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FOLIATIONS AND THE GENERALIZED COMPLEX MONGE–AMPÈRE EQUATIONS

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Introduction

The generalized complex Monge–Ampère equations arise when looking for a complex analogue of the principles of Dirichlet and Thomson, including the inhomogeneity (weight) functions on the space (complex manifold) and the hermitian structure. These equations were introduced by the second named author in 1975 [10] and then studied by him, partially together with A. Andreotti [2], [3]. A considerable part of the results included in this paper is due to the first named author. His Theorem 3 ensures the existence of a foliation corresponding to a generalized complex Monge–Ampère equation, and even to a more general equation. This enables him to obtain a weak maximum principle and some corollaries.

We begin with the formulation of well known Dirichlet's and Thomson's principles. Then we introduce on hermitian manifolds the capacities due to the second named author [10], [11], [12], give their basic properties and explain their connection with the generalized complex Monge–Ampère equations. Before formulating Theorems 3 and 4 we give some preliminaries on foliations.

1. The principles of Dirichlet and Thomson (the case of R^2)

We begin with the formulation of Dirichlet's and Thomson's principles (cf. [16] and [14]).

DIRICHLET'S PRINCIPLE. The energy of a constant electric field in a smooth condenser (D, γ_0, γ_1) has the minimal value among the energies of all irrotational fields $\tilde{E} \in \mathscr{E}$, $\mathscr{E}$ being the class of all functions of the form $\tilde{E} = -\operatorname{grad}\tilde{V}$, $\tilde{V} \in \mathscr{V}$, and $\mathscr{V}$ consisting of all $\tilde{V} \in C^2(\operatorname{cl} D)$, such that $\tilde{V}|\gamma_0 = V_0$, $\tilde{V}|\gamma_1 = V_1$, and the normal derivative of $\tilde{V}$ along $D\setminus\gamma_0\setminus\gamma_1$ vanishes. In other words, we have

$$W = \tfrac{1}{2}\inf_{\tilde{E}\in\mathscr{E}} \iint_D \varepsilon_0\, \varepsilon \tilde{E}^2 dx\, dy,$$

where $\tilde{E}^2 = \tilde{E}\cdot\tilde{E}$ and $\varepsilon' = \varepsilon_0\varepsilon$ denotes the electric permeability.

THOMSON'S PRINCIPLE. The energy of a constant electric field in a smooth condenser (D, γ_0, γ_1) has the minimal value among the energies of all source-free (solenoidal) fields $\tilde{\boldsymbol{D}} \in \mathscr{D}$ with the energy density $\frac{1}{2}\tilde{\boldsymbol{E}} \cdot \tilde{\boldsymbol{D}} = \frac{1}{2}(1/\varepsilon_0\varepsilon)\tilde{\boldsymbol{D}} \cdot \tilde{\boldsymbol{D}}$, $\mathscr{D}$ being the class of all functions $\tilde{\boldsymbol{D}} \in [C^2(\mathrm{cl}D)]^2$, whose integrals along the oriented curves γ_0 and γ_1 are equal Q (the electric charge). In other words, we have

$$W = \tfrac{1}{2}\inf_{\tilde{D}\in\mathscr{D}} \iint_D \frac{1}{\varepsilon_0\,\varepsilon}\tilde{D}^2 dx\,dy.$$

Thus the Dirichlet principle is connected with a variation of the intensity of an electric field, whereas the Thomson principle — with a variation of the vector of electric induction. The principles of Dirichlet and Thomson may be reformulated in the terms of capacity:

$$\mathrm{cap}(D, \varepsilon') = \frac{\varepsilon_0}{(V_1 - V_0)^2} \inf_{\tilde{E}\in\mathscr{E}} \iint_D \varepsilon \tilde{E}^2 dx\,dy$$

(*Dirichlet's principle*),

$$\mathrm{cap}(D, \varepsilon') = \frac{Q^2}{\varepsilon_0^{-1}} \sup_{\tilde{D}\in\mathscr{D}} \left[1 \Big/ \iint_D \frac{1}{\varepsilon}\tilde{D}^2 dx\,dy\right]$$

(*Thomson's principle*).

