PARTIAL DIFFERENTIAL EQUATIONS
BANACH CENTER PUBLICATIONS, VOLUME 27
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 1992

HYPERLOGARITHMIC EXPANSION AND THE VOLUME OF A HYPERBOLIC SIMPLEX

K. AOMOTO

Department of Mathematics, Nagoya University
Nagoya, Japan

0. Introduction. Hyperlogarithmic functions (or higher logarithmic functions) are multivalued analytic functions defined on complex projective varieties, with unipotent monodromy and with regular singularity. It is known that they can be expressed by the use of iterated integrals of suitable logarithmic 1-forms in the sense of K. T. Chen (see [A1], [H1]). Recently these functions have played a considerable role in various problems of geometry and arithmetic (for example, see [H2], [B1], [G2], [V], etc.). These are a special case of hypergeometric functions on a Grassmannian manifold (see [A2], [G1], [V]).

However, there are other kinds of hyperlogarithmic functions which are related to the configuration of hyperplanes and a hyperquadric (see [A3]). The volume of a simplex in a hyperbolic space is a hyperlogarithmic function of basic algebraic invariants, as a simple consequence of the Schläfli formula. However, there remains the problem of divergence in the case where the vertices lie on the boundary.

In this note we want to *derive a modified Schläfli formula in such a degenerate case and give a hyperlogarithmic expansion for the volume, by using a technique developed in* [A3]. A similar result has been obtained by Kellerhals [K4]. Her method is to decompose a simplex into several orthoschemes and to obtain an explicit formula for each orthoscheme by using the Lobachevskiĭ function $Л(x)$.

In the appendix we discuss a relation between the volume and Appell's hypergeometric functions of type F_4.

1. The Schläfli formula. A geodesic simplex Δ in the n-dimensional hyperbolic space $H = \{t_0^2 - t_1^2 - \ldots - t_n^2 = 1,\ t_0 > 0\}$ is defined by the inequalities $f_j(t) \geq 0$ for $n+1$ linear functions $f_j(t) = u_{j,0} + \sum_{\nu=1}^{n} u_{j,\nu} t_\nu$, $1 \leq j \leq n+1$. Its volume $V_n(\varDelta)$ is given by the integral

$$V_n(\varDelta) = \int\limits_{f_1\geq 0,\ldots,f_{n+1}\geq 0} \varPhi\, dt_0 \wedge \ldots \wedge dt_{n+1}\,, \tag{1.1}$$

for $\varPhi = \exp[-\frac{1}{2}(t_0^2 - t_1^2 - \ldots - t_n^2)]$. This is also equal to $2^{(n-1)/2}\varGamma((n+1)/2)V_n(\widehat{\varDelta})$ for a geodesic simplex $\widehat{\varDelta}$ in the disc $D = \{x_1^2 + \ldots + x_n^2 < 1\}$ defined by the inequalities $\widehat{f}_j(x) \geq 0$ for the inhomogeneous linear functions $\widehat{f}_j(x) = u_{j,0} + \sum_{\nu=1}^n u_{j,\nu}x_\nu$. The volume $V_n(\widehat{\varDelta})$ is defined by the integral

$$V_n(\widehat{\varDelta}) = \int\limits_{\widehat{\varDelta}} (1 - x_1^2 - \ldots - x_n^2)^{-(n+1)/2}\, dx_1 \wedge \ldots \wedge dx_n\,. \tag{1.2}$$

First we assume that $\widehat{\varDelta}$ lies in D. Then $u_{j,0}^2 - \sum_{\nu=1}^n u_{j,\nu}^2 < 0$. We may normalize it so that $u_{j,0}^2 - \sum_{\nu=1}^n u_{j,\nu}^2 = -1$. Because of conformal invariance, $V_n(\varDelta)$ or equivalently $V_n(\widehat{\varDelta})$ depends only on the inner products $a_{j,k} = u_{j,0}u_{k,0} - \sum_{\nu=1}^n u_{j,\nu}u_{k,\nu}$ for $1 \leq j,k \leq n+1$. $a_{j,k}$, $j \neq k$, can also be expressed as $\cos\langle j,k\rangle$, where $\langle j,k\rangle$ denotes the dihedral angle subtended by $\widehat{\varDelta}$ between the hyperplanes $F_j = \{\widehat{f}_j(x) = 0\}$ and $F_k = \{\widehat{f}_k(x) = 0\}$. We denote by A the symmetric $(n+1)\times(n+1)$ matrix $((a_{j,k}))_{1\leq j,k\leq n+1}$. Note that $a_{j,j} = -1$. We denote by $A\binom{i_1\ldots,i_p}{j_1,\ldots,j_p}$ the subdeterminant of A with lines $i_1,\ldots,i_p$ and columns $j_1,\ldots,j_p$ for $\{i_1,\ldots,i_p\},\{j_1,\ldots,j_p\} \subset \{1,2,\ldots,n+1\}$. We abbreviate $A\binom{i_1\ldots,i_p}{j_1,\ldots,j_p}$ to $A(i_1,\ldots,i_p)$.

One can show that $\widehat{\varDelta}$ defines a simplex lying in D if and only if

$$(-1)^p\, A(i_1,\ldots,i_p) > 0 \quad \text{for } 1 \leq p \leq n, \text{ and} \tag{1.3}$$

$$(-1)^{n+1}\, A(1,2,\ldots,n+1) < 0\,. \tag{1.4}$$

We denote by $v_1,\ldots,v_{n+1}$ the vertices of $\widehat{\varDelta}$ such that $v_j \in \widehat{\varDelta} \cap F_1 \cap \ldots \cap F_{j-1} \cap F_{j+1} \cap \ldots \cap F_{n+1}$. Then v_j is on the boundary ∂D of D if and only if $A(1,2,\ldots,j-1,j+1,\ldots,n+1) = 0$. The Schläfli formula says that, as a function of the variables $a_{j,k}$, $V_n(\varDelta)$ satisfies the variational formula

$$dV_n(\widehat{\varDelta}) = -\frac{1}{2}\sum_{\substack{1\leq j,k\leq n+1\\ j\neq k}} V_{n-2}(\widehat{\varDelta}_{j,k})\, d\langle j,k\rangle\,, \tag{1.5}$$

where $\widehat{\varDelta}_{j,k}$ denotes the $(n-2)$-dimensional subsimplex $\widehat{\varDelta}_{j,k} = \widehat{\varDelta} \cap F_j \cap F_k$. $d\langle j,k\rangle$ is equal to the logarithmic 1-form

$$\theta\begin{pmatrix}\emptyset\\ j,k\end{pmatrix} = \frac{1}{2i}d\log\left(\frac{a_{j,k} + i\sqrt{A(j,k)}}{a_{j,k} - i\sqrt{A(j,k)}}\right).$$

