is harmonic.

2. Some related equations. The second group of terms of the equation: $d\phi.\nabla e(\phi)$ has a life of its own. Indeed, in the case of maps from $A \subset \mathbb{R}^m$ to $\mathbb{R}$, the equation $d\phi.\nabla e(\phi) = 0$, which reads

$$\sum_{i,j=1}^{m} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} = 0\,,$$

was studied in detail by G. Aronsson [1], [2], and in particular interpreted by a limiting process as an Euler–Lagrange equation for the functional

$$E_\infty(\phi) = \sup\{|d\phi(x)| : x \in A\} = \lim_{p \to \infty} \Big(\int_A |d\phi|^{2p} v_g \Big)^{1/(2p)}\,.$$

In another direction, we can include the equation of exponentially harmonic maps in a family of problems as follows.

Define a *density* $\varrho : M \times \mathbb{R}^+ \to \mathbb{R}^+$ and the *ϱ-energy density* of a map $\phi : (M,g) \to (N,h)$ by

$$e_\varrho(\phi)(x) = \frac{1}{2} \int_0^{|d\phi(x)|^2} \varrho(x,\xi)\, d\xi\,.$$

The Euler–Lagrange equation of $E_\varrho(\phi) = \int_M e_\varrho(\phi) v_g$ is

$$\tau_\varrho(\phi) = \operatorname{div}(\varrho(x, |d\phi|^2).d\phi) = 0\,,$$

and exponentially harmonic maps appear as a special case.

For $\dim N = 1$, the problem $\tau_\varrho(\phi) = 0$ appears in [12] as follows. If ϕ is a map from (M,g) to S^1, then $d\phi = w$ is a 1-form on M, and this establishes a bijection between the homotopy classes of maps $M \to S^1$ and the elements of the first integral cohomology group of M. The map ϕ satisfies $\tau_\varrho(\phi) = 0$ iff the form satisfies

$$dw = 0\,, \qquad d^*(\varrho(x, |w|^2)w) = 0\,.$$

In [12], L. and R. Sibner interpret these equations both as a non-linear version of Hodge theory and as a problem of gas dynamics, w representing the velocity vector of a gas.

They say that the density ϱ is *admissible* if there exist $A > 0$ and $k > 0$ such that for all $x \in M$ and $0 \le \xi < A$, we have

$$1/k < \varrho(x,\xi) < k$$

and $0 < (\partial/\partial\xi)(\xi.\varrho^2(x,\xi))$.

This last condition for all ξ is equivalent to ellipticity of the equation.

The supremum of values of A for which these conditions can be realised is the *sonic value* of the problem. In particular, when the sonic value is infinite, the problem is always elliptic and the solution is called subsonic. This is the case for $\varrho(x,\xi) = \exp(\frac{1}{2}\xi)$.

They call the density *regular* if the sonic value is infinite and there is a K such that

$$\frac{1}{K} < \frac{\partial}{\partial \xi}(\xi.\varrho^2(x,\xi)) < K$$

for all ξ. This is not realised for $\exp(\frac{1}{2}\xi)$.

In [12], they prove existence of a solution of the Euler–Lagrange equation when ϱ is regular. This establishes at the same time the existence of the flow of a compressible gas on a manifold, with density ϱ, and the existence in each cohomology class of a ϱ-harmonic one-form, solution of $dw = 0$ and $d^*(\varrho(x, |w|^2)w) = 0$.

For an admissible problem with finite sonic value, one cannot expect existence in all classes: indeed, a flow on M of large prescribed circulation will have to reach a supersonic speed.

The problem with density $\varrho(x,\xi) = \exp(\frac{1}{2}\xi)$ is admissible with infinite sonic value—but not regular.

Nevertheless, existence of a smooth solution in each class was established by D. M. Duc and J. Eells [5].

3. The existence problem.

We immediately get

PROPOSITION. *Let (M,g) and (N,h) be compact Riemannian manifolds, $\mathcal{H}$ a homotopy class (relative to a Dirichlet problem, if M has a boundary). Then $\mathcal{H}$ contains an $\mathbb{E}$-minimising map, which is α-Hölder continuous for all $\alpha < 1$.*

This can be verified using the properties of the Sobolev spaces of maps from M to N, which are defined as follows.