The inf and 1/sup in the above formulae are essentially equal to an extremal length of Ahlfors and Beurling [1].

2. Capacities on hermitian manifolds

Let M be a complex manifold of complex dimension n endowed with an hermitian metric h and a C^1 tensor field H of type $(1, 1)$. In particular we may let H depend on h or take as H an almost complex structure of the tangent bundle TM, for instance the complex structure of M. Let further D be a *condenser* on M, i.e. a domain whose complement consists of two distinguished disjoint closed sets C_0 and C_1 (the *condenser plates*), $q\colon M \to C$ a continuous mapping (the *inhomogeneity function*), and p a real number $\geqslant 1$. Consider the class $\mathrm{Adm}\,D$ of all plurisubharmonic C^2-functions u on $\mathrm{cl}D$, satisfying the conditions $0 < u(z) < 1$ for $z \in D$, $u|\partial C_0 = 0$ and $u|\partial C_1 = 1$. Let

(1) $$\mathrm{Cap}_p(D, q) = \inf_{\tilde{u}\in\mathrm{Adm}D} \left|\int_D q[h(d^c\tilde{u}, d^c\tilde{u})]^{\frac{1}{2}p-1} \det H\, d\tilde{u} \wedge d^c\tilde{u} \wedge (dd^c\tilde{u})^{n-1}\right|,$$

where $h(d^c u, d^c u) = h^{j\bar{k}} u_{:j} u_{:\bar{k}}$, $u_{:j} = (u \circ \mu^{-1})_{|j} \circ \mu$ in any local coordinate system $\mu = (\mu^j)$ on M.

Let further Γ be a homology class of D with real coefficients and $\dim\Gamma = r$. Consider all currents of Γ (more precisely: corresponding to the elements of Γ) in the sense of de Rham, and a locally finite open covering $\mathscr{U} = \{U_j\colon j \in I\}$ of M. Denote by $\mathrm{adm}(D, \mathscr{U})$ the family of all plurisubharmonic C^2-functions u_j on $U_j \cap D$ defined in each member of the covering which satisfy the following conditions:

(i) the oscillation of u_j in $U_j \cap D$ is less than one,

(ii) $du_j = du_k$ in $U_j \cap U_k \cap D \neq \emptyset$.

Condition (ii) describes a closed real one-form in D. Similarly $d^c u_j$ and $dd^c u_j$ are also well-defined in D. Without ambiguity, we can denote them omitting the indices. Let [10]:

(2) $$\mathrm{cap}_p(D, q, \Gamma, \mathscr{U}) = \sup_{u\in\mathrm{adm}(D,\mathscr{U})} \inf_{T\in\Gamma} |T[q\{h(d^c u, d^c u)\}^{\frac{1}{2}p-1} \det H\, D^r u]|,$$

where

$$D^r u = \begin{cases} d^c u \wedge (dd^c u)^{\frac{1}{2}r-2} & \text{for } r \text{ odd}, \\ du \wedge d^c u \wedge (dd^c u)^{\frac{1}{2}r-1} & \text{for } r \text{ even}. \end{cases}$$

For a detailed description of the capacities (1) and (2) we refer to [12], and for an example of their application to [13].

3. The generalized complex Monge–Ampère equations

When looking for a complex analogue of the principle of Dirichlet a natural procedure is to take in (2) for Γ the $(2n-1)$-dimensional homology class of level hypersurfaces $\{z \in \mathrm{cl}D\colon u(z) = \mathrm{const}\}$. (In analogy, for a complex counterpart of the principle of Thomson we had to take in (2) for Γ the orthogonal 1-dimensional homology class.) One should expect that under some reasonable conditions, in particular if we take in (2) for admissible functions only u defined globally with $0 < u(z) < 1$ for $z \in D$ (we write $u \in \mathrm{adm}\,D$), both capacities (2) and (1) will coincide.