Further, for $I = \{i_1,\ldots,i_p\}$ and $J = \{i_1,\ldots,i_p,i_{p+1},i_{p+2}\}$ we define the logarithmic 1-form

$$\theta\binom{I}{J} = \frac{1}{2i} d\log\left(\frac{A\binom{I,i_{p+1}}{I,i_{p+2}} + i\sqrt{A(I)A(J)}}{A\binom{I,i_{p+1}}{I,i_{p+2}} - i\sqrt{A(I)A(J)}}\right) \tag{1.6}$$

for $p \le n-2$, and

$$\theta\binom{I}{J} = \frac{1}{2} d\log\left(\frac{A\binom{I,i_n}{I,i_{n+1}} + \sqrt{-A(I)A(J)}}{A\binom{I,i_n}{I,i_{n+1}} - \sqrt{-A(I)A(J)}}\right) \tag{1.7}$$

for $p = n-1$, n odd.

As a simple consequence of (1.5), $V_n(\widehat{\Delta})$ can be expressed as a hyperlogarithm (sometimes called higher logarithm) (see [A5]):

$$V_n(\widehat{\Delta}) = \sum_{\emptyset \subset I_1 \subset \ldots \subset I_\nu} \int_*^A \theta\binom{\emptyset}{I_1} \ldots \theta\binom{I_{\nu-1}}{I_\nu}, \tag{1.8}$$

for a sequence of increasing subsets $I_1, \ldots, I_\nu$ of $\{1, 2, \ldots, n+1\}$, $I_r = \{i_1, \ldots \ldots, i_{2r}\}$. ν is equal to $(n+1)/2$ or $n/2$ according as n is odd or even. The integration on the right hand side means K. T. Chen's iterated integrals along a path from the base point $*$ to A. As special cases we have

$$V_1(\widehat{\Delta}) = \int_\alpha^\beta (1-x^2)^{-1}\, dx = \frac{1}{2}\log\frac{a_{1,2} + \sqrt{a_{1,2}^2 - 1}}{a_{1,2} - \sqrt{a_{1,2}^2 - 1}} \tag{1.9}$$

for $\alpha = -u_{1,0}/u_{1,1}$, $\beta = -u_{2,0}/u_{2,1}$ and $a_{1,2} = u_{1,0}u_{2,0} - u_{1,1}u_{2,1}$, while

$$V_2(\widehat{\Delta}) = \pi - \langle 1,2\rangle - \langle 2,3\rangle - \langle 3,1\rangle\,. \tag{1.10}$$

The following is an immediate consequence of (1.7).

LEMMA 1. *The hyperbolic distance between v_n and v_{n+1} is given by*

$$\frac{1}{2}\log\frac{A\binom{1,\ldots,n-1,n}{1,\ldots,n-1,n+1} + \sqrt{A(1,\ldots,n-1)A(1,\ldots,n+1)}}{A\binom{1,\ldots,n-1,n}{1,\ldots,n-1,n+1} - \sqrt{A(1,\ldots,n-1)A(1,\ldots,n+1)}}\,. \tag{1.11}$$

We see at the same time that

$$A\binom{i_1,\ldots,i_{n-1},i_n}{i_1,\ldots,i_{n-1},i_{n+1}} > 0\,. \tag{1.12}$$

This inequality will be used later for $n = 3$.

2. Regularization of divergent integrals. When one of the vertices lies on ∂D, $V_n(\Delta)$ is well defined and continuous in $a_{j,k}$, while $V_1(\Delta)$ diverges. *The formula* (1.5) *holds for $n \ge 4$ but not for $n = 3$.* We want to derive a modified version of the Schläfli formula for $V_3(\Delta)$. To do this, we use the technique of regularization of divergent integrals which has been frequently used since the times of J. Hadamard. We consider the integral

$$V_n(\widehat{\Delta}|\mu) = \int\limits_{\widehat{\Delta}} (1-|x|^2)^{-(n+1+\mu)/2} dx_1 \wedge \ldots \wedge dx_n \,, \tag{2.1}$$

for $\mu > 0$. When $\mu = 0$, it reduces to $V_n(\widehat{\Delta})$. (2.1) is no more conformally invariant. It cannot be expressed as a function of the variables $a_{j,k}$ for $1 \le j,k \le n+1$. We denote by $\widetilde{A}$ the enlarged symmetric $(n+2)\times(n+2)$ matrix $((a_{j,k}))_{0\le j,k\le n+1}$ with $a_{0,0} = 1$, and $a_{0,j} = a_{j,0} = u_{j,0}$. Obviously $a_{j,0}$ is not conformally invariant.

The following variational formula has been proved in [A3] (see the formula (3.7) *loc. cit.* for $\lambda_1, \ldots, \lambda_{n+1} \to 0$):

LEMMA 2.1. *For an arbitrary $n \ge 1$,*

$$\begin{aligned}(n-1+\mu)\, dV_n(\widehat{\Delta}|\mu) &= -\frac{1}{2} \sum_{\substack{1\le j,k\le n+1\\ j\ne k}} d\langle j,k\rangle \left\{\frac{A(j,k)}{A(0,j,k)}\right\}^{-\mu/2} V_{n-2}(\widehat{\Delta}_{j,k}|\mu)\\ &\quad + \mu \sum_{k=1}^{n+1} da_{0,k} \left\{\frac{-1}{A(0,k)}\right\}^{-\mu/2} \frac{1}{\sqrt{A(0,k)}} V_{n-1}(\widehat{\Delta}_k|\mu-1)\,,\end{aligned} \tag{2.2}$$

where $\widehat{\Delta}_{j,k} = \widehat{\Delta} \cap F_j \cap F_k$ and $\widehat{\Delta}_k = \widehat{\Delta} \cap F_k$. $V_1(\widehat{\Delta}_{j,k}|\mu)$ has a definite meaning and gives a function meromorphic in μ at least with a pole at $\mu = 0$.