Choose a finite atlas on M and a Riemannian embedding of (N,h) in some Euclidean space V (Nash's theorem). Let $\mathcal{L}_1^p(M,V)$ be the Sobolev space of L^p functions from M to V whose first partial derivatives are also L^p. Then set

$$\mathcal{L}_1^p(M,N) = \{\phi \in \mathcal{L}_1^p(M,V) : \phi(x) \in N \text{ a.e.}\}\,.$$

Set $W = \bigcap_{p\geq 1} \mathcal{L}_1^p(M,N)$, and consider in $W \cap \mathcal{H}$ a minimising sequence (ϕ_n) for $\mathbb{E}$. Since

$$\mathbb{E}(\phi) = \int_M \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{|d\phi|^2}{2}\right)^k v_g\,,$$

(ϕ_n) is bounded in each $\mathcal{L}_1^p(M,N)$. Using the compactness of various Sobolev embeddings and a diagonal argument, we deduce that a subsequence converges weakly in each $\mathcal{L}_1^p$, strongly in each $\mathcal{L}^p$, and in C^α for each $\alpha < 1$ (indeed, for each such α, choose p with $0 < \alpha < 1 - \dim M/p$ to get a compact embedding $\mathcal{L}_1^p \hookrightarrow C^\alpha$). In particular, the convergence is uniform and the limit ϕ belongs to the homotopy class $\mathcal{H}$. Convexity in P of $\exp(|P|^2/2)$ insures lower-semicontinuity of $\mathbb{E}$ for that convergence (see e.g. [8], Th. 2.3), so that $\mathbb{E}(\phi) \leq \liminf \mathbb{E}(\phi_n)$, and ϕ is a C^α minimiser.

However, in general, we do not know yet if ϕ is smooth or if it satisfies the Euler–Lagrange equation of the problem, even in a weak sense. Indeed, the fact that $\mathbb{E}(\phi)$ is finite does not imply a priori that its first variation is finite.

Work on that question is in progress.

In the case $\dim N = 1$, D. M. Duc and J. Eells have solved the problem in [5] by proving *the existence of a smooth minimum of* $\mathbb{E}$ *for the Dirichlet problem in the case* $(M,g) \to \mathbb{R}$ *or in all homotopy classes in the case* $(M,g) \to S^1$.

In the case $\dim M = 1$, we consider maps from the circle or an interval into (N,h); the Euler–Lagrange equation reduces to

$$\nabla_{\partial/\partial t}\frac{\partial\phi}{\partial t} + \left\langle \nabla_{\partial/\partial t}\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial t}\right\rangle \frac{\partial\phi}{\partial t} = 0\,.$$

Denoting by T and N the tangential and normal components of a vector with respect to $\partial\phi/\partial t$, this equation reads

$$2\left(\nabla_{\partial/\partial t}\frac{\partial\phi}{\partial t}\right)^T + \left(\nabla_{\partial/\partial t}\frac{\partial\phi}{\partial t}\right)^N = 0\,,$$

i.e.

$$\nabla_{\partial/\partial t}\frac{\partial\phi}{\partial t} = 0\,.$$

Thus, the C^2 exponentially harmonic curves are simply geodesics parametrised proportionally to arc length, and their existence in each homotopy class is well known.

The existence of smooth minimisers for $\mathbb{E}$ was also established directly by M. Carpenter as a special case of the following result [4]: *if* $\phi : S^1 \to (N,h)$ *is a minimiser of a convex, coercive and autonomous variational problem, then* ϕ *is smooth.* His approach involves the classical Tonelli theory. In the case of $\mathbb{E}$, as we noted, it follows that every $\mathbb{E}$-minimising curve is a smooth geodesic.

4. Jensen's inequality. This last observation also follows from the following considerations.

Jensen's inequality for convex functions ([9], p. 21) takes here the form

PROPOSITION. *Let* $\phi : (M,g) \to (N,h)$ *be a map in* W. *Then*

$$\exp\left(\frac{1}{\operatorname{vol} M}E(\phi)\right) \le \frac{1}{\operatorname{vol} M}\mathbb{E}(\phi)\,.$$

Equality is valid iff $e(\phi)$ *is constant a.e.*

COROLLARY. *Let* $\mathcal{H}$ *be a homotopy class of maps between compact Riemannian manifolds in which the minimum of* E *is realised by a harmonic map of constant energy density. Then the same map minimises* $\mathbb{E}$, *and any* $\mathbb{E}$-*minimum has constant energy density a.e.*