The above idea as well as both definitions in the case where $H = J$ (the complex structure of M), $p = 2$, and $q = \mathrm{const}$ is due to Chern, Levine and Nirenberg [7], but the affirmative answer is known only in very special subcases [7], [9], [5]. In the case mentioned the functional minimized attains its minimum for $\tilde{u} = u$ if and only if u fulfils the *complex Monge–Ampère equation*

(3) $$(dd^c u)^n = 0.$$

Since $(dd^c u)^n = 4^n n! \det[(\partial^2/\partial z_j \partial \bar{z}_k)u]\,(2i)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$, equation (3) is the complex analogue of the *real Monge–Ampère equation*

(4) $$\det[(\partial^2/\partial x_j \partial x_k)u] = 0.$$

Equation (3) is a special case of the *generalized complex Monge–Ampère equations*

(5) $$dd^c(Fu) \wedge (dd^c u)^{n-1} = 0, \quad F \in C^2(\mathrm{cl}D)$$

or

(6) $$d(Gd^c u) \wedge (dd^c u)^{n-1} = 0, \quad G \in C^1(\mathrm{cl}D)$$

which play an analogous role for the general capacity (1) with

(7) $$d^c(Fu) = Gd^c u, \quad G = q[h(d^c u, d^c u)]^{\frac{1}{2}p-1} \det H.$$

In the general case the function u is replaced by a system satisfying the condition (ii). We quote the following two theorems due to Andreotti and Ławrynowicz [2], [3]:

THEOREM 1. *Suppose that*:

(a) D *has a piecewise* C^1*-smooth boundary and compact closure*,

(b) $n \geqslant 2$, $p = 2$, *and* q *is of the class* C^1,

(c) u *belongs to* $\operatorname{adm} D$ *and satisfies* (6), *where* $Q = q \det H$,

(d) $d(Gd^c u) = f dd^c u$, $f \geqslant (n-1)^{-1} G$, f *being of the class* C^1.

Then the infimum in (1) *is attained for the* u *in question.*

Remark 1. Theorem 1 remains valid for $n = 1$. In this case the condition (d) is superfluous.

THEOREM 2. *Suppose that* (a) *and* (b) *hold and that*:

(c′) *the infimum in* (1) *is attained for some* u,

(d′) $d(Gd^c u) = f dd^c u$, $G = q \det H$, f *being continuous*.

Then u *satisfies* (7).

Remark 2. Theorem 2 remains valid for $n = 1$. In this case the condition (d′) is superfluous.

4. Foliations

We are going to give some preliminaries on foliations. Here we refer to [6] and [15].

By a p-dimensional C^r-*foliation* of an m-dimensional C^r-differentiable manifold M we mean a decomposition of M into a union of disjoint connected subsets $\{L_j : j \in I\}$ (I always uncountable) called the *leaves* of the foliation, with the following property: Every point of M has a neighbourhood U and a system of local C^r-differentiable coordinates $x = (x^1, \ldots, x^m)$: $U \to \boldsymbol{R}^m$ such that for each leaf L_j the components of $U \cap L_j$ are described by the equations $x^{p+1} = \text{const}, \ldots, x^m = \text{const}$.

Foliations arise naturally in various situations in mathematics and it would be instructive to give some examples.

EXAMPLE 1: *Submersions.* Let M and N be C^r-differentiable manifolds of dimension m and n, $m \geqslant n$, respectively, and let f: $M \to N$ be a submersion, that is, suppose that $\operatorname{rank}(df) \equiv n$. It follows from the Implicit Function Theorem that f induces on M a C^r-foliation of codimension n whose leaves are defined as the components of $f^{-1}(\{y\})$ for $y \in N$. Also differentiable fibre bundles are examples of this sort.