The following lemma can be seen by a computation.

LEMMA 2.2. $A(0,i) < 0$, $A(0,i,j) > 0$, $A(0,i,j,k) < 0$ *for any* $i,j,k \in \{1,2,3,4\}$ *and* $A(0,1,2,3,4) = 0$.

When $\beta = 1$ in (1.9), $V_1(\widehat{\Delta}|\mu)$ has a Laurent expansion at $\mu = 0$:

$$V_1(\widehat{\Delta}|\mu) = -\frac{1}{\mu} + \left\{\log 2 - \frac{1}{2}\log\frac{1+\alpha}{1-\alpha}\right\} + O(\mu)\,, \tag{2.3}$$

with $\alpha = -a_{0,2}/\sqrt{-A(0,2)}$, where the constant term (denoted by C.T. $V_1(\widehat{\Delta}|\mu)$) represents the regular part of the divergent integral $V_1(\widehat{\Delta})$:

$$\operatorname{reg} V_1(\widehat{\Delta}) = \text{C.T.}\, V_1(\widehat{\Delta}|\mu) = \log 2 - \frac{1}{2}\log\frac{1+\alpha}{1-\alpha}\,. \tag{2.4}$$

When $\alpha = \beta = -1$, we have

$$V_1(\widehat{\Delta}|\mu) = -\frac{2}{\mu} + 2\log 2 + O(\mu)\,, \tag{2.5}$$

whence $\operatorname{reg} V_1(\widehat{\Delta}) = 2\log 2$.

3. Modified Schläfli formula for $n = 3$. Because of symmetry, we only have to consider the following 4 cases: (i) $v_4 \in \partial D$, (ii) $v_3, v_4 \in \partial D$, (iii) $v_2, v_3, v_4 \in \partial D$ and (iv) $v_1, v_2, v_3, v_4 \in \partial D$.

(1) *Assume that v_4 lies on ∂D and $v_1, v_2, v_3 \in D$.* This is equivalent to saying that $A(1,2,3) = 0$, i.e. $\langle 1,2\rangle + \langle 2,3\rangle + \langle 3,1\rangle = \pi$. Then $V_1(\widehat{\Delta}_{1,2}), V_1(\widehat{\Delta}_{2,3})$ and $V_1(\widehat{\Delta}_{3,1})$ diverge, while $V_1(\widehat{\Delta}_{1,4})$ $V_1(\widehat{\Delta}_{2,4}), V_1(\widehat{\Delta}_{3,4})$ are well defined. For $j,k = 1,2,3$, as μ tends to 0, the coefficient of $\langle j,k\rangle$ on the right hand side of (2.2) has a Laurent expansion

$$\left\{\frac{A(j,k)}{A(0,j,k)}\right\}^{-\mu/2} V_1(\widehat{\Delta}_{j,k}|\mu) = -\frac{1}{\mu} + \left\{\log 2 - \frac{1}{2}\log\frac{1+\alpha}{1-\alpha} + \frac{1}{2}\log\frac{A(j,k)}{A(0,j,k)}\right\} + O(\mu)\,, \tag{3.2}$$

i.e.

$$\text{C.T.}\left[\left\{\frac{A(j,k)}{A(0,j,k)}\right\}^{-\mu/2} V_1(\widehat{\Delta}_{j,k}|\mu)\right] = \log 2 - \frac{1}{2}\log\frac{1+\alpha}{1-\alpha} + \frac{1}{2}\log\frac{A(j,k)}{A(0,j,k)}\,. \tag{3.3}$$

Here α denotes $-A\binom{4,j,k}{0,j,k}/\sqrt{-A(j,k)A(0,j,k,4)}$. We set

$$W_{j,k} = \frac{A(j,k)(1+\alpha)}{A(0,j,k)(1-\alpha)}\,.$$

Then by taking the constant term of (2.2) in μ, we have

$$2dV_3(\widehat{\Delta}) = d\langle 1,2\rangle \log W_{1,2} + d\langle 2,3\rangle \log W_{2,3} + d\langle 3,1\rangle \log W_{3,1} + d\langle 1,4\rangle \log W_{1,4} + d\langle 2,4\rangle \log W_{2,4} + d\langle 3,4\rangle \log W_{3,4}\,, \tag{3.4}$$

for $i,j = 1,2,3$, since $\langle 1,2\rangle + \langle 2,3\rangle + \langle 3,1\rangle = \pi$, i.e.

$$W_{i,j} = \frac{A(0,i,j)}{A(i,j)} \cdot \frac{\sqrt{-A(i,j)A(0,i,j,4)} - A\binom{4,i,j}{0,i,j}}{\sqrt{-A(i,j)A(0,i,j,4)} + A\binom{4,i,j}{0,i,j}} \tag{3.5}$$

and

$$W_{i,4} = \frac{A\binom{i,4,j}{i,4,k} - \sqrt{-A(i,4)A(i,j,k,4)}}{A\binom{i,4,j}{i,4,k} + \sqrt{-A(i,4)A(i,j,k,4)}} \tag{3.6}$$

for the complement $\{j,k\} = \{1,2,3,4\} - \{i,4\}$. Since $d\langle 1,2\rangle = -d\langle 1,3\rangle - d\langle 2,3\rangle$, (3.4) can be expressed as

$$2dV(\widehat{\Delta}) = d\langle 1,3\rangle \log W_{1,3}/W_{1,2} + d\langle 2,3\rangle \log W_{2,3}/W_{1,2} + d\langle 1,4\rangle \log W_{1,4} + d\langle 2,4\rangle \log W_{2,4} + d\langle 3,4\rangle \log W_{3,4}\,. \tag{3.7}$$

We want to express the quantities $W_{1,3}/W_{1,2}$ and $W_{2,3}/W_{1,2}$ in terms of the variables $a_{j,k}$, $1 \le j,k \le 4$. By a conformal change of variables we may assume that $v_4=(0,0,1)\in\partial D\cap F_1\cap F_2\cap F_3$ and that

$$f_1=x_1\,,\quad f_2=u_{2,1}x_1+u_{2,2}x_2\,,\qquad (3.8)$$
$$f_j=u_{j,1}x_1+u_{j,2}x_2+u_{j,3}x_3+u_{j,0}\,,\quad \text{for } j=3,4\,,$$

where $1=u_{2,1}^2+u_{2,2}^2=u_{3,1}^2+u_{3,2}^2=u_{4,1}^2+u_{4,2}^2+u_{4,3}^2-u_{4,0}^2$ and $u_{3,3}+u_{3,0}=0$. We can further assume that $u_{2,2}>0$, $u_{3,2}<0$, $u_{3,3}<0$ and $u_{4,3}>0$. We then have $a_{0,1}=a_{0,2}=0$ and