Indeed, let ϕ_0 be a minimiser of E in $\mathcal{H}$ with $e(\phi_0) = K$, a constant. For each $\phi \in \mathcal{H}$, we have

$$\exp\left(\frac{1}{\operatorname{vol} M}E(\phi_0)\right) \le \exp\left(\frac{1}{\operatorname{vol} M}E(\phi)\right) \le \frac{1}{\operatorname{vol} M}\mathbb{E}(\phi)\,,$$

so that

$$\exp\left(\frac{1}{\operatorname{vol} M} E(\phi_0)\right) \le \inf_{\mathcal{H}} \frac{1}{\operatorname{vol} M} \mathbb{E}\,.$$

On the other hand, since $e(\phi_0) = K$,

$$\frac{1}{\operatorname{vol} M} \mathbb{E}(\phi_0) = \exp K = \exp\left(\frac{1}{\operatorname{vol} M} E(\phi_0)\right)$$

so that ϕ_0 is $\mathbb{E}$-minimising.

This corollary applies in particular to any homotopy class of curves.

5. Stress-energy tensor. In analogy with [3], we consider the

Definition. Let $\phi : (M, g) \to (N, h)$ be a C^2 map. The *exponential stress-energy tensor* of ϕ is the 2-covariant symmetric tensor on M given by

$$S(\phi) = (\exp e(\phi)).(g - \phi^* h)\,.$$

Proposition.

$$\operatorname{div} S(\phi) = -\langle \not{\tau}(\phi), d\phi \rangle\,,$$

where the divergence of $S(\phi)$ is expressed in local coordinates by

$$(\operatorname{div} S(\phi))_i = g^{jk} \nabla_j S_{ik}(\phi)\,.$$

The proof of this formula is similar to that of the corresponding statement for the classical harmonic maps: for a vector field X on M, we write

$$\begin{aligned} 0 &= \int_M d[i(X)(\exp e(\phi).v_g)] = \int_M L_X(\exp e(\phi).v_g) \\ &= \int_M \langle \not{\tau}(\phi), d\phi.X\rangle + \operatorname{div}(\exp e(\phi).(g - \phi^* h))\,, \qquad X > v_g \end{aligned}$$

(see [6], §6).

Thus, for any C^2 exponentially harmonic map, we get $\operatorname{div} S(\phi) = 0$.

However, we doubt that this proposition will have many applications. Indeed, in the case of harmonic maps, the stress-energy tensor is defined as $e(\phi).g - \phi^* h$ and, for instance, vanishes iff $\dim M = 2$ and ϕ is conformal. In the present case, the stress-energy tensor vanishes only in the case of isometries. Note in particular that neither $\mathbb{E}$ nor the Euler–Lagrange equations are invariant under a homothetic transformation of g or h. This is quite unusual in the framework of geometry, but more natural in comparison with gas dynamics.

6. Second variation of $\mathbb{E}$. Let $\phi_{s,t}$ be a smooth 2-parameter family of maps from (M, g) to (N, h), where $s, t \in (-\varepsilon, \varepsilon)$. A computation similar to that of [13] shows that the Hessian of $\mathbb{E}$ at $\phi = \phi_{0,0}$ is given by

$$H_\phi(v, w) = \left.\frac{\partial^2 \mathbb{E}(\phi_{s,t})}{\partial s \partial t}\right|_{s,t=0} = \int_M \exp e(\phi).[\langle \nabla^\phi v, d\phi\rangle \langle \nabla^\phi w, d\phi\rangle$$

$$+\langle\nabla^\phi v, \nabla^\phi w\rangle - \langle R^N(d\phi, v)d\phi, w\rangle - \langle\nabla^\phi_{\partial/\partial t} v, \mathrm{Trace}\,\nabla(\exp e(\phi).d\phi)\rangle]v_g\,.$$

Here, $v = \partial\phi_{s,t}/\partial s|_{s,t=0}$, $w = \partial\phi_{s,t}/\partial t|_{s,t=0}$, ∇^ϕ is the pull-back connection and R^N the curvature tensor of N (see [6], §4, for details).