EXAMPLE 2: *Subbundles of the tangent bundle of a given* C^r*-differentiable manifold* M. We say that a smooth subbundle $E \subset TM$ is *integrable* if and only if for any two smooth sections (resp. vector fields) X and Y of E the section (resp. vector field) $[X, Y]$ is also a section (resp. vector field) of E. By the Frobenius theorem the set of all maximal *integrals* E_0 of E (i.e. submanifolds of M such that $T_x E_0$ is contained in the fibre over x of the subbundle E for every $x \in E_0$) forms a foliation of M.

A foliation always appears as the family of solutions for some nonsingular systems of differential equations (cf. Example 2). The study of foliations consists of studying the global behaviour of solutions. For instance a nonsingular system of ordinary differential equations, when reduced to the first order system, becomes a non-vanishing vector field. The local solutions (orbits of the local flow generated by the vector field) form together a 1-dimensional foliation.

One can consider analogously ordinary differential equations in the complex case (where dependence on the variables is holomorphic). One obtains non-singular holomorphic vector fields and corresponding foliations by complex curves.

Let now M be a $2n$-dimensional C^∞-differentiable manifold and let TM be its tangent bundle. Let J denote an almost complex structure on M. The spaces $TM^{1,0}$ and $TM^{0,1}$ may be defined by the splitting $TM \otimes_{\boldsymbol{R}} \boldsymbol{C} \simeq TM^{1,0} \oplus TM^{0,1}$, where $\alpha = \frac{1}{2}(\alpha - iJ\alpha) \oplus \frac{1}{2}(\alpha + iJ\alpha)$. Hence also $T^*M \otimes_{\boldsymbol{R}} \boldsymbol{C} \simeq T^*M^{1,0} \oplus T^*M^{0,1}$. Under this splitting, $d = \partial + \bar{\partial}$. We shall use the notation $\partial_j = \partial/\partial z_j$ and $\bar{\partial}_j = \partial/\partial \bar{z}_j$. We denote by

$$\Lambda^{p,q} M = \Lambda^p(T^*M^{1,0}) \wedge \Lambda^q(T^*M^{0,1})$$

the spaces of forms of type (p, q) on M, and the spaces of k-forms on M are given by

$$\Lambda^k M = \bigoplus_{p+q=k} \Lambda^{p,q} M.$$

We also extend J^* (the adjoint of J) to

$$\Lambda M = \bigoplus_k \Lambda^k M$$

by the rule $J^* f = f$ if f is a 0-form and, in general, by $J^*(\xi \wedge \eta) = J^*\xi \wedge J^*\eta$. If $X \in TM \otimes_{\boldsymbol{R}} \boldsymbol{C}$ is any tangent vector, then $\lrcorner X$: $\Lambda^k \to \Lambda^{k-1}$ is the *contraction* by X defined by

$$(\omega \lrcorner X)(Y_1, \ldots, Y_{k-1}) = \omega(X, Y_1, \ldots, Y_{k-1}),$$

where $\omega \in \Lambda^k M$ and $\lrcorner$ stands for the inner product. If $\mathscr{F} \in \Lambda M$ is an ideal, then

$$\operatorname{Ann} \mathscr{F} = \{X \in TM : \omega \lrcorner X \in \mathscr{F} \text{ for all } \omega \in \mathscr{F}\}.$$

The following lemma can easily be established [4]:

LEMMA 1. *Let* $\mathscr{F} = (\omega_1, \ldots, \omega_k)$ *be the ideal of* ΛM *generated by* q*-forms* $\omega_1, \ldots, \omega_k$. *If* $d\mathscr{F} \subset \mathscr{F}$, *then* $\operatorname{Ann} \mathscr{F}$ *is involutive, i.e. for* $X, Y \in \operatorname{Ann} \mathscr{F}$ *we have* $[X, Y] \in \operatorname{Ann} \mathscr{F}$.