LEMMA 3.1.

$$W_{1,2}=\frac{a_{0,3}^2A(1,2,4)}{A(1,2,3,4)}\,.$$

Proof. Since

$$0=A(0,1,2,3,4) \qquad (3.9)$$
$$=A(1,2,3,4)-a_{0,3}^2A(1,2,4)+2a_{0,3}a_{0,4}A\binom{1,2,3}{1,2,4}-a_{0,4}^2A(1,2,3)\,,$$

we have

$$a_{0,4}=\frac{A(1,2,4)a_{0,3}^2-A(1,2,3,4)}{2a_{0,3}A\binom{1,2,3}{1,2,4}} \qquad (3.10)$$

from the equality $A(1,2,3)=0$. Since $A(1,2)A(1,2,3,4)-A\binom{1,2,3}{1,2,4}^2=0$, we have

$$A(0,1,2,4)=A(1,2,4)-A(1,2)a_{0,4}^2 \qquad (3.11)$$
$$=-\frac{A(1,2)\{a_{0,3}^2A(1,2,4)+A(1,2,3,4)\}^2}{4a_{0,3}^2A\binom{1,2,3}{1,2,4}^2}\,,\quad \text{i.e.}$$

$$\sqrt{-A(1,2)A(0,1,2,4)}=-\frac{A(1,2)\{a_{0,3}^2A(1,2,4)+A(1,2,3,4)\}}{2a_{0,3}A\binom{1,2,3}{1,2,4}}\,. \qquad (3.12)$$

Note that $a_{0,3}>0$, $A(1,2)>0$, $A(1,2,4)<0$, $A\binom{1,2,3}{1,2,4}>0$ and $A(1,2,3,4)<0$. Again from (3.10),

$$\sqrt{-A(1,2)A(0,1,2,4)}-A\binom{4,1,2}{0,1,2}=\frac{-a_{0,3}A(1,2,4)A(1,2)}{A\binom{1,2,3}{1,2,4}}\,. \qquad (3.13)$$

In the same way

$$\sqrt{-A(1,2)A(0,1,2,4)}+A\binom{4,1,2}{0,1,2}=-\frac{A(1,2,3,4)A(1,2)}{a_{0,3}A\binom{1,2,3}{1,2,4}}\,, \qquad (3.14)$$

whence Lemma 3.1 is proved.

LEMMA 3.2.

$$W_{1,3}=a_{0,3}^2\frac{A(1,2)A(1,3,4)}{A(1,3)A(1,2,3,4)}\,, \qquad (3.15)$$

$$W_{2,3} = a_{0,3}^2 \frac{A(1,2)A(2,3,4)}{A(2,3)A(1,2,3,4)} . \tag{3.16}$$

Proof. First remark $a_{0,3} = u_{3,0} > 0$, $A\binom{1,3,2}{1,3,4} = u_{2,2}u_{3,2}u_{3,3}(u_{4,0} + u_{4,3}) < 0$, $A\binom{1,2}{1,3} = u_{2,2}u_{3,2} < 0$ and $A\binom{0,1,3,2}{0,1,3,4} = -u_{2,2}u_{3,3}(u_{3,3}u_{4,2} - u_{3,2}u_{4,3}) > 0$. By the Jacobi identity

$$\begin{aligned} 0 &= A(0,1,2,3,4)A(0,1,3) \\ &= A(0,1,3,4)A(0,1,2,3) - A\begin{pmatrix}0,1,3,2\\0,1,3,4\end{pmatrix}^2 \\ &= -A(0,1,3,4)A(1,2)a_{0,3}^2 - A\begin{pmatrix}0,1,3,2\\0,1,3,4\end{pmatrix}^2 \end{aligned} \tag{3.17}$$

since $A(0,1,2,3) = -a_{0,3}^2 A(1,2)$, whence

$$A(0,1,3,4) = -\frac{A\binom{0,1,3,2}{0,1,3,4}^2}{a_{0,3}^2 A(1,2)} . \tag{3.18}$$

From (1.3) and the above,

$$\sqrt{-A(1,3)A(0,1,3,4)} = \sqrt{\frac{A(1,3)}{A(1,2)}}\,\frac{A\binom{0,1,3,2}{0,1,3,4}}{a_{0,3}} , \tag{3.19}$$

where $A\binom{0,1,3,2}{0,1,3,4}$ equals

$$\frac{1}{2}A\begin{pmatrix}1,3,2\\1,3,4\end{pmatrix} - a_{0,3}^2 \frac{A\binom{1,2}{1,4}A\binom{1,2,3}{1,2,4} - \frac{1}{2}A\binom{1,2}{1,3}A(1,2,4)}{A\binom{1,2,4}{1,2,3}} , \tag{3.20}$$

in view of the Jacobi identities $A\binom{1,2}{1,3}A\binom{1,2,3}{1,2,4} = -A(1,2)A\binom{1,3,2}{1,3,4}$ and $A(1,2)A(1,3) - A\binom{1,2}{1,3}^2 = 0$. Hence

$$\begin{aligned} &\sqrt{-A(1.3)A(0,1,3,4)} + A\begin{pmatrix}4,1,3\\0,1,3\end{pmatrix} \\ &\quad = \sqrt{\frac{A(1,3)}{A(1,2)}}\,\frac{A\binom{1,3,2}{1,3,4}}{a_{0,3}} + a_{0,3}\frac{A\binom{1,2,3}{1,2,4}}{A(1,2)} \\ &\quad = -\frac{A\binom{1,2,3}{1,2,4}}{A(1,2)}\left\{-\frac{A(1,3)}{a_{0,3}} - a_{0,3}\right\} = \frac{A\binom{1,2,3}{1,2,4}A(0,1,3)}{a_{0,3}A(1,2)} , \end{aligned} \tag{3.21}$$

since $A\binom{1,2,3}{1,2,4} = A\binom{1,2}{1,4}A\binom{1,2}{1,3} - A(1,2)A\binom{1,3}{1,4}$ and $A(0,1,3) = A(1,3) + a_{0,3}^2$. Similarly

$$\sqrt{-A(1,3)A(0,1,3,4)} - A\begin{pmatrix}4,1,3\\0,1,3\end{pmatrix} = -a_{0,3}\frac{A(1,2)A(1,3,4)}{A\binom{1,2,4}{1,2,3}} . \tag{3.22}$$