We first note that when the sectional curvature of N is non-positive ($\mathrm{Riem}^N \leq 0$), this expression has the same consequences as in the case of harmonic maps. In particular:

Definition. An exponentially harmonic map ϕ is $\mathbb{E}$-*stable* (or stable) if its index is zero, i.e. its Hessian is positive semi-definite.

Corollary 1. *If* $\mathrm{Riem}^N \leq 0$, *any exponentially harmonic map is stable.*

Indeed, in the expression of $H_\phi(v,v), \mathrm{Trace}\,\nabla(\exp e(\phi).d\phi) = 0$ because ϕ is exponentially harmonic and all the other terms are positive or zero by hypothesis.

Corollary 2. *Let* $\mathrm{Riem}^N \leq 0$ *and let* $\phi_0, \phi_1 : M \to N$ *be two exponentially harmonic maps. If* ∂M *is non-void and* ϕ_0 *and* ϕ_1 *are homotopic relatively to a Dirichlet problem*, *then* $\phi_0 = \phi_1$. *If* ∂M *is empty*, $\mathrm{Riem}^N < 0$, ϕ_0 *and* ϕ_1 *are homotopic and* ϕ_0 *has rank* ≥ 2 *somewhere*, *then* $\phi_0 = \phi_1$.

The proof follows the technique of R. Schoen [10]: construct a homotopy ϕ_t from ϕ_0 to ϕ_1 such that $t \mapsto \phi_t(x)$ is a geodesic for each x. Then in the expression of $d^2\mathbb{E}(\phi_t)/dt^2$, the last term vanishes because $\nabla_{(\partial/\partial t)} v = 0$, and all remaining terms are non-negative. Therefore $d^2\mathbb{E}(\phi_t)/dt^2 \geq 0\ \forall t$.

Since ϕ_0 and ϕ_1 are exponentially harmonic, we have also

$$\frac{d\mathbb{E}}{dt}(\phi_0) = 0 \quad \text{and} \quad \frac{d\mathbb{E}}{dt}(\phi_1) = 0\,.$$

These three conditions imply that $\mathbb{E}(\phi_t)$ is constant.

In particular, $d^2\mathbb{E}(\phi_t)/dt^2 = 0$, so that $\nabla^\phi v = 0$ and $\langle R^N(d\phi, v)d\phi, v\rangle = 0$ everywhere.

The first equation implies $(d/dt)\|v\|^2 = 0$, and $\|v\| = \text{constant}$. If ϕ_0 and ϕ_1 coincide on ∂M, we get $v = 0$ and $\phi_0 = \phi_1$ on M.

In the case $\partial M = \phi$, we use the condition on the curvature and the rank to get $v = 0$ as well.

Corollary 3. *Let* $\mathrm{Riem}^N \leq 0$ *and let* ϕ_0 *be an exponentially harmonic map from* M *to* N. *Then* ϕ_0 *minimizes* $\mathbb{E}$ *in its homotopy class.*

Indeed, for any map ϕ_1 in the same homotopy class, we construct a homotopy ϕ_t as above and get $(d\mathbb{E}/dt)(\phi_0) = 0$ and $d^2\mathbb{E}(\phi_t)/dt^2 \geq 0$.

In general, we cannot expect to obtain such results when Riem^N is not restricted. Indeed, examples of closed geodesics on spheres or warped product manifolds show that exponentially harmonic maps need not be stable or unique in their homotopy class.

However, the behaviour of the identity map $I : (M, g) \to (M, g)$ is strikingly different when viewed as an extremal of $\mathbb{E}$ or E. In fact, we have the

PROPOSITION. *The identity map I of a compact manifold to itself is always $\mathbb{E}$-stable, and the only Jacobi fields for $\mathbb{E}$ are the Killing fields.*

Indeed, since $e(I)$ is constant, we have for $\phi_0 = I$

$$\frac{d^2\mathbb{E}}{dt^2}(\phi_t)|_{t=0} = \exp e(I).\int_M (|\operatorname{div} v|^2 + |\nabla^I v|^2 - \operatorname{Ricci}^M(v,v))v_g\,.$$

By [6], (4.14), this reduces to

$$\frac{d^2\mathbb{E}}{dt^2}(\phi_t)\bigg|_{t=0} = \frac{1}{2}\exp e(I).\int_M |L_v g|^2 v_g\,,$$

where L is the Lie derivative. The proposition follows immediately.

Note that for E, the identity map is not always stable (see [7], (6.1)–(6.12)).

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