Proof. The assertion easily follows from the identities

$$\lrcorner [X, Y] = [L_X, \lrcorner Y] \quad \text{and} \quad L_X = d \circ \lrcorner X + \lrcorner X \circ d.$$

We also have [4]:

LEMMA 2. *If* $\mathscr{F} = (\omega_1, \ldots, \omega_k)$ *is an ideal of* ΛD, $D \subset \boldsymbol{C}^n$, *generated by real* (1, 1)*-forms, then* $\operatorname{Ann} \mathscr{F}$ *is* J*-invariant.*

Proof. The conclusion follows from the observation that every real (1, 1)-form ω may be diagonalized into the form $\omega = \pm b_1 \wedge Jb_1 \pm \ldots \pm b_{n'} \wedge Jb_{n'}$, where $b_1, \ldots \ldots, b_{n'}$ are real 1-forms.

The following lemma is basic for our considerations:

LEMMA 3. *Let* ω_1 *and* ω_2 *be real* (1, 1)*-forms with* $\omega_2 \geqslant 0$, *and let there exist* $a_j, b_j \in T^*D$, $D \subset C^n$, *such that* $\omega_1 = \sum \pm a_j \wedge Ja_j$, $\omega_2 = -\sum b_j \wedge Jb_j$. *Then* $\dim \operatorname{span}\{a_j, Ja_j, b_j, Jb_j\} = 2p$ *if and only if* $(\omega_1 + i\omega_2)^p \neq 0$ *and* $(\omega_1 + i\omega_2)^{p+1} = 0$.

For the proof we refer to [4].

Remark 3. There are situations in which the condition $\omega_2 \geqslant 0$ is unnecessary. For instance, if ω_1 and ω_2 can be simultaneously diagonalized, then the conclusion remains valid.

5. Foliations and certain equations involving the complex hessian

Using the foliation technique we are going to study certain nonlinear partial differential equations involving the complex hessian $[(\partial^2/\partial z_j \partial \bar{z}_k)u]$, which can be specified as (5) or (6). With the help of Lemma 3 we can obtain a generalization of Theorem 5.1 in [4]:

THEOREM 3. *Suppose that* $u\colon D \to C$, D *being a bounded domain in* C^n, *belongs to* $C^3(D)$ *and* $\operatorname{im} u$ *is plurisubharmonic in* D. *Let further* ω *be a real* (1, 1)*-form of the class* $C^3(D)$ *such that the rank of* ω *is equal to* k, $1 \leqslant k \leqslant n-1$, *at every point of* D *and such that the following conditions hold*:

(8) $$\omega^k \wedge (dd^c u)^p = 0, \quad 3 \leqslant k+p \leqslant n,$$

(9) $$\omega^k \wedge (dd^c u)^{p-1} \neq 0,$$

(10) $$d\omega \in \mathscr{F} = \operatorname{ideal}(\omega, dd^c \operatorname{re} u, dd^c \operatorname{im} u).$$

Then there exists a foliation $\mathscr{L}_{p+k-1}$ *of* D *by complex manifolds of codimension* $p+k-1$ *with the property that for every leaf* $M \in \mathscr{L}_{p+k-1}$ *the functions* $\operatorname{im} u|M$ *and* $\operatorname{re} u|M$ *are pluriharmonic, but* $\partial(\operatorname{im} u)/\partial z_j|M$ *and* $\partial(\operatorname{re} u)/\partial z_j|M$ *are holomorphic on* M *for each* j, $1 \leqslant j \leqslant n$.

Remark 4. If the function u in Theorem 3 is, in addition, continuous on $\operatorname{cl} D$, then the functions $\operatorname{re} u$, $|\partial(\operatorname{re} u)/\partial z_j|$, and $|\partial(\operatorname{im} u)/\partial z_j|$ satisfy the *weak maximum principle* in $\operatorname{cl} D$, i.e. the maximum of $\operatorname{re} u$ on $\operatorname{cl} D$ is equal to the maximum of $\operatorname{re} u$ on ∂D etc.