Now (3.21) and (3.22) imply

$$(3.23)\quad \frac{\sqrt{-A(1,3)A(0,1,3,4)} - A\binom{4,1,3}{0,1,3}}{\sqrt{-A(1,3)A(0,1,3,4)} + A\binom{4,1,3}{0,1,3}} = -\frac{a_{0,3}^2 A(1,2)^2 A(1,3,4)}{A\binom{1,2,4}{1,2,3}^2 A(0,1,3)} = \frac{A(1,2)A(1,3,4)a_{0,3}^2}{A(1,2,3,4)A(0,1,3)},$$

which proves (3.15); (3.16) follows by symmetry.

Corollary.

$$(3.24)\quad W_{1,3}/W_{1,2} = \frac{A(1,2)A(1,3,4)}{A(1,3)A(1,2,4)},$$

$$(3.25)\quad W_{2,3}/W_{1,2} = \frac{A(1,2)A(2,3,4)}{A(2,3)A(1,2,4)}.$$

As a result we have

Proposition (*modified Schläfli formula*).

$$(3.26)\quad 2dV_3(\widehat{\Delta}) = d\langle 1,3\rangle \log(W_{1,3}/W_{1,2}) + d\langle 2,3\rangle \log(W_{2,3}/W_{1,2}) + d\langle 1,4\rangle \log W_{1,4} + d\langle 2,4\rangle \log W_{2,4} + d\langle 3,4\rangle \log W_{3,4},$$

where $W_{1,3}/W_{1,2}$ *and* $W_{2,3}/W_{1,2}$ *are given by* (3.24)–(3.25) *and* $W_{i,4}$ *are given by* (3.6).

(2) *Suppose that* $v_3, v_4 \in \partial D$ *and* $v_1, v_2 \in D$. Then $A(1,2,3) = A(1,2,4) = 0$, or equivalently $\langle 1,2\rangle + \langle 2,3\rangle + \langle 3,1\rangle = \langle 1,2\rangle + \langle 2,4\rangle + \langle 4,1\rangle = \pi$. One can choose as independent variables $\langle 1,3\rangle, \langle 2,3\rangle, \langle 1,4\rangle$ and $\langle 3,4\rangle$, so that

$$(3.27)\quad 2dV_3(\widehat{\Delta}) = d\langle 1,3\rangle \log(W_{2,4}W_{1,3}/W_{1,2}) + d\langle 2,3\rangle \log(W_{2,4}W_{2,3}/W_{1,2}) + d\langle 1,4\rangle \log(W_{1,4}/W_{2,4}) + d\langle 3,4\rangle \log(W_{3,4}).$$

We must express each coefficient on the right hand side as a function of $a_{j,k}$, $1 \le j,k \le 4$. As functions of μ,

$$(3.28)\quad \text{C.T.}\left\{\frac{A(1,2)}{A(0,1,2)}\right\}^{-\mu/2} V_1(\widehat{\Delta}_{1,2}|\mu) - 2\log 2 = -\log\frac{A(1,2)}{A(0,1,2)} = 0,$$

i.e. $W_{1,2} = 1$, since it is assumed that $a_{0,1} = a_{0,2} = 0$. As for $W_{1,3}, W_{2,3}$, Lemma 3.2 is valid. For $W_{1,4}$ and $W_{2,4}$, similarly,

$$(3.29)\quad W_{1,4} = a_{0,4}^2 \frac{A(1,2)A(1,4,3)}{A(1,4)A(1,2,3,4)},$$

$$(3.30)\quad W_{2,4} = a_{0,4}^2 \frac{A(1,2)A(2,4,3)}{A(2,4)A(1,2,3,4)}.$$

On the other hand, $W_{3,4}$ equals (3.6). (3.10) reduces to $2a_{0,3}a_{0,4}A\binom{1,2,3}{1,2,4} =$

$-A(1,2,3,4)$. Hence

$$W_{1,3}W_{2,4}/W_{1,2} = -\frac{1}{4}\frac{A(1,2)A(1,3,4)A(2,3,4)}{A(2,4)A(1,3)A(1,2,3,4)}, \tag{3.31}$$

$$W_{2,4}W_{2,3}/W_{1,2} = -\frac{1}{4}\frac{A(1,2)A(2,3,4)^2}{A(2,4)A(2,3)A(1,2,3,4)}, \tag{3.32}$$

$$W_{1,4}/W_{2,4} = \frac{A(2,4)A(1,3,4)}{A(1,4)A(2,3,4)}, \tag{3.33}$$

$$W_{3,4} = \frac{A\binom{3,4,1}{3,4,2} - \sqrt{-A(3,4)A(1,2,3,4)}}{\binom{3,4,1}{3,4,2} + \sqrt{-A(3,4)A(1,2,3,4)}}, \tag{3.34}$$

since $A\binom{1,2,3}{1,2,4}^2 = -A(1,2)A(1,2,3,4)$.