Proof of Theorem 3. The ideal $\mathscr{F}$ is d-closed and invariant under J. It follows from (9) that $\operatorname{Ann}\mathscr{F}$ has the complex codimension at least $k+p-1$. To show that the codimension of $\operatorname{Ann}\mathscr{F}$ is $k+p-1$, we select forms $c_{k'}, a_{k'}, b_{k'} \in T^*D$ such that

(11) $$\omega = \sum_{j=1}^{k} \pm c_j \wedge Jc_j, \quad dd^c \operatorname{re} u = \sum \pm a_{k'} \wedge Ja_{k'}, \quad dd^c \operatorname{im} u = \sum -b_{k'} \wedge Jb_{k'}.$$

We claim that the real dimension of the span of

$$\{c_j, Jc_j, a_j, Ja_j, b_j, Jb_j\} = V^*$$

is at most $2(p+k-1)$. Let us choose a J-invariant complementary subspace of $V_0^* = \operatorname{span}\{c_j, Jc_j\}$, i.e. $V^* = V_1^* \oplus V_0^*$. Thus we have

$$a_{k'} = a'_{k'} + c'_{k'}, \quad b_{k'} = b'_{k'} + c''_{k'}, \quad \text{where} \quad a'_{k'}, b'_{k'} \in V_1^* \text{ and } c'_{k'}, c''_{k'} \in V_0^*.$$

By the definition of V^*, $\operatorname{span}\{a'_{k'}, Ja'_{k'}, b'_{k'}, Jb'_{k'}\} = V^*$. Therefore

$$dd^c \operatorname{re} u = \sum \pm a'_{k'} \wedge Ja'_{k'} + \sum_{j=1}^{k} (c_j \wedge s_j^{(1)} + Jc_j \wedge s_j^{(2)}),$$

$$dd^c \operatorname{im} u = \sum -b'_{k'} \wedge Jb'_{k'} + \sum_{j=1}^{k} (c_j \wedge t_j^{(1)} + Jc_j \wedge t_j^{(2)}).$$

Now, if $\dim_R V^* \geqslant 2(p+k)$, then $\dim_R V_1^* \geqslant 2p$. Thus $\left(\sum \pm a'_{k'} \wedge Ja'_{k'} - i\sum b'_{k'} \wedge Jb'_{k'}\right)^p \neq 0$ since $\{a'_{k'}, Ja'_{k'}, b'_{k'}, Jb'_{k'}\}$ span V_1^*. But this implies that

$$\begin{aligned}
\omega^k \wedge (dd^c u)^p &= \Big(\sum_{j=1}^{k} \pm c_j \wedge Jc_j\Big)^k \wedge \Big[\sum \pm a'_{k'} \wedge Ja'_{k'} - i\sum b'_{k'} \wedge Jb'_{k'} + \\
&\qquad + \sum (c_j \wedge s_j^{(1)} + Jc_j \wedge s_j^{(2)} + ic_j \wedge t_j^{(1)} + iJc_j \wedge t_j^{(2)})\Big]^p \\
&= \pm k! \sum_{j=1}^{k} c_j \wedge Jc_j \wedge \Big(\sum \pm a'_{k'} \wedge Ja'_{k'} - i\sum b'_{k'} \wedge Jb'_{k'}\Big)^p \\
&\neq 0,
\end{aligned}$$

which contradicts (8). Since $\operatorname{Ann}\mathscr{F}$ has a constant dimension, it is integrable and this gives the foliation $\mathscr{L}_{p+k+1}$.

If we let $\iota\colon M \to D$ denote the inclusion mapping, then $\iota^* u = u|M$ and, consequently,

$$\begin{aligned}
\partial_M \bar{\partial}_M (u|M) &= \partial_M \bar{\partial}_M (\operatorname{re} u|M) + i\partial_M \bar{\partial}_M (\operatorname{im} u|M) \\
&= \iota^* \partial\bar{\partial} \operatorname{re} u + i(\iota^* \partial\bar{\partial} \operatorname{im} u) = 0
\end{aligned}$$