(3) *We assume that* $v_2, v_3, v_4 \in \partial D$, *and* $v_1 \in D$. Then $A(1,2,3) = A(1,2,4) = A(1,3,4) = 0$, or equivalently $\langle 1,2\rangle + \langle 2,3\rangle + \langle 3,1\rangle = \langle 1,2\rangle + \langle 2,4\rangle + \langle 4,1\rangle = \langle 1,3\rangle + \langle 3,4\rangle + \langle 4,1\rangle = \pi$. One can choose as independent variables $\langle 1,2\rangle, \langle 1,3\rangle$ and $\langle 1,4\rangle$. (3.7) reduces to

$$\begin{aligned} 2dV_3(\widehat{\Delta}) &= d\langle 1,2\rangle \log(W_{1,2}/(W_{2,3}W_{2,4})) \\ &\quad + d\langle 1,3\rangle \log(W_{1,3}/(W_{2,3}W_{3,4})) + d\langle 1,4\rangle \log(W_{1,4}/(W_{2,4}W_{3,4})). \end{aligned} \tag{3.35}$$

By using the relation $2a_{0,3}a_{0,4} = -A(1,2,3,4)/A\binom{1,2,3}{1,2,4}$, (3.16) and (3.30), we deduce (3.36) below. (3.37) and (3.38) are obtained by symmetry.

$$W_{1,2}/(W_{2,3}W_{2,4}) = -4\frac{A(2,3)A(2,4)A(1,2,3,4)}{A(1,2)A(2,3,4)^2}, \tag{3.36}$$

$$W_{1,3}/(W_{2,3}W_{3,4}) = -4\frac{A(2,3)A(3,4)A(1,2,3,4)}{A(1,3)A(2,3,4)^2}, \tag{3.37}$$

$$W_{1,4}/(W_{3,4}W_{2,4}) = -4\frac{A(3,4)A(2,4)A(1,2,3,4)}{A(1,4)A(2,3,4)^2}. \tag{3.38}$$

(4) *Case where all the vertices* $v_1, v_2, v_3, v_4 \in \partial D$. Then $A(i,j,k)$ vanishes for any i,j,k, or equivalently $\langle i,j\rangle + \langle j,k\rangle + \langle k,i\rangle = \pi$. One can choose the vertices as $v_1 = (\xi_1, \xi_2, \xi_3)$, $v_2 = (0,1,0)$, $v_3 = (0,0,-1)$, and $v_4 = (0,0,1)$ respectively. The point (ξ_1,ξ_2,ξ_3) in the unit sphere is related to the complex number $z = x+iy$ by stereographic projection:

$$\xi_1 = \frac{2y}{1+|z|^2}, \qquad \xi_2 = \frac{2x}{1+|z|^2}, \qquad \xi_3 = \frac{1-|z|^2}{1+|z|^2}. \tag{3.39}$$

Then from (2.2) and (2.5),

$$dV_3(\widehat{\Delta}) = \sum_{1\le i<j\le 4} d\langle i,j\rangle W_{i,j}, \tag{3.40}$$

where $W_{i,j}$ equals $A(0,i,j)/A(i,j)$. Actually $W_{1,2} = \frac{1}{2}W_{1,3} = \frac{1}{2}W_{1,4} = 1$, $W_{2,3} = 1+|z|^2$, $W_{2,4} = (1+|z|^2)/|z|^2$ and $W_{3,4} = 2(1+|z|^2)/|1-z|^2$. Moreover, $\langle 1,2\rangle =$

$\arg z$, $\langle 2,3\rangle = \arg \bar z(z-1)$ and $\langle 3,1\rangle = \arg(1-\bar z)$. (3.4) becomes

$$dV_3(\widehat{\Delta}) = 2(\log|z|\, d\arg(z-1) - \log|z-1|\, d\arg z)\,, \tag{3.41}$$

i.e. $V_3(\widehat{\Delta})$ is the Bloch–Wigner function represented by

$$V_3(\widehat{\Delta}) = \frac{1}{i}\left\{\operatorname{dilog} z - \operatorname{dilog}\bar z + \log|z|\log\frac{1-z}{1-\bar z}\right\}. \tag{3.42}$$

This function and its polylogarithmic extension have been investigated by many authors (see [M1], [M2], [G2], [W], [Z]).

Summarizing all the results in Sections 1 and 3, we have

THEOREM. *For $v_1,\ldots,v_{n+1}\in\partial D\cup D, V_n(\widehat{\Delta})$ has a hyperlogarithmic (higher logarithmic) expansion*:

$$V_n(\widehat{\Delta}) = \sum_{\emptyset\subset I_1\subset\ldots\subset I_{\nu-2}} \int_*^A \theta\binom{\emptyset}{I_1}\theta\binom{I_1}{I_2}\ldots\theta\binom{I_{\nu-3}}{I_{\nu-2}}V_3(\widehat{\Delta}_{I_{\nu-2}}) \tag{3.43}$$

for $n=2\nu-1$, and

$$V_n(\widehat{\Delta}) = \sum_{\emptyset\subset I_1\subset\ldots\subset I_{\nu-1}} \int_*^A \theta\binom{\emptyset}{I_1}\theta\binom{I_1}{I_2}\ldots\theta\binom{I_{\nu-2}}{I_{\nu-1}}V_2(\widehat{\Delta}_{I_{\nu-1}}) \tag{3.44}$$

for $n=2\nu$, where $V_3(\widehat{\Delta}_J)$ and $V_2(\widehat{\Delta}_J)$ are given by (3.26), (3.27), (3.35), (3.41) *respectively. $V_2(\widehat{\Delta}_J)$ is given by* (1.8). *$I_r=\{i_1,\ldots,i_r\}$ denotes a subset of $\{1,2,\ldots,n+1\}$.*

4. Appendix. Appell's hypergeometric integrals of type F_4 and the hyperbolic volume. The integral

$$\begin{aligned} J(\lambda) &= \int_\Delta \Phi f_1^{\lambda_1-1}f_2^{\lambda_2-1}f_3^{\lambda_3-1}f_4^{\lambda_4-1}\,dt_0\wedge dt_1\wedge dt_2\wedge dt_3 \\ &= \frac{1}{\sqrt{-A(1,2,3,4)}}\int_{y_1\ge0,y_2\ge0,y_3\ge0,y_4\ge0}\exp[-\tfrac12\,{}^t yBy] \\ &\qquad\qquad \times y_1^{\lambda_1-1}y_2^{\lambda_2-1}y_3^{\lambda_3-1}y_4^{\lambda_4-1}\,dy_1\wedge dy_2\wedge dy_3\wedge dy_4 \\ &= \frac12\frac{1}{\sqrt{-A(1,2,3,4)}}\Gamma\left(\frac{\lambda_1-\lambda_2+\lambda_3+\lambda_4}{2}\right)\Gamma(\lambda_3) \\ &\quad \times\int_{\eta_1\ge0,\eta_2\ge0}\eta_1^{\lambda_1-1}\eta_2^{\lambda_3-1}(b_{4,2}+b_{1,2}\eta_1+b_{2,3}\eta_2)^{-\lambda_2} \\ &\qquad\qquad \times(b_{4,1}\eta_1+b_{4,3}\eta_2+b_{1,3}\eta_1\eta_2)^{-\lambda_4'}\,d\eta_1\wedge d\eta_2 \end{aligned} \tag{4.1}$$

with $2\lambda_4'=\lambda_1+\lambda_2+\lambda_4-\lambda_3$, where $B=((b_{r,s}))_{1\le r,s\le4}$ denotes the inverse A^{-1}. By the definition we have the homogeneity

$$J(\lambda|\{b_{r,s}\varrho_r\varrho_s\}) = \varrho_1^{-\lambda_1}\varrho_2^{-\lambda_2}\varrho_3^{-\lambda_3}\varrho_4^{-\lambda_4}J(\lambda|\{b_{r,s}\})\,, \tag{4.2}$$

for$\varrho_j \in \mathbb{C}^*$. One can choose ϱ_r such that $\varrho_1\varrho_2 b_{1,3} = -1$, $\varrho_1\varrho_4 b_{1,4} = \varrho_2\varrho_4 b_{2,4} = \varrho_3\varrho_4 b_{3,4} = 1$. For $b_{1,3} = -1$, $b_{1,4} = b_{2,4} = b_{3,4} = 1$, $J(\lambda|\{b_{r,s}\})$ has an integral expression similar to Appell's hypergeometric function of type F_4 (see [K1]):

$$F_4(\alpha,\beta,\gamma,\gamma' \mid u,v) = \sum_{l\geq 0, m\geq 0} \frac{(\alpha)_{l+m}(\beta)_{l+m}}{(\gamma)_l(\gamma')_m}\frac{u^l v^m}{l!m!} \tag{4.3}$$

for $u = -b_{1,2}$, $v = -b_{2,3}$, $\alpha = \lambda_2$, $\beta = \lambda_3$, $\gamma = 1 + (\lambda_3 + \lambda_1 - \lambda_2 - \lambda_4)/2$ and $\gamma' = \lambda_3 - \lambda_1 + 1$ respectively. They both satisfy the following holonomic system of partial differential equations $(\mathcal{E})$ (see [K3], Chap. XI):

$$\begin{aligned} &u(1-u)R - v^2T - 2uvS \\ &\qquad + \{\gamma - (\alpha+\beta+1)u\}P - (\alpha+\beta+1)vQ - \alpha\beta J = 0\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} &v(1-v)T - u^2R - 2uvS \\ &\qquad + \{\gamma' - (\alpha+\beta+1)v\}Q - (\alpha+\beta+1)uP - \alpha\beta J = 0 \end{aligned} \tag{4.5}$$

for $R = \partial^2 J/\partial u^2$, $S = \partial^2 J/\partial u\partial v$, $T = \partial^2 J/\partial v^2$, $P = \partial J/\partial u$ and $Q = \partial J/\partial v$. The change of variables

$$u = w_1 w_2\,, \qquad v = (1-w_1)(1-w_2)\,, \tag{4.6}$$

which we call the *Burchnall–Chaundy transformation* or simply *B.C. transformation* has an integral representation associated with a line configuration (see [B2], and [K2] for an extension):

$$\begin{aligned} F_4(\alpha,\beta,\gamma,\gamma' \mid w_1w_2', w_1'w_2) &= \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma'-\beta)} \\ \times \int_0^1\int_0^1 x^{\alpha-1}y^{\beta-1}&(1-x)^{\gamma-\alpha-1}(1-y)^{\gamma'-\beta-1}(1-xw_1)^{\alpha-\gamma-\gamma'}(1-yw_2)^{\beta-\gamma-\gamma'} \\ &\times(1-w_1x-w_2y)^{\gamma+\gamma'-\alpha-\beta-1}\,dx\wedge dy\,, \end{aligned} \tag{4.7}$$

where we put $w_1' = 1 - w_1$ and $w_2' = 1 - w_2$. However, we do not know whether $J(\lambda)$ itself is given by a similar representation through the B.C. transformation.

The holonomic system $(\mathcal{E})$ has an alternative expression, i.e., the Gauss–Manin connection by using the additional integrals $\widetilde{\varphi}(i,j)$ and $\widetilde{\varphi}(1,2,3,4)$. Indeed, we put

$$\widetilde{\varphi}(i,j) = \int \Phi\frac{d\tau}{f_i f_j}\,, \tag{4.8}$$

$$\widetilde{\varphi}(1,2,3,4) = \int \Phi\frac{d\tau}{f_1f_2f_3f_4}\,. \tag{4.9}$$

Then as functions of the variables $(a_{i,j}))_{1\leq i,j\leq 4}$, $\widetilde{\varphi}(\emptyset)$, $\widetilde{\varphi}(i,j)$, $\widetilde{\varphi}(1,2,3,4)$ satisfy a variational formula in closed form (Gauss–Manin connection $(\mathcal{E}')$) (see [A3], Proposition 2.4_p):

$$d\widetilde{\varphi}(\emptyset)=\frac{1}{2}\sum_{i\neq j}d\langle i,j\rangle\lambda_i\lambda_j\widetilde{\varphi}(i,j)\,, \tag{4.10}$$

$$\begin{aligned}&A(i,j)d\widetilde{\varphi}(i,j)\\&\quad=dA\binom{k,i,j}{l,i,j}\lambda_k\lambda_l\,\varphi(1,2,3,4)+da_{i,j}\widetilde{\varphi}(\emptyset)\\&\qquad+\lambda_k\Big\{-dA\binom{i,j}{k,j}\widetilde{\varphi}(k,j)+dA\binom{i,j}{k,i}\widetilde{\varphi}(k,i)\Big\}\\&\qquad+\lambda_l\Big\{-dA\binom{i,j}{l,j}\widetilde{\varphi}(l,j)+dA\binom{i,j}{l,i}\widetilde{\varphi}(l,i)\Big\}\,,\end{aligned} \tag{4.11}$$

$$\begin{aligned}A(1,2,3,4)d\widetilde{\varphi}\,(1,2,3,4)&=\frac{1}{2}\sum_{i\neq j}(-1)^{i+j}dA\binom{k,i,j}{l,i,j}\widetilde{\varphi}(i,j)\\&\quad+\frac{1}{2}dA(1,2,3,4)\{-1+\lambda_1+\lambda_2+\lambda_3+\lambda_4\}\widetilde{\varphi}(1,2,3,4)\,,\end{aligned} \tag{4.12}$$