since $TM \subset \operatorname{Ann}(dd^c \operatorname{re} u, dd^c \operatorname{im} u)$. Finally we have to show that $(\operatorname{im} u)_{|j}|M$ and $(\operatorname{re} u)_{|j}|M$ are holomorphic. Let

$$X = \sum_{j=1}^{n} C^j \partial_j \in TM^{1,0}$$

be any vector field. Since TM, $JTM \subset \operatorname{Ann}(dd^c \operatorname{re} u, dd^c \operatorname{im} u)$, it follows that

$$(dd^c \operatorname{re} u) \lrcorner X = (dd^c \operatorname{im} u) \lrcorner X = \sum_{j=1}^{n} C^j u_{|j\bar{k}} = \sum_{j=1}^{n} C^j \operatorname{im} u_{|j\bar{k}} = 0,$$

and this proves that $(\operatorname{re} u)_{|j}|M$ and $(\operatorname{im} u)_{|j}|M$ are holomorphic indeed.

EXAMPLE 1. Suppose that F and u are real-valued C^3-smooth functions on

$D \subset C^n$ such that u is plurisubharmonic on D. Let further $\operatorname{rank}(dd^c(Fu)) = 1$ and the following conditions hold:

$$dd^c(Fu)\wedge(dd^cu)^{n-1} = 0, \quad dd^c(Fu)\wedge(dd^cu)^{n-2} \neq 0.$$

Owing to Theorem 3 we have the foliations of D by complex manifolds of codimension $n-1$.

EXAMPLE 2. Let u be a real-valued C^3-smooth plurisubharmonic function on D such that the following conditions hold:

$$(dd^cu)^{p+1} = 0, \quad (dd^cu)^p \neq 0.$$

Owing to Theorem 3 we have the foliations of D by complex manifolds of codimension p [4].

EXAMPLE 3. Let u be as in Theorem 3, satisfying the conditions

$$\partial\bar{u}\wedge\bar{\partial}u\wedge(\partial\bar{\partial}u)^p = 0, \quad \partial\bar{u}\wedge\bar{\partial}u\wedge(\partial\bar{\partial}u)^{p-1} \neq 0.$$

A simple calculation shows that the assumptions of Theorem 3 hold with $\omega = i\partial\bar{u}\wedge\wedge\bar{\partial}u$ and $k = 1$. This example was investigated in detail in [4].

THEOREM 4. *Let D be a bounded domain in C^n and u a continuous function on* $\operatorname{cl} D$ *satisfying the additional assumptions of Theorem* 3. *Then the functions* $\operatorname{re} u$, $|(\operatorname{re} u)_{|j}|$, *and* $|(\operatorname{im} u)_{|j}|$ *satisfy the weak maximum principle in D.*

Proof. We are going to prove this theorem indirectly. Assume that, for instance, the function $\operatorname{re} u$ does not satisfy the maximum principle, i.e. there exists a point $z_0 \in D$ such that

$$\max_{z\in \operatorname{cl} D} \operatorname{re} u(z) = \operatorname{re} u(z_0), \quad \max_{z\in\partial D} \operatorname{re} u(z) < \operatorname{re} u(z_0).$$

Further, let $M \in \mathscr{L}_{p+k-1}$ be the leaf of the foliation given by Theorem 3, passing through the point z_0. By Theorem 3 the function $\operatorname{re} u|M$ is pluriharmonic, so it satisfies the maximum principle on M and, by connectedness of M, the function $\operatorname{re} u|M$ would be identically equal to $\operatorname{re} u(z_0)$. Since further $\operatorname{cl} M$ is not a compact subset of D (E. Bedford and J. E. Fornaess, to appear), we can find a sequence of $z_n \in M$ which is convergent to some $\tilde{z} \in \partial D$. By the continuity of $\operatorname{re} u|M$ we would have then $\operatorname{re} u(\tilde{z}) = \operatorname{re} u(z_0)$ which contradicts our assumption. Similarly we prove the remaining part of our theorem.

The last two theorems are due to J. Kalina.

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