with the fundamental relations

$$0=\lambda_j\widetilde{\varphi}(1,2,3,4)-\sum_{k=1,k\neq j}^{4}b_{k,j}\widetilde{\varphi}(j,k)\,, \tag{4.13}$$

for each j, $1\le j\le 4$. Hence $\widetilde{\varphi}(1,2,3,4)$, $\widetilde{\varphi}(1,4)$, $\widetilde{\varphi}(2,4)$ and $\widetilde{\varphi}(3,4)$ are expressed by linear combinations of $\widetilde{\varphi}(1,2)$, $\widetilde{\varphi}(2,3)$ and $\widetilde{\varphi}(3,1)$:

$$\begin{aligned}&2\lambda_4 b_{2,3}\widetilde{\varphi}(1,4)\\&\quad=(\lambda_2+\lambda_3+\lambda_4-1)b_{1,4}\widetilde{\varphi}(2,3)+(\lambda_2+\lambda_3-\lambda_1-\lambda_4)\widetilde{\varphi}(1,3)\\&\qquad+(\lambda_2+\lambda_3-\lambda_1-\lambda_4)b_{3,4}\widetilde{\varphi}(1,2)\,,\qquad\text{etc.}\end{aligned} \tag{4.14}$$

The volume $V_3(\widehat{\Delta})$ given by the formula

$$y\int_{\eta_1\ge 0,\eta_2\ge 0}(1+b_{2,3}\eta_1+b_{2,1}\eta_2)^{-1}(\eta_1+\eta_2-\eta_1\eta_2)^{-1}d\eta_1\wedge d\eta_2 \tag{4.15}$$

is a special case of the hypergeometric integrals of Appell's type F_4 for $\alpha=\beta=\gamma=\gamma'=1$. The equations $(\mathcal{E}')$ reduce to (3.41).

The B.C. transformation gives

$$w,\overline{w}=\frac{1+b_{2,3}-b_{1,2}\pm\sqrt{B(1,2,3,4)}}{2} \tag{4.16}$$

for $b_{i,i}=0$, $b_{1,3}=-1$, $b_{1,4}=b_{2,4}=b_{3,4}=1$ and

$$b_{1,2}=-\frac{1-\xi_2}{2(1+\xi_3)}\,,\qquad b_{2,3}=-\frac{1-\xi_3}{2(1+\xi_3)}\,.$$

$B(1,2,3,4)$ equals $1+b_{1,2}^2+b_{2,3}^2+2b_{2,3}+2b_{1,2}-2b_{1,2}b_{2,3}=-\xi_1^2/(1+\xi_3)^2=y^2$.

On the other hand,

$$z = \frac{\xi_2 + i\xi_1}{1+\xi_3} = 1 - \frac{1}{w}\,. \tag{4.17}$$

Hence *the B.C. transformation*

$$w\overline{w} = -b_{1,2}, \qquad (1-w)(1-\overline{w}) = -b_{2,3} \tag{4.18}$$

is the composite of the linear fractional transformation (4.17) *and the correspondence* (3.39) *between the configuration matrix* B *and the point* $z \in \mathbb{C}$ *which represents the vertex* v_1.

References

[A1] K. Aomoto, *Fonctions hyper-logarithmiques et groupes de monodromie unipotents*, J. Fac. Sci. Univ. Tokyo 25 (1973), 149–156.

[A2] —, *On structure of integrals of power product linear functions*, Sci. Papers Col. Gen. Edu. Univ. Tokyo 27 (1977), 49–61.

[A3] —, *Configurations and invariant Gauss–Manin connections of integrals I*, Tokyo J. Math. 5 (1982), 249–287; *II*, ibid. 6 (1983), 1–24.

[A4] —, *Vanishing of certain* 1*-form attached to a configuration*, ibid. 9 (1986), 453–455.

[A5] —, *Analytic structure of Schläfli function*, Nagoya Math. J. 68 (1977), 1–16.

[B1] A. A. Beilinson, A. Varchenko and V. Schechtman, *Projective geometry and K-theory*, Algebra and Anal. 2 (1990), 78–130.

[B2] J. L. Burchnall and T. W. Chaundy, *Expansions of Appell's double hypergeometric functions*, Quart. J. Math. 11 (1940), 249–270.

[G1] I. M. Gelfand and A. V. Zelevinski, *Algebraic and combinatorial aspect of general theory of hypergeometric functions*, Funct. Anal. Appl. 20 (1986), 17–34.

[G2] A. B. Goncharov, *The classical trilogarithm, algebraic K-theory of fields and Dedekind zeta functions*, preprint, 1990.

[H1] R. Hain, *The geometry of the mixed Hodge structure on the fundamental group*, in: Proc. Sympos. Pure Math. 46, Amer. Math. Soc., 1987, 247–282.

[H2] R. Hain and R. MacPherson, *Higher logarithms*, Illinois J. Math. 34 (1990), 392–475.

[K1] J. Kaneko, *Monodromy group of Appell's system* (F_4), Tokyo J. Math. 4 (1981), 35–54.

[K2] —, *Selberg integrals and hypergeometric functions*, preprint, 1990.

[K3] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Chap. XI, Springer, 1990.

[K4] R. Kellerhals, *On the volume of hyperbolic polyhedra*, Math. Ann. 285 (1989), 541–569.

[K5] —, *The dilogarithm and volumes of hyperbolic polytopes*, preprint.

[M1] J. Milnor, *Hyperbolic geometry. The first* 150 *years*, Bull. Amer. Math. Soc. 6 (1982), 9–24.

[M2] —, *On polylogarithms, Hurwitz zeta functions, and the Kubert identities*, Enseign. Math. 29 (1983), 281–322.

[V] A. Varchenko, *Multidimensional hypergeometric functions and their appearance in conformal field theory. Algebraic K-theory, algebraic geometry, etc.*, congress talk in Kyoto, 1990.

[W] Z. Wojtkowiak, *A note on functional equations of polylogarithms*, preprint, 1990.

[Z] D. Zagier, *Hyperbolic manifolds and special values of Dedekind zeta functions*, Invent. Math. 83 (1986), 285–301.